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MEASURE ALGEBRAS ON SOME TOPOLOGICAL GROUPS

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Introduction. Throughout this paper we will denote by G a locally compact abelian group and by $M(G)$ the measure algebra consisting of all bounded regular Borel measures on G. Measure algebras on Raikov systems (see Definition 2.1) are important objects in a study of the structure of the measure algebra $M(G)$. But they are unclarified even now ([9], [12], [13]). In this paper we shall consider measure algebras on some topological groups as a kind of measure algebras on Raikov systems. To be exact, we shall investigate how a set of bounded measures on G with a general group topology stronger than the original topology of G can be imbedded into $M(G)$ as a measure algebra on a Raikov system and what properties it has in $M(G)$.

In the first section of this paper, we shall construct a measure algebra on G with a group topology stronger than the original topology of G. The method of such a construction is based on [1], [6] and [7]. Further we shall show that the measure algebra constructed above can be imbedded into $M(G)$ as a measure algebra on a Raikov system. This fact is well-known in such a case that its stronger group topology is locally compact ([10] p. 496). In the second section, we shall see that the set of all measure algebras on Raikov systems with one single generators coincides with the set of all measure algebras on certain topological groups. Furthermore, we shall investigate a measure algebra on an inductive limit group of topological groups. In the final section, we shall consider whether a measure algebra determines its group topology uniquely or not.

We will follow [2] and [4] for terminology.

1. Construction of a measure algebra on a stronger topological group. Let G_0 be G with a stronger topology than the original topology of G and let $C(G_0)$ be the Banach space of all bounded continuous functions on G_0 under the supremum norm. $C^*(G_0)$ means the set of limits $f = \lim_\alpha \uparrow f_\alpha$ of increasing generalized sequences $\{f_\alpha\}$ in $C^+(G_0)$, the set of all non-negative functions in $C(G_0)$. $\mathscr{M}$ means the smallest σ-field which renders all of elements in $C^*(G_0)$ measurable.

REMARK 1. $\mathscr{M}$ has following properties;

(i) $\mathscr{M}$ is contained in the Borel field of G_0.

(ii) $\mathscr{M}$ contains the Baire field of G_0.

(iii) $\mathscr{M}$ contains the Borel field of G.

(iv) Especially, $\mathscr{M}$ contains all compact subsets in G_0.

(v) For each compact set K in G_0, two classes of $K \cap \mathscr{M} = \{K \cap E: E \in \mathscr{M}\}$ and the Borel field in K coincide.

(vi) In such a case that G_0 is completely regular, $\mathscr{M}$ coincides with the Borel field of G_0.

A measure μ on $\mathscr{M}$ will be said to be regular if

$$|\mu|(E) = \sup\{|\mu|(C): C \text{ is compact in } G_0 \text{ and } E \supset C\}$$

for each $E \in \mathscr{M}$. We denote by $M(G_0)$ the space of all bounded regular measures on $\mathscr{M}$ and $M^+(G_0)$ means the set of all non-negative measures in $M(G_0)$. Let $\mathscr{L}(G_0)$ be the set of all bounded linear functionals L on $C(G_0)$ whose restrictions on the unit ball of $C(G_0)$ are continuous for the compact convergence topology. $\mathscr{L}^+(G_0)$ is the set of all positive functionals L in $\mathscr{L}(G_0)$ in the sense that $L(f) \geqq 0$ for all $f \in C^+(G_0)$. Note that $\mathscr{L}(G_0)$ is a Banach space under the operator norm. Especially, if G_0 is locally compact, $\mathscr{L}(G_0)$ is the Banach dual of $C_0(G_0)$ the set of all continuous functions vanishing at infinity. In fact, let L be an element of $\mathscr{L}(G_0)$ vanishing on $C_0(G_0)$ and let $\varepsilon > 0$. Then there exists a compact set K in G_0 such that if $f \in C(G_0)$ vanishes on K and $\|f\| \leqq 1$ then $|L(f)| < \varepsilon$ ([1] p. 53, Corollary of Theorem 2). We choose a function $g \in C_0(G_0)$ such that $g = 1$ on K, $0 \leqq g \leqq 1$ and g has compact support. For any $f \in C(G_0)$, $f = fg + (1 - g)f$. Therefore,

