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LIMIT SETS OF SOME KLEINIAN GROUPS

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1. It is well known that the limit set of the so-called Schottky group, whose fundamental domain is bounded by finitely many mutually disjoint circles, has always 2-dimensional measure zero. Recently Abikoff [1] proved that there exists an infinitely generated Kleinian group whose fundamental domain is bounded by infinitely many mutually disjoint circles and whose limit set is of positive 2-dimensional measure. In this note we shall give a sufficient condition in order that the limit sets of such groups have 2-dimensional measure zero.

2. Let $\{C_i, C_i'\}_{i=1}^{N}$ $(N \leqq +\infty)$ be an at most countable number of mutually disjoint circles in the complex plane $\boldsymbol{C}$ and assume, in the case $N = +\infty$, that these circles cluster to a totally disconnected compact set E in $\boldsymbol{C}$ and that these circles together with the set E bound an unbounded domain F. Let T_i be a hyperbolic or loxodromic linear transformation of $\hat{\boldsymbol{C}}$ onto itself which maps the outside of C_i onto the inside of C_i', where $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ is the Aleksandrov compactification of $\boldsymbol{C}$. Then $G^* = \{T_i\}_{i=1}^{N}$ generates a free discontinuous group G whose fundamental domain is F. In what follows, we call such a group G an S-group. If $N < +\infty$, then an S-group is finitely generated and is a Schottky group. The set $\Lambda(G)$ of all accumulation points of a set $\{\zeta \in \boldsymbol{C} \mid \zeta = V(\infty) \text{ for some } V \in G\}$ is the limit set of G. Unless $N = 1$, the set $\Lambda(G)$ for an S-group contains more than two points and G is (non-elementary) Kleinian by definition (Ford [2]). In the following we shall deal with an infinitely generated S-group, that is, the case $N = +\infty$.

For two elements V and W in G, we denote by VW the composite transformation $VW(z) = V(W(z))$ belonging to G. Since G is free, any $V \in G$ is uniquely represented in the form $V = S_{i_n} S_{i_{n-1}} \cdots S_{i_1}$, where $S_{i_j} \in G^* \cup G^{*-1}$ $(G^{*-1} = \{T_i^{-1}\}_{i=1}^{\infty})$ and $S_{i_{j+1}}^{-1} \neq S_{i_j}$ $(1 \leqq j \leqq n-1)$. Here we call the number n the grade of V. An element of grade n in G is often denoted by $S_{(n)}$. The element $S_{(0)} \in G$ is the identity I of G.

For an S-group G generated by $G^* = \{T_i\}_{i=1}^{\infty}$, let us denote by G_p the Schottky subgroup of G generated by $G_p^* = \{T_i\}_{i=1}^{p}$. The grade of $V^{(p)} \in G_p$ can be defined in the same way as in the case of G. The unbounded domain $F_p(\subset \hat{C})$ surrounded by $\{C_i, C_i'\}_{i=1}^p$ is a fundamental domain of G_p. Consider the image $S_{(n)}^{(p)}(F_p)$ of F_p by $S_{(n)}^{(p)}$ $(\in G_p)$ with grade n. It is easily seen that $S_{(n)}^{(p)}(F_p)\,(n \neq 0)$ is bounded by an outer boundary circle and $(2p-1)$ inner boundary circles which are images of C_i or C_i' $(1 \leqq i \leqq p)$. We shall call a closed disc bounded by an inner boundary circle of $S_{(n)}^{(p)}(F_p)$ a closed disc of grade n with respect to the group G_p. The disc of grade 0 with respect to G_p is a closed disc $[C_i]$ or $[C_i']$ bounded by C_i or C_i', $(1 \leqq i \leqq p)$. It is obvious that the number of all closed discs of grade n with respect to G_p is equal to $q(p, n) = 2p(2p-1)^n$ and every one of them can be represented by $S_{(n)}^{(p)}([C_i])$ or $S_{(n)}^{(p)}([C_i'])$ for some $S_{(n)}^{(p)} \in G_p$ and for some C_i or C_i' $(1 \leqq i \leqq p)$.

3. Let G be an S-group and let $V \in G$ be of the form

$$V: z \mapsto \frac{a_V z + b_V}{c_V z + d_V}, \quad a_V d_V - b_V c_V = 1\,.$$

Since no $V \in G$ $(V \neq I)$ fixes the point $\infty \in \hat{C}$, we see $c_V \neq 0$. Further, $|c_V|^{-1}$ equals the radius of the isometric circle $|c_V z + d_V| = 1$ of V in the sense of Ford [2]. Hence the following lemma holds (cf. Ford [2]).

