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THE UNSTABLE DIFFERENCE BETWEEN HOMOLOGY COBORDISM AND PIECEWISE LINEAR BLOCK BUNDLES

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0. Introduction and statement of results. N. Martin and C. R. F. Maunder [9] developed the theory of homology cobordism bundles which is an adequate bundle theory in the category of polyhedral homology manifolds. They introduced certain Δ-sets $H(n)$ which play the role of "structure groups" in the bundle theory. A typical k-simplex of $H(n)$ is a homology cobordism bundle-automorphism of the product bundle $\Delta^k \times S^{n-1}$, or equivalently, a homology cobordism bundle over $\Delta^k \times I$ which is the product bundle over $\Delta^k \times \{0, 1\}$. According to N. Martin [10], the structure groups $\widetilde{PL}(n)$ of PL n-block bundles are homotopically equivalent to sub-Δ-sets $\overline{PL}(n)$ of $H(n)$. By definition a typical k-simplex of $\overline{PL}(n)$ is a PL n-block bundle over $\Delta^k \times I$ which is the product bundle over $\Delta^k \times \{0, 1\}$.

Our main result is the following

THEOREM 1. *If $n \geqq 3$, we have*

$$\pi_k(H(n), \overline{PL}(n)) = \begin{cases} 0 & (k \neq 3) \\ \mathscr{H}^3 & (k = 3) \end{cases},$$

where $\mathscr{H}^3$ is the abelian group of PL H-cobordism classes of oriented PL homology 3-spheres.

This improves the result of [10] in the unstable ranges. Theorem 1 will be proved in §1.

Now for the case $n = 2$, let $\mathscr{G}_k$ be the ordinary knot cobordism group of PL $(k, k+2)$-sphere pairs and let $\mathscr{G}_k^H$ be the knot cobordism group of PL homology $(k, k+2)$-sphere pairs; any element of $\mathscr{G}_k^H$ is represented by a locally flat pair (M^k, N^{k+2}) consisting of oriented PL homology k- and $(k+2)$-spheres. Such pairs (M_1^k, N_1^{k+2}) and (M_2^k, N_2^{k+2}) represent the same element of $\mathscr{G}_k^H$ if and only if the connected sum $(M_1^k \# - M_2^k, N_1^{k+2} \# - N_2^{k+2})$ bounds a locally flat pair of acyclic manifolds (V^{k+1}, W^{k+3}). Also $\mathscr{G}^{AH}$ denotes the subgroup of $\mathscr{G}_1^H$ whose element is represented by a pair $(M^1,$

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N^3) such that N^3 bounds an acyclic 4-manifold. It is easy to show that $0 \to \mathscr{G}^{AH} \to \mathscr{G}_1^H \to \mathscr{H}^3 \to 0$ is a split exact sequence. Then our result for $n = 2$ is stated as follows.

THEOREM 2. *We have*

$$\pi_k(H(2), \overline{PL}(2)) \cong \begin{cases} \mathscr{G}_k & (k \geqq 4) \\ 0 & (k = 2) \\ \mathscr{G}^{AH} & (k = 1) \end{cases}$$

and for $k = 3$, there is an exact sequence

$$0 \to \mathscr{G}_3 \to \pi_3(H(2), \overline{PL}(2)) \to \mathscr{H}^3 \to 0 .$$

We shall prove Theorem 2 in §2 after studying some kinds of knot cobordism groups. Note that $\mathscr{G}_k = 0$ for even k and the following proposition.

PROPOSITION 3. *Suppose $k \geqq 2$ and $k \neq 3$, then the natural homomorphism $\psi_k : \mathscr{G}_k \to \mathscr{G}_k^H$ is an isomorphism. If $k = 3$, we have an exact sequence*

$$0 \to \mathscr{G}_3 \to \mathscr{G}_3^H \to \mathscr{H}^3 \to 0 .$$

REMARK 4. *For the case $n = 1$, it is easy to see that $\pi_k(H(1), \overline{PL}(1)) = 0$ for any $k \geqq 1$.*

In §3 we shall introduce a $\varDelta$-set RN_2 which plays the role of the "structure $\varDelta$-set" of the bundle theory of codimension 2 regular neighbourhoods. This bundle theory has been considered by Cappell and Shaneson [2]. Then the $\varDelta$-set RN_2 will be regarded as an intermediate $\varDelta$-set between $H(2)$ and $\overline{PL}(2)$.