$$|L(f)| = |L(fg) + L((1-g)f)| = |L((1-g)f)| < \varepsilon$$

because $fg \in C_0(G_0)$ and $(1 - g)f = 0$ on K. Hence $L = 0$ on $C(G_0)$. By the Hahn-Banach theorem, $\mathscr{L}(G_0) = C_0(G_0)^*$.

THEOREM 1.1 (cf. [6], [1] p. 52, Theorem 2). *There is an isometric isomorphism between $M(G_0)$ and $\mathscr{L}(G_0)$ as Banach spaces such that corresponding elements L and μ satisfy the identity;*

$$L(f) = \int f d\mu \qquad \text{for every } f \in C(G_0) .$$

Furthermore, this isomorphism preserves order.

PROOF. To see that every $\mu \in M(G_0)$ determines an element L of $\mathscr{L}(G_0)$, let $\varepsilon > 0$ and fix non-zero $\mu \in M(G_0)$. We choose a compact set K in G_0 such that $|\mu|(K^c) < \varepsilon/2$ where K^c is the complement of K in G_0.

We put

$$V(K, \varepsilon) = \{f \in C(G_0): ||f|| \leqq 1, \sup_{x \in K} |f(x)| < \varepsilon/2||\mu||\} .$$

Then

$$|L(f)| = \left|\int f d\mu\right| \leqq \left|\int_K f d\mu\right| + \left|\int_{K^c} f d\mu\right|$$
$$< (\varepsilon/2||\mu||) \cdot |\mu|(K) + |\mu|(K^c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for each $f \in V(K, \varepsilon)$. Hence, $L \in \mathscr{L}(G_0)$.

For any $L \in \mathscr{L}(G_0)$ we will next find a measure $\mu \in M(G_0)$ with the form:

$$L(f) = \int f d\mu \qquad \text{for every } f \in C(G_0) .$$

First we consider the case $L \in \mathscr{L}^+(G_0)$. To do it we shall now imitate the Daniell's construction ([7] p. 65). If f_0 is a limit of an increasing generalized sequence $\{f_\alpha\}$ in $C^+(G_0)$ then $L(f_0) = \lim_\alpha \uparrow L(f_\alpha)$ unambiguously defines an extension on $C^*(G_0)$. For, by the Dini's theorem,

$$\lim_\alpha \downarrow L(f_\alpha) = 0$$

for every generalized sequence $\{f_\alpha\}$ decreasing to 0 in $C(G_0)$. We put

$$\zeta = \{E: \chi_E \in C^*(G_0)\} ,$$

where χ_E is the characteristic function of E. We define the set function

$$\mu^*(F) = \inf \{L(\chi_E): E \in \zeta, E \supset F\}$$

for each subset F in G_0. Then the class $\mathscr{D} = \{E: \mu^*(E) + \mu^*(E^c) = \mu^*(G_0)\}$ is a σ-field and the restriction of μ^* to $\mathscr{D}$ is a measure. $\mathscr{D}$ contains ζ and therefore, also the σ-field which ζ generates. The σ-field generated coincides with the smallest σ-field $\mathscr{M}$ which renders all of elements in $C^*(G_0)$ measurable. We denote by μ the restriction to $\mathscr{M}$ of μ^*. Then μ is a bounded non-negative measure such that