LEMMA 1. *Let G be an S-group. Then the series*

$$\sum_{I \neq V \in G} \frac{1}{|c_V|^{\mu}}$$

converges for any real number $\mu \geqq 4$.

Next we shall prove another lemma.

LEMMA 2. *Let D be a closed disc with radius r in F. Suppose that, for a linear transformation $V \in G$, c_V is not equal to zero and the pole $V^{-1}(\infty) = -d_V c_V^{-1}$ of V lies outside D. If ρ is the distance of D from $V^{-1}(\infty)$, then 2-dimensional measure $m^2(V(D))$ of the image $V(D)$ of D by V satisfies*

$$m^2(V(D)) = \frac{\pi}{|c_V|^4}\left[\frac{r}{(\rho + r)^2 - r^2}\right]^2.$$

PROOF. Let us denote by $C: |z - z_0| = r$ the boundary circle of D. Obviously $V(D)$ is also bounded by the image circle $V(C)$ of C by V. Letting L be the length of $V(C)$ and putting $\theta = \arg\{(z - z_0)/(V^{-1}(\infty) - z_0)\}$, we have

$$\begin{aligned} L &= \int_C \left|\frac{dV(z)}{dz}\right| |dz| = \int_C \frac{|dz|}{|c_V z + d_V|^2} \\ &= \frac{1}{|c_V|^2}\int_0^{2\pi} \frac{r\,d\theta}{(\rho + r)^2 - 2(\rho + r)r\cos\theta + r^2} = \frac{2\pi}{|c_V|^2}\cdot\frac{r}{(\rho + r)^2 - r^2}\,. \end{aligned}$$

Evidently $m^2(V(D))$ is the area of $V(D)$ and is equal to $L^2/4\pi$. Therefore we have our lemma.

4. Let G be an infinitely generated S-group whose fundamental domain F is an unbounded domain bounded by mutually disjoint circles $\{C_i, C_i'\}_{i=1}^{\infty}$ in $\boldsymbol{C}$ which cluster to only one point $z=0$, the origin of $\boldsymbol{C}$. Denote by $r(C)$ the radius of a circle C in $\boldsymbol{C}$. Then we can prove the following

THEOREM. *Suppose that there exists a numerical constant K satisfying*

$$\sup\left\{\frac{r(C)}{l(C)}\ ;\quad C\in\{C_i, C_i'\}_{i=1}^{\infty}\right\}=K<\infty\ ,$$

where $l(C)=\inf|z-\zeta|$ and the infimum is taken for all $z\in C$ and for all $\zeta\in\{C_i, C_i'\}_{i=1}^{\infty}-C$. Then the limit set $\Lambda(G)$ of the group G has 2-dimensional measure zero.

PROOF. Describe a closed disc $D_{\eta_1}\colon |z|\leqq\eta_1$ in $\boldsymbol{C}$ and pick up all pairs (C_i, C_i') such that at least one of $[C_i]$ and $[C_i']$ contains a point lying outside D_{η_1}. We may assume that all pairs picked up as above are $\{(C_i, C_i')\}_{i=1}^{p_1}$, where p_1 depends on η_1. Put $G_{p_1}^*=\{T_i\}_{i=1}^{p_1}$ and denote by G_{p_1} the group generated by $G_{p_1}^*$. Clearly G_{p_1} is a Schottky subgroup of G. We call G_{p_1} the Schottky subgroup of G associated with η_1. Let us denote by $\{\delta_j^{(p_1,n)}\}_{j=1}^{q(p_1,n)}$, $q(p_1, n)=2p_1(2p_1-1)^n$, the set of discs of grade n with respect to G_{p_1}.

Now we put $k_0=(4K^2+1)/(2K+1)^2$ and take a constant k satisfying $k_0<k<1$. Here K is the numerical constant appeared in the assumption of Theorem. For a given number ε as such as $0<\varepsilon<k/k_0-1$, we determine a positive integer $n_1=n(\eta_1, \varepsilon)$ such that

$$(1)\qquad \sum_{j=1}^{q(p_1,n_1)} m^2(\delta_j^{(p_1,n_1)})=m^2\left(\bigcup_{j=1}^{q(p_1,n_1)}\delta_j^{(p_1,n_1)}\right)<\varepsilon\ .$$

In fact, every $\delta_j^{(p_1,n_1)}$ has the form $S_{(n_1)}^{(p_1)}([C_i])$ or $S_{(n_1)}^{(p_1)}([C_i'])$ for some $S_{(n_1)}^{(p_1)}\in G_{p_1}$ and for a suitable C_i or C_i' and $\delta_j^{(p_1,n_1)}=S_{(n_1)}^{(p_1)}([C_i])$, for instance, implies $S_{(n_1)}^{(p_1)-1}(\infty)\notin[C_i]$. Hence we can apply Lemma 2 to estimate $m^2(\delta_j^{(p_1,n_1)})$ and Lemma 1 yields (1).