THEOREM 5. *We have*

$$\pi_k(RN_2, \overline{PL}(2)) \cong \mathscr{G}_k$$

and

$$\pi_k(H(2), RN_2) \cong \begin{cases} 0 & (k \geqq 4) \\ \mathscr{H}^3 & (k = 3) \\ \mathscr{K} & (k = 2) \\ \mathscr{K}' & (k = 1) , \end{cases}$$

where $\mathscr{K}$ and $\mathscr{K}'$ are the kernel group and the cokernel group of the natural homomorphism $\psi : \mathscr{G}_1 \to \mathscr{G}^{AH}$ respectively. (*We do not know whether $\mathscr{K}$ or $\mathscr{K}'$ are trivial or not.*)

1. Proof of Theorem 1. Throughout this paper, we use the same notation as N. Martin [10]. Let $PLH(n)$ be an intermediate Kan Δ-set of which a typical k-simplex is a block-preserving H-cobordism by PL-manifolds between $\Delta^k \times S^{n-1}$ and itself (See [10], pp. 200–201.).

LEMMA 1.1. *For $n \geqq 2$ we have*

$$\pi_k(H(n), PLH(n)) \cong \begin{cases} 0 & (k \neq 3 \quad and \quad n+k \neq 4) \\ \mathscr{H}^3 & (k = 3 \quad or \quad n+k=4) \, . \end{cases}$$

Moreover unless $k = 3$, the natural homomorphism

$$\pi_k(H(n), \overline{PL}(n)) \longrightarrow \pi_k(H(n), PLH(n))$$

is a zero map.

PROOF. (Cf. [10], Lemma 2.) According to Martin [10], any element α of $\pi_k(H(n), PLH(n))$ is representable as a homology cobordism S^{n-1}-bundle over $\Delta^k \times I$ with the total space G, which is a block preserving PL H-cobordism over $\partial\Delta^k \times I$ and is the product bundle over $\Delta^k \times \{0, 1\} \cup \Delta^{k-1} \times I$ where Δ^{k-1} is a $(k-1)$-face of Δ^k. G is an oriented connected homology $(n+k)$-manifold with PL boundary. Recall that there is the obstruction theory to resolving the singularities of G to make it a PL manifold [13], [1], [14]. It tells us that there exists a well-defined obstruction element $\lambda(G, \partial G) \in H^4(G, \partial G; \mathscr{H}^3)$ which vanishes if and only if G is H-cobordant relative the boundary to a PL manifold G'. (For the obstruction theory in this form we refer to Proposition in [10] at p. 199.)

N. Martin proved that $\pi_k(H(n), PLH(n)) = 0$ assuming that $k \neq 3$ and $n + k \neq 4$. Indeed under this assumption we have $H^4(G, \partial G; \mathscr{H}^3) = 0$, so G is H-cobordant relative the boundary to a PL H-cobordism, that is, $\alpha = [G] = 0$.

Now we assume that $k = 3$ and $n \geqq 2$. Then, given a fixed orientation on $\Delta^3 \times S^{n-1} \times \{0\}$, the obstruction theory gives an element $\lambda(\alpha) = \lambda(G, \partial G)$ of $\mathscr{H}^3 = H^4(G, \partial G; \mathscr{H}^3)$. This homomorphism $\lambda: \pi_3(H(n), PLH(n)) \longrightarrow \mathscr{H}^3$ is proved to be surjective because $C\Sigma^3 \times S^{n-1}$ represents an element of $\pi_3(H(n), PLH(n))$ with $\lambda([C\Sigma^3 \times S^{n-1}]) = [\Sigma^3]$ (See [10], p. 203.). On the other hand, $\lambda(G, \partial G) = 0$ implies that G is H-cobordant relative the boundary to a PL H-cobordism, so λ is injective, and hence bijective.

In order to complete the proof, it remains to show Lemma in the case when $n + k = 4$. Now suppose that $n + k = 4$, then the singularities of G to be resolved consist of a finite number of points $p_1, \cdots, p_n$ in $\operatorname{Int} G$. Let $St(p_i, G)$ be a star neighbourhood of p_i in $\operatorname{Int} G$. Construct a boundary connected sum of them within G along suitable arcs:

$$St(p_1, G)\natural \cdots \natural St(p_r, G) .$$

Denote the resulting manifold by M^4. The boundary ∂M^4, which is an oriented PL homology 3-sphere, represents $\lambda([G]) \in \mathscr{H}^3 \cong H^4(G, \partial G; \mathscr{H}^3)$.