$$L(f) = \int f d\mu$$

for every $f \in C^*(G_0)$ and therefore, also for every $f \in C(G_0)$. To see μ is regular, let $\varepsilon > 0$ and then there exists a compact set K in G_0 such that if $f \in C(G_0)$ vanishes on K and $||f|| \leqq 1$ then $|L(f)| < \varepsilon/2$ ([1] p. 53, Corollary of Theorem 2). Since $C(G_0) \supset C(G)$, it follows that $K^c \in \zeta$ and $\mu(K^c) = \sup \{L(f): f \in C(G_0)$ vanishes on K and $0 \leqq f \leqq 1\}$. Therefore $\mu(K^c) \leqq \varepsilon/2$. From the fact that elements in ζ are all open, for any $B \in \mathscr{M}$, there exists a closed set F in G_0 such that $\mu(B) < \mu(F) + \varepsilon/2$ and $B \supset F$. Hence we have an inequality:

$$\mu(B) < \mu(F \cap K) + \varepsilon .$$

As $F \cap K$ is compact in G_0, it holds that

$$\mu(B) = \sup \{\mu(C) : C \text{ is compact in } G_0 \text{ and } B \supset C\} .$$

Therefore μ is regular.

For an arbitrary element L of $\mathscr{L}(G_0)$, we will show the existence of the corresponding measure. L can be expressed as the form:

$$L = (L_1^1 - L_2^1) + i(L_1^2 - L_2^2) ,$$

where each $L_j^k(j = 1, 2; k = 1, 2)$ is a positive bounded linear functional on $C(G_0)$ and

$$L_1^k(f) = \sup \{(L_1^k - L_2^k)(g) : 0 \leqq g \leqq f,\ g \in C(G_0)\} , \quad \text{for} \quad k = 1, 2 ,$$

$$L_2^k(f) = -\inf \{(L_1^k - L_2^k)(g) : 0 \leqq g \leqq f,\ g \in C(G_0)\} , \quad \text{for} \quad k = 1, 2 ,$$

for each $f \in C^+(G_0)$ ([4] (B. 34), (B. 37) and (B. 38)). To show $L_j^k \in \mathscr{L}(G_0)$ $(j = 1, 2; k = 1, 2)$, let $\{f_\alpha\}$ be a generalized sequence in $C(G_0)$ converging to 0 uniformly on each compact set in G_0 and $\|f_\alpha\| \leqq 1$ for all α. Then by the definition of L_1^k, there exists a generalized sequence $\{h_\alpha\}$ in $C(G_0)$ satisfying:

$$0 \leqq h_\alpha \leqq |f_\alpha| , \quad \text{and} \quad L_1^k(|f_\alpha|) \leqq 2(L_1^k - L_2^k)(h_\alpha) \leqq 2|L(h_\alpha)| .$$

Then $h_\alpha \to 0$ uniformly on each compact set in G_0, and therefore $L(h_\alpha) \to 0$. Hence $L_1^k(f_\alpha) \to 0$, that is, $L_1^k \in \mathscr{L}^+(G_0)$. Similarily, $L_2^k \in \mathscr{L}^+(G_0)$. From the preceding assertion there exists $\mu_j^k \in M(G_0)$ for $j = 1, 2$; $k = 1, 2$, such that

$$L_j^k(f) = \int f d\mu_j^k \qquad (j = 1, 2; k = 1, 2)$$

for every $f \in C(G_0)$. If we put in $M(G_0)$

$$\mu = \mu_1^1 - \mu_2^1 + i(\mu_1^2 - \mu_2^2) ,$$

then we have

$$\begin{aligned} L(f) &= L_1^1(f) - L_2^1(f) + i(L_1^2(f) - L_2^2(f)) \\ &= \int f d\mu_1^1 - \int f d\mu_2^1 + i\Big(\int f d\mu_1^2 - \int f d\mu_2^2\Big) \\ &= \int f d\mu \end{aligned}$$

for each $f \in C(G_0)$. This implies that the mapping in Theorem 1.1 is onto.

Since $L(f) = \int f d\mu \geqq 0$ for every $\mu \in M^+(G_0)$ and every $f \in C^+(G_0)$, the mapping in Theorem 1.1 is order preserving.