Put $\widetilde{D_{\eta_1}}=\bigcup_{i=p_1+1}^{\infty}([C_i]\cup[C_i'])\cup\{0\}$, which is a closure of the set $\bigcup_{i=p_1+1}^{\infty}\{[C_i]\cup[C_i']\}$. Obviously

$$(2)\qquad \Lambda(G)\subset\left(\bigcup_{j=1}^{q(p_1,n_1)}\delta_j^{(p_1,n_1)}\right)\cup\left(\bigcup_{n=0}^{n_1}\bigcup_{(p_1)}S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}})\right).$$

Here $\bigcup_{(p_\lambda)}$ means the union taken over all $S_{(n)}^{(p_\lambda)}\in G_{p_\lambda}$ and this abreviation is used throughout the paper. We choose a number M satisfying

$$(3)\qquad \max\Big(m^2\Big(\bigcup_{n=0}^{n_1}\bigcup_{(p_1)} S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}})\Big), 1\Big) < M .$$

The existence of such an M follows from the fact that the set $S_{(0)}^{(p_1)}(\widetilde{D_{\eta_1}})$ coincides with $\widetilde{D_{\eta_1}}$ and $S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}})$ $(n \neq 0)$ lies inside some C_i or C'_i, $1 \leqq i \leqq p_1$.

Now choose a positive number η_2 $(< \eta_1)$ so small that

i) there is a circle C_i $(p_1 < i)$ outside the open disc $|z| < \eta_2$,

ii) $(d^2\eta_2/r([C_{p_1+1}])l_{p_1}^2)^2 \leqq 1/(2K+1)^2$ for the diameter d of the set $\bigcup_{i=1}^{\infty}\{[C_i] \cup [C'_i]\}$ and the distance l_{p_1} of $\bigcup_{i=1}^{p_1}\{[C_i] \cup [C'_i]\}$ from $\bigcup_{i=p_1+1}^{\infty}\{[C_i] \cup [C'_i]\}$.

Let us determine the number p_2 in the following way: $\{(C_i, C'_i)\}_{i=1}^{p_2}$ is the set of all pairs (C_i, C'_i) such that at least one of $[C_i]$ and $[C'_i]$ contains a point lying outside the closed disc $|z| \leqq \eta_2$.

Let $n_2 = n(\eta_2, k\varepsilon)$ be such a number that

$$(4)\qquad m^2\Big(\bigcup_{j=1}^{q(p_2,n_2)} \delta_j^{(p_2,n_2)}\Big) < k\varepsilon , \qquad q(p_2, n_2) = 2p_2(2p_2-1)^{n_2} ,$$

where $\delta_j^{(p_2,n_2)}$ is a disc of grade n_2 with respect to G_{p_2} and k satisfies $\varepsilon < k/k_0 - 1$ as stated already. By the same reasoning as in the case for G_{p_1}, we have the inclusion relation

$$\Lambda(G) \subset \Big(\bigcup_{j=1}^{q(p_2,n_2)} \delta_j^{(p_2,n_2)}\Big) \cup \Big(\bigcup_{n=0}^{n_2}\bigcup_{(p_2)} S_{(n)}^{(p_2)}(\widetilde{D_{\eta_2}})\Big) ,$$

similar to (2).

For the sake of brevity we put

$$A_\lambda = \bigcup_{j=1}^{q(p_\lambda,n_\lambda)} \delta_j^{(p_\lambda,n_\lambda)} , \qquad B_\lambda = \bigcup_{n=0}^{n_\lambda}\bigcup_{(p_\lambda)} S_{(n)}^{(p_\lambda)}(\widetilde{D_{\eta_\lambda}}) ,$$

for $\lambda = 1, 2$.

It is not so difficult to certify that

$$A_{1,2} = \bigcup_{j=1}^{q(p_1,n_2)} \delta_j^{(p_1,n_2)} \subset A_1 , \qquad q(p_1, n_2) = 2p_1(2p_1-1)^{n_2} ,$$

$$B_{1,2} = \bigcup_{n=0}^{n_2}\bigcup_{(p_1)} S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}}) \subset A_1 \cup B_1 ,$$

$A_\lambda \cap B_\lambda = \varnothing$ $(\lambda = 1, 2)$ and $A_{1,2} \cap B_{1,2} = \varnothing$. Further we can see that $A_2 \cup B_2 \subset A_{1,2} \cup B_{1,2}$.