A boundary connected sum $G' = G \natural C\Sigma^3$ along a 3-disk over $(\partial\Delta^k - \Delta^{k-1}) \times I$ gives a new element

$$[G'] \in \pi_k(H(n), PLH(n))$$

with

$$\lambda([G']) = [\partial M^4] + [\Sigma] \in \mathscr{H}^3 \cong H^4(G', \partial G'; \mathscr{H}^3) .$$

Therefore, if $n + k = 4$, $\lambda\colon \pi_k(H(n), PLH(n)) \rightarrow \mathscr{H}^3$ is surjective and hence bijective, because λ is injective by the obstruction theory.

Suppose now that the element $[G] \in \pi_k(H(n), PLH(n))$ is in the image of $\pi_k(H(n), \overline{PL}(n)) \rightarrow \pi_k(H(n), PLH(n))$ with $n + k = 4$. Then the restriction $G|\partial(\Delta^k \times I)$ is a PL block S^{n-1}-bundle, and it is extended to a PL block n-disk bundle η over $\partial(\Delta^k \times I)$ with the total space $E(\eta)$. Gluing $E(\eta)$ to G along $\partial G = \partial E(\eta)$, we obtain a homology 4-sphere $X^4 = G \cup E(\eta)$. Let $W^4 = \mathrm{cl}[X^4 - M^4]$. Then W^4 is an acyclic PL manifold with $\partial W^4 = -\partial M^4$, so by the definition of $\mathscr{H}^3$, we have $\lambda([G]) = [\partial M^4] = 0$. Therefore, $[G] = 0$ by the bijectivity of λ. This completes the proof of Lemma 1.1. q.e.d.

LEMMA 1.2. (Cf. [10], Lemma 1.) *If $k \geqq 1$ and $n \geqq 3$, $\pi_k(PLH(n), \overline{PL}(n)) = 0$.*

PROOF. If $k = 1$, $n \geqq 3$, this lemma is an implication of Lemma 1 in [10]. Hereafter, we may suppose that $k \geqq 2$ and $n \geqq 3$. Any element α of $\pi_k(PLH(n), \overline{PL}(n))$ may be represented by a PL H-cobordism G between $\Delta^k \times S^{n-1}$ and itself, which is a PL block-bundle over $\partial\Delta^k \times I$ and which is the product bundle over $\Delta^{k-1} \times I$ for a $(k-1)$-face Δ^{k-1} of Δ^k. Let $P' = (p, 0) \times S^{n-1}$, which is contained in ∂G, where $(p, 0)$ is a point of $\Delta^k \times \{0\}$. By pushing P' slightly into the interior of G, $\mathrm{Int}\, G$, we obtain a submanifold P of $\mathrm{Int}\, G$ with a trivial normal bundle. Clearly the inclusion $i\colon P \subsetneq G$ induces an isomorphism of cohomology groups with arbitrary coefficients. Let w_2 denote the 2-nd Stiefel-Whitney class, then $i^*w_2(G) = w_2(P) = 0$. This implies $w_2(G) = 0$, for i^* is an isomorphism. Recall here the following lemma due to Kato, who proved it in a more general setting. For the proof, refer to Kato [6].

LEMMA 1.3. (Kato [6], Lemma 3.4.) *Let G be a compact PL q-manifold, P a connected sub-polyhedron of G with $\pi_1(P) = \{1\}$ or Z. Suppose $q \geqq 5$, $w_2(G) = 0$ and $H_i(G, P; Z) = 0$ for $i \leqq 2$. Then one can attach to* $\mathrm{Int}\, G \times \{1\} \subset G \times I$ *a finite number of handles of indices $\leqq 3$ to form* *a PL* $(q+1)$*-manifold* U *and a PL* q*-manifold* $G' = \mathrm{cl}\,[\partial U - G \times \{0\}]$ *such that* $\pi_1(G') \cong \pi_1(U) \cong \pi_1(P)$ *and* $H_i(U, G \times \{0\}) \cong H_i(U, G') = 0$ *for all* $i \geqq 0$.

REMARK. For our purpose in this section, it is sufficient to restrict ourselves to the case $\pi_1(P) = \{1\}$. However, in §2, we will have to consider the case when $\pi_1(P) \cong Z$.