We will show that the mapping in Theorem 1.1 is an isometry. It is clear that $\| L \| \leqq \| \mu \|$ for corresponding elements μ and L. To show the converse inequality, let $\varepsilon > 0$ and let $E_1, \cdots, E_n$ be disjoint sets such that

$$\sum_{i=1}^{n} | \mu(E_i) | \geqq \| \mu \| - \varepsilon/2 .$$

Let C_i be a compact set in G_0 such that

$$| \mu |(E_i \backslash C_i) < \varepsilon/4n , \quad \text{and} \quad E_i \supset C_i \qquad (i = 1, 2, \cdots, n) .$$

We can choose open sets $U_1, \cdots, U_n$ in G_0 such that

(i) $U_i \supset C_i$ $(i = 1, 2, \cdots, n)$,

(ii) U_i is open in G $(i = 1, 2, \cdots, n)$,

(iii) $U_1, U_2, \cdots, U_n$ are disjoint, and

(iv) $| \mu |(U_i \backslash C_i) < \varepsilon/4n$.

This choice is possible since μ is regular and the topology of G_0 is stronger than that of G. Hence there exist $f_1, \cdots, f_n$ in $C(G)$ such that $f_i(x) = 0$ outside U_i, $f_i(x) = 1$ on C_i and $0 \leqq f_i \leqq 1$ for $i = 1, 2, \cdots, n$. Let $\alpha_1, \cdots, \alpha_n$ be complex numbers such that $\alpha_i \mu(E_i) = | \mu(E_i) |$ and put $f_0 = \sum_{i=1}^{n} \alpha_i f_i$. Then

$$\begin{aligned} | L(f_0) - \| \mu \| | &\leqq \left| \sum_{i=1}^{n} \alpha_i \int f_i d\mu - \sum_{i=1}^{n} \alpha_i \mu(E_i) \right| + \left| \sum_{i=1}^{n} | \mu(E_i) | - \| \mu \| \right| \\ &< \sum_{i=1}^{n} \left| \int f_i d\mu - \mu(E_i) \right| + \varepsilon/2 \leqq \sum_{i=1}^{n} \{ | \mu(C_i) - \mu(E_i) | + | \mu |(U_i \backslash C_i) \} + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon , \end{aligned}$$

so that $\| \mu \| = \sup_{\|f\| \leqq 1} | L(f) | = \| L \|$. Hence the mapping is an isometry. It is obvious that the mapping is a homomorphism. Hence, the proof is complete.

THEOREM 1.2. *Let $i: G_0 \to G$ be the continuous identity mapping. Define a homomorphism $i^*: M(G_0) \to M(G)$, by $i^* \mu(E) = \mu(i^{-1}(E))$ for every Borel set E in G. Then i^* is an isometric $*$-isomorphism of $M(G_0)$ into $M(G)$.*

PROOF. It is clear from ([10] p. 493, Proposition 2).

THEOREM 1.3. *Let i^* be as in Theorem 1.2. Let $\mu \in M(G)$, then μ belongs to $i^* M(G_0)$ if and only if μ is concentrated on some σ-compact set in G_0 as a subset of G.*

PROOF. The sufficiency is trivial, so that we prove the necessity. We may assume that μ is positive. Let $\{K_n\}_{n=1}^{\infty}$ be an increasing sequence of compact sets in G_0 such that μ is concentrated on $\bigcup_{n=1}^{\infty} K_n$. Since two of the topology of G_0 and the topology of G coincide on each compact set in G_0, each restriction μ_n of μ to K_n is in $M(G_0)$ by Remark 1(v) and μ_n converges to μ with the norm. Therefore, $\mu \in i^*M(G_0)$.