We shall show that

$$m^2(B_2) \leqq k_0 m^2(B_{1,2}) .$$

For the purpose we consider all the sets $\{{}^jS^{(p_2)}_{(k_j)}(\widetilde{D_{\eta_2}})\}_{j=1}^{N_{n,p_1}}$ contained in a set $S^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}})(\subset B_{1,2})$, where an element $S^{(p_1)}_{(n)} \in G_{p_1}$ $(0 \leqq n \leqq n_2)$ is fixed and $N_{n,p_1} = N_{n,p_1}(S^{(p_1)}_{(n)})$ is a number of sets ${}^jS^{(p_2)}_{(k_j)}(\widetilde{D_{\eta_2}})$ contained in $S^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}})$. Necessarily, $n \leqq k_j \leqq n_2$, and a grade number k_j of some ${}^jS^{(p_2)}_{(k_j)}$ may coincide to each other.

If $n < k_j$, then every ${}^jS^{(p_2)}_{(k_j)}([C_i])$ and ${}^jS^{(p_2)}_{(k_j)}([C'_i])$ $(p_2 < i)$ are contained in a certain $S^{(p_1)}_{(n)}([C_{i'}])$ or $S^{(p_1)}_{(n)}([C'_{i'}])$ $(p_1 < i' \leqq p_2)$ which is a subset of $S^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}})$. Hence for a concentric disc Γ_i; $|z - z_i| \leqq r(C_i)(1 + 1/2K)$ of $[C_i]$ or Γ'_i; $|z - z'_i| \leqq r(C'_i)(1 + 1/2K)$ of $[C'_i]$, we easily see

$$ {}^jS^{(p_2)}_{(k_j)}([C_i]) \subset {}^jS^{(p_2)}_{(k_j)}(\Gamma_i) \subset S^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}}) \,, \qquad p_2 < i \,, $$

$$ {}^jS^{(p_2)}_{(k_j)}([C'_i]) \subset {}^jS^{(p_2)}_{(k_j)}(\Gamma'_i) \subset S^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}}) \,, \qquad p_2 < i \,, $$

and

$$ \begin{aligned} {}^jS^{(p_2)}_{(k_j)}(\Gamma_i) \cap {}^jS^{(p_2)}_{(k_j)}(\Gamma'_i) &= {}^jS^{(p_2)}_{(k_j)}(\Gamma_i) \cap {}^jS^{(p_2)}_{(k_j)}(\Gamma'_{i'}) \\ &= {}^jS^{(p_2)}_{(k_j)}(\Gamma_{i'}) \cap {}^jS^{(p_2)}_{(k_j)}(\Gamma'_{i'}) = \varnothing \,, \qquad p_2 < i \,. \end{aligned} $$

Further the pole of ${}^jS^{(p_2)}_{(k_j)}$ is outside of $\bigcup_{i=p_2+1}^{\infty} \{\Gamma_i \cup \Gamma'_i\} \cup \{0\}$. Hence from Lemma 2, we have

$$ \begin{aligned} &\frac{m^2({}^jS^{(p_2)}_{(k_j)}([C_i]))}{m^2({}^jS^{(p_2)}_{(k_j)}(\Gamma_i))} \\ &\quad = \left[\frac{r(C_i)}{(\rho + r(C_i))^2 - r(C_i)^2} \cdot \frac{(\rho + r(C_i))^2 - r(C_i)^2(1 + 1/2K)^2}{r(C_i)(1 + 1/2K)}\right]^2 \\ &\quad \leqq \frac{4K^2}{(2K+1)^2} \,. \end{aligned} $$

For $[C'_i]$ $(i > p_2)$, we obtain the quite similar estimate

$$ \frac{m^2({}^jS^{(p_2)}_{(k_j)}([C'_i]))}{m^2({}^jS^{(p_2)}_{(k_j)}(\Gamma'_i))} \leqq \frac{4K^2}{(2K+1)^2} \,. $$