PROOF OF LEMMA 1.2. (Continued) By Lemma 1.3, G is PL H-cobordant relative the boundary to a simply-connected PL manifold G'. Clearly, G' is a PL h-cobordism between $\Delta^k \times S^{n-1}$ and itself. Let η be the PL block n-disk bundle over $\partial(\Delta^k \times I)$ constructed by conical extension from the block S^{n-1}-bundle $G|\partial(\Delta^k \times I)$. As before, let $E(\eta)$ denote the total space of η. Gluing $E(\eta)$ to G' along $\partial G'$, we obtain a closed $(n+k)$-manifold $\Sigma = E(\eta) \cup G'$. By a simple calculation Σ is a PL homotopy sphere, and so by the h-cobordism theorem it is a natural sphere. (N.B. $n + k \geqq 5$.) The k-sphere $\partial(\Delta^k \times I)$ is regarded as a locally flat submanifold of $E(\eta)$ and hence of Σ. Since the codimension n is greater than or equal to 3, the sphere pair $(\Sigma, \partial(\Delta^k \times I))$ is PL homeomorphic to the standard sphere pair (Zeeman's unknotting theorem). Therefore, we may find a locally flat PL embedding $e\colon (\Delta^k \times I, \partial(\Delta^k \times I)) \to (D, \Sigma)$ extending the inclusion $\partial(\Delta^k \times I) \subsetneq \Sigma$, where D is an $(n+k+1)$-disk bounded by Σ. Let N be a normal PL block disk bundle of $(\Delta^k \times I)$ in D. It is easy to see that the associated PL block S^{n-1}-bundle N_0 represents the same element as α. However, clearly N_0 represents the zero element of $\pi_k(PLH(n), \overline{PL}(n))$. Thus $\alpha = 0$. This completes the proof of Lemma 1.2.

q.e.d.

PROOF OF THEOREM 1. Consider the exact sequence

$$\pi_k(PLH(n), \overline{PL}(n)) \xrightarrow{i} \pi_k(H(n), \overline{PL}(n)) \xrightarrow{j} \pi_k(H(n), PLH(n)) \to \pi_{k-1}(PLH(n), \overline{PL}(n)) .$$

By Lemma 1.2, the first group is a trivial group for $n \geqq 3$, $k \geqq 1$. On the other hand, Lemma 1.1 states that $j = 0$ for $k \neq 3$. Therefore, we have

$$\pi_k(H(n), \overline{PL}(n)) = 0 \quad \text{for} \quad n \geqq 3, k \neq 3 .$$

For the case $k = 3$ and $n \geqq 3$, Lemma 1.2 states that the first group and the last group are trivial. Therefore, we get that $\pi_3(H(n), \overline{PL}(n)) \cong \pi_3(H(n), PLH(n))$, while the latter group is isomorphic to $\mathscr{H}^3$ by Lemma 1.1.

q.e.d.

2. Some kinds of knot cobordism groups and proof of Theorem 2.

In the proof of Lemma 1.2, $\pi_k(PLH(n), \overline{PL}(n))$ is considered to be the knot cobordism group of pairs of a PL k-sphere locally flatly embeded in a PL homology $(k+n)$-sphere; any element of $\pi_k(PLH(n), \overline{PL}(n))$ is representable as a locally flat pair (Σ^k, N^{k+n}) consisting of oriented PL k-sphere and oriented PL homology $(k+n)$-sphere. Such pairs (Σ_1^k, N_1^{k+n}) and (Σ_2^k, N_2^{k+n}) represent the same element of $\pi_k(PLH(n), \overline{PL}(n))$ if and only if the connected sum $(\Sigma_1^k \# -\Sigma_2^k, N_1^{k+n} \# -N_2^{k+n})$ bounds a locally flat pair of $(k+1)$-disk and PL acyclic $(k+n+1)$-manifold (D^{k+1}, W^{k+n+1}).

Now we restrict ourselves to the case when $n=2$. (The above observation remains true in this case.)

LEMMA 2.1. *We have*

$$\pi_2(PLH(2), \overline{PL}(2)) = \mathscr{G}_2^H = 0 .$$

More precisely, let (M^2, N^4) *be any representative of an element of* $\pi_2(PLH(2), \overline{PL}(2))$ *or of* $\mathscr{G}_2^H$, *and let* W^5 *be a contractible manifold bounded by* N^4. *(Such* W *always exists.) Then, there exists a* 3*-disk* D^3 *which is embedded in* W^5 *locally flatly and such that* $\partial D^3 = M^2$.