Henceforth, let G_0 be the abstract group G with a stronger group topology than the original topology of G. We say such a topological group G_0 simply a stronger topological group G_0. Since a topological (Hausdorff) abelian group is completely regular ([4], (8.4)), $M(G_0)$ is the set of all bounded regular Borel measures by Remark 1(vi). By Theorem 1.1, $M(G_0)$ can be identified with $\mathscr{L}(G_0)$. Now, we will define a convolution in $M(G_0)$. We put

$$F(x) = \int f(x + y)d\mu(y) \text{ for } f \in C(G_0) \text{ and } \mu \in M(G_0) .$$

We fix non-zero $f \in C(G_0)$ and non-zero $\mu \in M(G_0)$, and let $\varepsilon > 0$. We choose a compact set K in G_0 such that $|\mu|(K^c) < \varepsilon/4\|f\|$. For fixed $x_0 \in G_0$, there exists a neighborhood V of x_0 in G_0 such that

$$\sup_{y \in K} |f(x + y) - f(x_0 + y)| < \varepsilon/2\|\mu\|$$

for every $x \in V$. Therefore, we have

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int f(x + y)d\mu(y) - \int f(x_0 + y)d\mu(y) \right| \\ &\leqq 2\|f\| \, |\mu|(K^c) + \|\mu\| \sup_{y \in K} |f(x + y) - f(x_0 + y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Hence $F(x) \in C(G_0)$. For fixed $\nu \in M(G_0)$, the mapping: $f \to \nu(F)$ is a bounded linear functional on $C(G_0)$. This functional will be called convolution of μ and ν, and it will be written $\mu * \nu$. It is clear that $\|\mu * \nu\| \leqq \|\mu\| \|\nu\|$, $\mu * \nu = \nu * \mu$ and $(\mu * \nu) * \tau = \mu * (\nu * \tau)$ for $\mu, \nu, \tau \in M(G_0)$. To show that $\mu * \nu \in \mathscr{L}(G_0)$, let $\varepsilon > 0$, then by Theorem 1.1 there exist $\delta > 0$ and a compact set K in G_0 such that if f is in $C(G_0)$ with $\|f\| \leqq 1$ and $|f(x)| < \delta$ on K then $|\nu(f)| < \varepsilon$. Moreover, there exist $\delta' > 0$ and a compact set K' in G_0 such that if f is in $C(G_0)$ with $\|f\| \leqq 1$ and $|f(x)| < \delta'$ on K' then $|\mu(f)| < \delta$. We put

$$V(K + K', \delta') = \{f \in C(G_0): \|f\| \leqq 1, |f(x)| < \delta' \text{ on } K + K'\} .$$

Then it holds that for $x \in K$ and for $f \in V(K + K', \delta')$

$$|F(x)| = \left|\int f(x+y)d\mu(y)\right| < \delta ,$$

so that

$$|\nu(F)| < \varepsilon \quad \text{for} \quad f \in V(K+K', \delta') .$$

Therefore, $\mu * \nu \in \mathscr{L}(G_0)$.

$M(G_0)$ becomes a commutative Banach algebra under this convolution and i^* as in Theorem 1.2 is an isometric $*$-isomorphism of $M(G_0)$ into $M(G)$ as Banach algebras.

Henceforth, we will not distinguish $M(G_0)$ from its image of i^*. We will call it the measure algebra on G_0.

We have a following result as a corollary of Theorem 1.3;

COROLLARY 1.4. *$M(G_0)$ is a measure algebra on some Raikov system in $M(G)$ (see Definition 2.1).*

PROOF. Since σ-compact subsets in G_0 form a Raikov system in G, it is clear from Theorem 1.3.

2. Raikov systems and an inductive limit topology of group topologies.

DEFINITION 2.1. A Raikov system is a collection $\mathscr{F}$ of σ-compact sets in G satisfying the following conditions;

(i) If $F \in \mathscr{F}$ and E is a σ-compact set in G with $E \subset F$ then $E \in \mathscr{F}$,

(ii) If $F_1, F_2 \in \mathscr{F}$ then $F_1 + F_2 \in \mathscr{F}$,

(iii) If $F_i \in \mathscr{F}$ for $i = 1, 2, \cdots$ then $\bigcup_{i=1}^{\infty} F_i \in \mathscr{F}$,

(iv) If $F \in \mathscr{F}$ and $x \in G$ then $F + x \in \mathscr{F}$.