Next we consider the case $n = k_j$. In this case, it is seen that ${}^jS^{(p_2)}_{(k_j)}(\widetilde{D_{\eta_2}}) = {}^jS^{(p_1)}_{(n)}(\widetilde{D_{\eta_2}})$. Since the pole of $S^{(p_1)}_{(n)}$ lies inside $\bigcup_{i=1}^{p_1} ([C_i] \cup [C'_i])$ and from the properties (i), (ii) of η_2, we have

$$ \begin{aligned} \frac{m^2({}^jS^{(p_2)}_{(n)}(\widetilde{D_{\eta_2}}))}{m^2({}^jS^{(p_1)}_{(n)}(\widetilde{D_{\eta_1}}))} &\leqq \frac{\sum_{p_2 < i} m^2({}^jS^{(p_2)}_{(n)}([C_i] \cup [C'_i])}{m^2(S^{(p_1)}_{(n)}([C_{p_1+1}]))} \\ &\leqq \left(\frac{d^2}{r([C_{p_1+1}])} \cdot \frac{\eta_2}{l^2_{p_1}}\right)^2 \leqq \frac{1}{(2K+1)^2} \,. \end{aligned} $$

Therefore, for a set $S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}})$ which appears in $B_{1,2}$ we get

$$\frac{m^2\left(\bigcup_{j=1}^{N_{n,p_1}} {}^jS_{(k_j)}^{(p_2)}(\widetilde{D_{\eta_2}})\right)}{m^2(S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}}))}$$

$$\leqq \frac{\sum_{k_j\neq n}\sum_{p_2<i} m^2({}^jS_{(k_j)}^{(p_2)}([C_i]\cup[C_i']))}{\sum_{k_j\neq n}\sum_{p_2<i} m^2({}^jS_{(k_j)}^{(p_2)}(\Gamma_i\cup\Gamma_i'))} + \frac{m^2({}^jS_{(n)}^{(p_2)}(\widetilde{D_{\eta_2}}))}{m^2(S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}}))} \leqq \frac{4K^2+1}{(2K+1)^2} = k_0 .$$

Since the set B_2 can be obtained as a union $\bigcup_{n=0}^{n_2}\bigcup_{(p_1)}\bigcup_{j=1}^{N_{n,p_1}} {}^jS_{(k_j)}^{(p_2)}(\widetilde{D_{\eta_2}})$, it follows that

$$\frac{m^2(B_2)}{m^2(B_{1,2})} = \frac{\sum_{n=0}^{n_2}\sum' m^2\left(\bigcup_{j=1}^{N_{n,p_1}} {}^jS_{(k_j)}^{(p_2)}(\widetilde{D_{\eta_2}})\right)}{\sum_{n=0}^{n_2}\sum' m^2(S_{(n)}^{(p_1)}(\widetilde{D_{\eta_1}}))} \leqq k_0 ,$$

where $\sum'$ means the sum for all $S_{(n)}^{(p_1)} \in G_{p_1}$ and $N_{n,p_1} = N_{n,p_1}(S_{(n)}^{(p_1)})$. Thus we can see

$$m^2(B_2) \leqq k_0 m^2(B_{1,2}) .$$

Therefore, it holds from (1), (3) and $\varepsilon < k/k_0 - 1$ that

$$\begin{aligned} m^2(B_2) &\leqq k_0 m^2(A_1\cup B_1) = k_0(m^2(A_1) + m^2(B_1)) \\ &< k_0(\varepsilon + M) < k - k_0 + k_0M = kM + (k-k_0)(1-M) \leqq kM , \end{aligned}$$

which together with (4) implies

$$m^2(A_2\cup B_2) = m^2(A_2) + m^2(B_2) < k(\varepsilon + M) .$$

Repeat the same procedure. Then we get the sequence $\{\eta_\lambda\}_{\lambda=1}^{\infty}$ of positive numbers such that $\eta_\lambda < \eta_{\lambda-1}$, $\lim_{\lambda\to\infty}\eta_\lambda = 0$ and such that for the Schottky subgroup G_{p_λ} of G associated with η_λ, the estimate

$$m^2(A_\lambda\cup B_\lambda) < k^{\lambda-1}(\varepsilon + M)$$

holds, where A_λ is the union $\bigcup_{j=1}^{q(p_\lambda,n_\lambda)} \delta_j^{(p_\lambda,n_\lambda)}$, $q(p_\lambda, n_\lambda) = 2p_\lambda(2p_\lambda - 1)^{n_\lambda}$, of discs with grade $n_\lambda = n_\lambda(\eta_\lambda, k^{\lambda-1}\varepsilon)$ with respect to G_{p_λ} and $B_\lambda = \bigcup_{n=0}^{n_\lambda}\bigcup_{(p_\lambda)} S_{(n)}^{(p_\lambda)}(\widetilde{D_{\eta_\lambda}})$. Clearly $\Lambda(G)\subset A_\lambda\cup B_\lambda$ so that

$$m^2(\Lambda(G)) < k^{\lambda-1}(\varepsilon + M) .$$

Since $0 < k < 1$ and λ is arbitrary, we have our Theorem.

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