PROOF OF LEMMA 2.1. The proof is essentially the same as that of THÉORÈME III. 6 in [7]. The argument of pp. 265–266 in [7] can be applied to our situation without any essential change: Let K^3 be a locally flat oriented submanifold of N^4 such that $\partial K^3 = M^2$, and let D^3 be a 3-disk. We construct an orientable closed 3-manifold L^3 from the disjoint union $K^3 \cup D^3$ by identifying the boundaries. L^3 bounds a parallelizable 4-manifold P^4 which admits a handle-body decomposition of the form

$$P^4 = L^3 \times I + \sum_i (\varphi_i^1) + \sum_j (\varphi_j^2) + \sum_k (\varphi_k^3) + (\varphi^4) .$$

We may assume that $\varphi_i^1, \varphi_j^2, \varphi_k^3$ are disjoint from $D^3 \times I \subset L^3 \times I$, and we obtain a manifold with corners

$$P_0^4 = K^3 \times I + \sum_i (\varphi_i^1) + \sum_j (\varphi_j^2) + \sum_k (\varphi_k^3) .$$

Let X_p denote the *sub*-handlebody of P_0^4 consisting of handles of indices $\leqq p$. By the general position argument, the embedding $K^3 \to N^4$ can be extended to the embedding $X_2 \to W^5$. The boundary ∂X_2 is the union of K^3, $\partial K^3 \times I$ and Y^3. Here Y^3 is PL homeomorphic with the connected sum of finite number of copies of $S^1 \times S^2$ minus a 3-disk. We may assume that $\partial Y^3 = M^2$. Again by the general position argument, it is shown that the spherical modification starting with the canonical system of generators of $\pi_1(Y^3)$ is realizable as a modification within W^5. After the modification, we obtain a desired 3-disk D^3 in W^5 such that $\partial D^3 = M^2$. This completes the proof of Lemma 2.1. q.e.d.

LEMMA 2.2. *If $k \geqq 2$, we have*

$$\pi_k(PLH(2), \overline{PL}(2)) \cong \mathscr{G}_k .$$

We consider here $\pi_k(PLH(2), \overline{PL}(2))$ to be the knot cobordism group of pairs of PL k-spheres embedded locally flatly in PL homology $(k+2)$-spheres. Take the natural homomorphisms, $\varphi_k: \mathscr{G}_k \to \pi_k(PLH(2), \overline{PL}(2))$ and $\tau_k: \pi_k(PLH(2), \overline{PL}(2)) \to \mathscr{G}_k^H$. Remark that $\psi_k = \tau_k \circ \varphi_k$. Now we prove Lemma 2.2. and Proposition 3 of § 0 simultaneously.

PROOF OF LEMMA 2.2 AND PROPOSITION 3. Since $\mathscr{G}_2 = \pi_2(PLH(2), \overline{PL}(2)) = \mathscr{G}_2^H = 0$ by Lemma 2.1., we may assume that $k \geqq 3$. The proof is devided into several steps.

1) *If $k \geqq 3$, ψ_k is injective and hence so is φ_k.* Since $\mathscr{G}_k \cong 0$ for even k [7], we may assume $k = 2n-1$. Let $(\Sigma^{2n-1}, S^{2n+1})$ be a representative of an element of $\mathscr{G}_{2n-1}$ which belongs to the kernel of ψ_{2n-1}. Then it bounds a locally flat pair (V^{2n}, W^{2n+2}) of acyclic manifolds. Let K^{2n} be the oriented submanifold of S^{2n+1} bounded by Σ^{2n-1}, and let L^{2n} be the manifold obtained from the union $K^{2n} \cup V^{2n}$ by identifying the boundaries. L^{2n} bounds a submanifold Y^{2n+1} of W^{2n+2} by the Pontrjagin-Thom construction. Let $\theta: H_n(K^{2n}) \times H_n(K^{2n}) \to Z$ be the pairing defined by Levine [8] from which the Seifert matrix A is defined. Then the same argument as in § 8 of [8, pp. 232–233] works equally well in our situation, and one can prove that θ vanishes on the subspace Ker (inclusion$_*$: $H_n(K^{2n}) \to H_n(Y^{2n+1})$), and that the subspace has half a rank of $H_n(K^{2n})$. Therefore, the associated Seifert matrix A is null-cobordant in the sense of Levine, and by Lemmas 4 and 5 in [8], $(\Sigma^{2n-1}, S^{2n+1})$ is null-cobordant in the usual sense.

REMARK. Step 1) may be proven more formally by making use of the results of [11].

2) *If $k \geqq 4$, τ_k is surjective.* Let (M^k, N^{k+2}) be a representative of an element of $\mathscr{G}_k^H$. Since $k \geqq 4$, M^k is PL H-cobordant to a natural k-sphere Σ^k, so by virtue of the cobordism extension property, M^k itself may be assumed to be the k-sphere Σ^k.