If this system also satisfies the following

(v) If $F \in \mathscr{F}$ then $-F \in \mathscr{F}$,

we shall call it a symmetric Raikov system. If a Raikov system is contained properly in the Raikov system consisting of all σ-compact subsets in G, we say that it is proper. If a Raikov system $\mathscr{F}$ is the minimal (symmetric) Raikov system containing a collection $\{H_\alpha\}$ of σ-compact sets in G, then we say that each H_α is a generator of $\mathscr{F}$. If a Raikov system has a countable set of generators then it has a single generator ([12]). In a symmetric Raikov system, any generators may be assumed to be σ-compact subgroups of G. Let $\mathscr{F}$ be a Raikov system. Then bounded measures in $M(G)$ concentrated on $\mathscr{F}$ form a subalgebra $M(\mathscr{F})$ of $M(G)$ ([9]). We shall call it the measure algebra on $\mathscr{F}$.

Let $\mathscr{F}_0$ be the symmetric Raikov system consisting of all σ-compact sets in G_0. Corollary 1.4 asserts that $M(G_0) = M(\mathscr{F}_0)$.

A topological group is said to be locally σ-compact provided that every point has a base of neighborhoods which are σ-compact.

PROPOSITION 2.2. *The set of all measure algebras on symmetric Raikov systems with each single generator coincides with the set of all measure algebras on G with locally σ-compact group topologies stronger than the original topology of G.*

PROOF. Let $\mathscr{F}$ be a symmetric Raikov system with a single generator H which is a σ-compact subgroup in G. We introduce into G a topology which has a neighborhood basis at the unit in H as a neighborhood basis at the unit. The group G with such a topology is denoted by G_H. G_H is a locally σ-compact group stronger than the original topology of G, and H is an open σ-compact subgroup in G_H. As H generates $\mathscr{F}$, every set belonging to $\mathscr{F}$ is σ-compact in G_H. On the other hand, if $\alpha: G_H \to G_H/H$ is the continuous canonical mapping, then α transfers σ-compact sets in G_H to σ-compact sets in G_H/H. Since H is open in G_H, G_H/H is discrete, so that every σ-compact set in G_H is covered by a countable set of translates of H, and therefore, every σ-compact set in G_H belongs to $\mathscr{F}$. By Theorem 1.3, this proves that $M(\mathscr{F}) = M(G_H)$. Conversely, let G_0 be a stronger locally σ-compact group and take a σ-compact symmetric neighborhood V at the unit in G_0. We put

$$H = \bigcup_{n=1}^{\infty} nV, \text{ where } nV = \{x_1 + x_2 + \cdots + x_n : x_1, x_2, \cdots, x_n \in V\} .$$

Then H is an open σ-compact subgroup of G_0. Hence, $G_0 = G_H$ using the previous notation, and the Raikov system $\mathscr{F}$ consisting of all σ-compact sets in G_0 is generated by H. By Theorem 1.3, $M(G_0) = M(\mathscr{F})$.

DEFINITION 2.3. Let $\{G_\alpha\}_{\alpha \in I}$ be a family of stronger topological groups. The inductive limit group $G_0 = \varinjlim_{\alpha} G_\alpha$ is defined to be the abstract group G with the strongest group topology weaker than that of G_α for every α.

REMARK 2 ([10] p. 467–471). Concerning the definition above,

(1) We can, without loss of generality, assume that $\{G_\alpha\}_{\alpha \in I}$ is an inductive system, by which we mean

(i) I is a directed set,

(ii) If $\alpha < \beta$ then the topology of G_α is stronger than that of G_β.

(2) If the index I is countable then a basic set of neighborhoods of G_0 is given in forms

$$V = \bigcup_{n=1}^{\infty} (V_1 + V_2 + \cdots + V_n) ,$$

where V_n runs through a basic set of neighborhoods of G_n; $n = 1, 2, \cdots$. In this case, an inductive limit group of locally σ-compact groups is also locally σ-compact. Further, an inductive limit group of locally compact groups $\{G_n\}_{n=1}^{\infty}$ such that the topology of G_n is strictly stronger than that of G_{n+1} for each n, is locally σ-compact, but not locally compact.