3) *If $k \geqq 3$, φ_k is surjective.* Let U be the regular neighbourhood of Σ^k in N^{k+2}, and E the exterior of U in N^{k+2}; $E = \mathrm{cl}\,[N^{k+2} - U]$. By Kato's lemma (Lemma 1.3), E is PL H-cobordant relative the boundary to a PL-manifold E' with $\pi_1(E') \cong Z$. Identifying the boundaries, we obtain a PL homotopy $(k+2)$-sphere $E' \cup U$ which is, by the h-cobordism theorem, a natural sphere S^{k+2}. Hence $(\Sigma^k, N^{k+2}) = \varphi_k([\Sigma^k, S^{k+2}])$.

Remark that $\psi_k = \tau_k \circ \varphi_k$ is surjective for $k \geqq 4$ by 2) and 3).

4) *There is an exact sequence*: $0 \to \mathscr{G}_3 \xrightarrow{\psi_3} \mathscr{G}_3^H \to \mathscr{H}^3 \to 0$. A homomorphism $\sigma: \mathscr{G}_3^H \to \mathscr{H}^3$ is defined by sending an element $[(M^3, N^5)] \in \mathscr{G}_3^H$ to the element of $\mathscr{H}^3$ represented by M^3. From Step 1) and the arguments in 2) and 3), the exactness of the sequence $0 \to \mathscr{G}_3 \xrightarrow{\psi_3} \mathscr{G}_3^H \to \mathscr{H}^3$ follows immediately. However, any homology 3-sphere can be embedded in S^5 (See for example [4].), so σ is surjective. The proof of 4) is completed. Lemma 2.2 follows from 1) and 3), and Proposition 3 follows from 1), 2), 3) and 4). q.e.d.

For the case $k = 1$, since a PL homology 1-sphere is an 1-sphere and a PL acyclic 2-manifold is a 2-disk, the knot cobordism interpretation of $\pi_1(PLH(2), \overline{PL}(2))$ coincides with $\mathscr{G}_1^H$, that is,

LEMMA 2.3.

$$\pi_1(PLH(2), \overline{PL}(2)) \cong \mathscr{G}_1^H .$$

Now we are in a position to prove Theorem 2.

PROOF OF THEOREM 2. We consider the homotopy long exact sequence of a triple, $(H(2), PLH(2), \overline{PL}(2))$.

1) First for $k \geqq 4$, since $\pi_k(H(2), PLH(2)) = 0$ by Lemma 1.1, we get an exact sequence

$$0 \to \pi_k(PLH(2), \overline{PL}(2)) \to \pi_k(H(2), \overline{PL}(2)) \to 0 .$$

Therefore, $\pi_k(H(2), \overline{PL}(2)) \cong \mathscr{G}_k$ for $k \geqq 4$ by Lemma 2.2.

2) For the case $k = 3$, since $\pi_4(H(2), PLH(2)) = 0$ and $\pi_3(H(2), PLH(2)) \cong \mathscr{H}^3$ by Lemma 1.1 and $\pi_2(PLH(2), \overline{PL}(2)) = 0$ by Lemma 2.1, we get an exact sequence

$$0 \to \pi_3(PLH(2), \overline{PL}(2)) \to \pi_3(H(2), \overline{PL}(2)) \to \mathscr{H}^3 \to 0 .$$

Replacing $\pi_3(PLH(2), \overline{PL}(2))$ with $\mathscr{G}_3$ by virtue of Lemma 2.2, we get the desired exact sequence

$$0 \to \mathscr{G}_3 \to \pi_3(H(2), \overline{PL}(2)) \to \mathscr{H}^3 \to 0 .$$

3) For $k = 2$, we consider the following exact sequence

$$\pi_2(PLH(2), \overline{PL}(2)) \xrightarrow{i} \pi_2(H(2), \overline{PL}(2)) \xrightarrow{j} \pi_2(H(2), PLH(2)) .$$

Then, since the first group is a trivial group because of Lemma 2.1 and j is a zero map by Lemma 1.1, we get that

$$\pi_2(H(2), \overline{PL}(2)) = 0 .$$

4) For $k = 1$, by Lemma 1.1 and Lemma 2.3 we get a following

commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to \pi_2(H(2), PLH(2)) & \to & \pi_1(PLH(2), \overline{PL}(2)) & \to & \pi_1(H(2), \overline{PL}(2)) \to 0 \\ \lambda \downarrow \cong & & \Big| \cong & & \\ 0 \longrightarrow \mathscr{H}^3 & \xrightarrow{\quad i \quad} & \mathscr{G}_1^H . & & \end{array}$$

Since $\lambda([\Delta^2 \times I \times S^1 \natural C\Sigma]) = [\Sigma]$, we know that $i([\Sigma])$ is the class of the trivial knot connected summed with Σ in the ambient space. We define a map $j: \mathscr{G}_1^H \to \mathscr{G}^{AH}$ by $j(\Sigma^1 \subset \Sigma^3) = \Sigma^1 \subset \Sigma^3 \# -\Sigma^3$, then, $0 \to \mathscr{H}^3 \xrightarrow{i} \mathscr{G}_1^H \xrightarrow{j} \mathscr{G}^{AH} \to 0$ is an exact sequence, because $j \circ i$ is clearly a zero map and $\Sigma^1 \subset \Sigma^3 \# - \Sigma^3 = 0$ means that $[\Sigma^1 \subset \Sigma^3] - i([\Sigma^3]) = 0$.

Therefore, there exists a natural homomorphism: $\pi_1(H(2), \overline{PL}(2)) \to \mathscr{G}^{AH}$ which is seen to be an isomorphism by the 5-Lemma.

Note that the natural inclusion $i_0: \mathscr{G}^{AH} \to \mathscr{G}_1^H$ makes the above sequence split because $j \circ i_0 = id$. q.e.d.

3. Bundle theory for codimension two regular neighbourhoods. In this section, we will briefly describe a block-bundle theory for codimension two regular neighbourhoods. A definition of a Δ-set RN_2 will be given, and the relationship between RN_2 and $H(2)$ will be studied. RN_2 plays the role of the structure Δ-set for the block-bundle theory. (Cf. Cappell and Shaneson [2].)

The definition of the block-bundle is quite analogous to the usual one given in [5] or [12].

Let K be a PL cell complex.

DEFINITION 3.1. An *RN_2-bundle* ξ over K consists of a polyhedron $E(\xi)$ called the total space, the base complex K and a PL embedding $\iota: |K| \to E(\xi)$ called a cross section. The following conditions are to be satisfied:

(i) For each n-cell $\sigma_i \in K$, there exists an $(n+2)$-ball $\beta_i \subset E(\xi)$ such that $\iota(\sigma_i, \partial\sigma_i) \subset (\beta_i, \partial\beta_i)$, and such that the restriction $\iota|(\sigma_i, \partial\sigma_i): (\sigma_i, \partial\sigma_i) \to (\beta_i, \partial\beta_i)$ is a proper PL embedding. (N.B. ι is not necessarily locally flat.) β_i is called the *block* over σ_i.

(ii) $E(\xi)$ is the union of the blocks β_i.

(iii) The interiors of the blocks are disjoint.

(iv) Let $L = \sigma_i \cap \sigma_j$, then $\beta_i \cap \beta_j$ is the union of the blocks over the cells of L.

DEFINITION 3.2. Two RN_2-bundles ξ, η over K are *isomorphic* if there exists a PL homeomorphism $h: E(\xi) \to E(\eta)$ such that $h \circ \iota_\xi = \iota_\eta$, and such that for each cell $\sigma_i \in K$, $h(\beta_i(\xi)) = \beta_i(\eta)$. Notation: $\xi \cong \eta$ or $h: \xi \cong \eta$.

DEFINITION 3.3. Two RN_2-bundles ξ, η over K are *concordant* if there exists an RN_2-bundle ζ over the cell complex $K \times I$ such that $\zeta | K \times \{0\} \cong \xi$, $\zeta | K \times \{1\} \cong \eta$. Notation: $\xi \sim \eta$ or $\zeta: \xi \sim \eta$.

The "isomorphism" and the "concordance" relations are obviously equivalence relations. Let $C(K)$ denote the set of concordance classes of RN_2-bundles over K. All of our definitions can be carried over in the category of Δ-sets, and we can define the notion of induced bundles. Then $C(K)$ is a contravariant homotopy functor from the category of Δ-sets to the category of sets. It is proved to be representable, and one can construct the classifying space BRN_2 and the natural equivalence of functors $T: [\ \ , BRN_2] \to C(\ \ )$. (Cf. [9], [12].)

The proof of the following proposition is not difficult.

PROPOSITION 3.4. *Let M be an m-manifold properly embedded in an $(m+2)$-manifold Q. Suppose M and Q are triangulated so that M is a full subcomplex of Q. Let E be the derived neighbourhood of M in Q. (Note that $E \cap \partial Q$ is the derived neighbourhood of ∂M in ∂Q.) Then E is the total space of an RN_2-bundle ν over the dual cell complex K of M. In fact the block over a dual cell $D(\sigma, M)$ (or $D(\sigma, \partial M)$) is the dual cell $D(\sigma, Q)$ (or $D(\sigma, \partial Q)$), where σ is a simplex of M. The cross section $\iota: M \to E$ is defined by the inclusion. Moreover, the concordance class of ν depends only on the concordance class of the embedding of M in Q.*

DEFINITION 3.5. The RN_2-bundle ν constructed in Proposition 3.4 is called a *normal RN_2-bundle of M in Q.*

Now we will construct a Δ-set RN_2: A typical k-simplex of RN_2 is an RN_2-bundle ξ over the cell complex $\Delta^k \times I$ which over $\Delta^k \times \{0, 1\} \cup \Delta^{k-1} \times I$ is the product bundle. It is easy to see that RN_2 is a Kan Δ-set and is considered to be the fiber of the universal principal RN_2-bundle over BRN_2.

By considering the "associated S^1-bundle" of ξ as a homology cobordism bundle with the fiber S^1, we have a Δ-map $i: RN_2 \to H(2)$. With this map i, we regard RN_2 as a subcomplex of $H(2)$.

We are now in a position to prove Theorem 5. *Proof of that $\pi_k(RN_2, \overline{PL}(2)) \cong \mathscr{G}_k$.*

An element $\alpha \in \pi_k(RN_2, \overline{PL}(2))$ is represented by an RN_2 disk bundle with total space $E(\xi)$ over $\Delta^k \times I$ which is a PL block disk bundle over $\partial\Delta^k \times I$ and which is the product bundle over $\Delta^k \times \{0, 1\} \cup \Delta^{k-1} \times I$. Let η be the PL block bundle $\xi | \partial(\Delta^k \times I)$ and $\Sigma^k \subset E(\eta)$ be the section of this PL block disk bundle. Since $E(\xi)$ is a $(k+3)$-disk, $\partial E(\xi)$ is a $(k+2)$-sphere. Therefore, we get a knot $\Sigma^k \subset S^{k+2} = \partial E(\xi)$. (The construction of the ambient sphere is the same as in the case of $\pi_k(PLH(2), \overline{PL}(2))$ if we use an RN_2 sphere bundle as a representative of $\pi_k(RN_2, \overline{PL}(2))$.)

Clearly a concordance between the representatives gives a concordance between the induced knots. So we get a map: $\pi_k(RN_2, \overline{PL}(2)) \rightarrow \mathscr{G}_k$, which is easily seen to be a homomorphism. Assume that the induced knot $\Sigma^k \subset S^{k+2}$ is cobordant to zero, that is, there exists a locally flat disk pair $D^{k+1} \subset D^{k+3}$ which bounds the knot $\Sigma^k \subset S^{k+2}$. Take a sufficiently fine subdivision of the cone $CD^{k+1} \subset CD^{k+3}$ so that CD^{k+1} is a full subcomplex of CD^{k+3}. Then we get a normal RN_2 disk bundle over the dual cell complex of CD^{k+1} by Proposition 3.4. By an appropriate amalgamation, we get a concordance between a normal RN_2 disk bundle of $C\Sigma^k$ in CS^{k+2} which is concordant to $E(\xi)$ and a normal PL block disk bundle of D^{k+1} in D^{k+3}.

q.e.d.

PROOF OF THE LATTER PART OF THEOREM 5. We consider the homotopy long exact sequence of the triple $(H(2), RN_2, \overline{PL}(2))$. Then, by taking account of the following commutative diagram and noting that $\mathscr{G}_2 = \pi_2(H(2), \overline{PL}(2)) = 0$, we get easily the results.

$$\begin{array}{ccc} \pi_k(RN_2, \overline{PL}(2)) & \longrightarrow & \pi_k(H(2), \overline{PL}(2)) \\ & {\scriptstyle \cong}\searrow \quad \nearrow{\scriptstyle \varphi_k} & \\ & \mathscr{G}_k & \end{array}$$

The only rather non-trivial part is the surjectivity of the map: $\pi_1(H(2), \overline{PL}(2)) \rightarrow \pi_1(H(2), RN_2)$. But since any tame embedding of S^1 into PL s-manifold is locally flat, any element of $\pi_1(H(2), RN_2)$ has an element of $\mathscr{G}^{AH} = \pi_1(H(2), \overline{PL}(2))$ as its representative. q.e.d.

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