PROPOSITION 2.4. *Let G_0 be an inductive limit group of locally σ-compact groups $\{G_n\}_{n=1}^{\infty}$ stronger than the original topology of G. Then $M(G_0)$ is a closed subalgebra in $M(G)$ generated by $\{M(G_n)\}_{n=1}^{\infty}$.*

PROOF. It is clear that $M(G_0)$ contains the closed subalgebra generated by $\{M(G_n)\}_{n=1}^{\infty}$. To see the converse inclusion, we may assume by Remark 2(1) that

$$M(G_1) \subset M(G_2) \subset \cdots \subset M(G_n) \subset \cdots \subset M(G_0) .$$

Let $\mu \in M(G_0)$. Since $M(G_0) = M(G_n) + M(G_n)^{\perp}$ ([9]), μ can be decomposed uniquely with the form

$$\mu = \mu_n + \mu'_n ,$$

where $\mu_n \in M(G_n)$ and $\mu'_n \in M(G_n)^{\perp}$ for each n. We have

$$\| \mu_{n+1} - \mu_n \| = \| \mu_{n+1} \| - \| \mu_n \| \text{ for each } n ,$$

and

$$\| \mu_1 \| \leqq \| \mu_2 \| \leqq \cdots \leqq \| \mu_n \| \leqq \cdots \leqq \| \mu \| ,$$

so that μ_n converges to a measure ν in the closed subalgebra generated by $\{M(G_n)\}_{n=1}^{\infty}$. Put $\mu' = \mu - \nu$. Then $\mu' \perp M(G_n)$ for all n. By Remark 2(2), the Raikov system consisting of all σ-compact sets in G_0 coincides with the symmetric Raikov system generated by all of σ-compact sets in G_n for all n, so that $\mu' \perp M(G_0)$. But μ' belongs to $M(G_0)$, so that $\mu' = 0$, that is, μ belongs to the closed subalgebra generated by $\{M(G_n)\}_{n=1}^{\infty}$.

We don't know that Proposition 2.4 is true when an index is uncountable (and further, for an inductive limit group of stronger general topological groups). But this problem is reduced to the following problem; "The Raikov system consisting of all σ-compact sets in an inductive limit group of stronger topological groups $\{G_\alpha\}_{\alpha \in I}$ (not necessarily countable!) is generated by all of σ-compact sets in G_α for all α."

If this problem can be solved affirmatively, the following extension of Proposition 2.2 holds good; "Every symmetric Raikov system is a measure algebra on some stronger topological group."

3. Uniqueness of a topological group determined by a measure algebra. Generally, a measure algebra does not determine a topological group uniquely, that is, there exist stronger topological groups G_1, G_2 such that G_1 and G_2 are different, but $M(G_1) = M(G_2)$.

EXAMPLE. Let Z be the set of all integers. We define two different topologies on Z;

(a) the weakest topology on Z for which all of elements in the torus T are continuous as characters of Z, and

(b) the discrete topology on Z.

$Z_w(Z_d)$ will mean the abstract group Z with the topology of (a) ((b), respectively). Then Z_w is a non-discrete pre-locally compact Hausdorff group ([10] p. 480 and [4] (4.23)) and the completion $\tilde{Z}$ of Z_w is a locally compact Hausdorff group ([5] p. 212). Let $\tilde{Z}_w(\tilde{Z}_d)$ be $\tilde{Z}$ with the topology which has a neighborhood basis at zero in Z_w (Z_d, respectively) as a neighborhood basis at zero. Since Z is a countable set, Z is σ-compact in $\tilde{Z}$. Therefore, by Proposition 2.2, $M(\tilde{Z}_w) = M(\tilde{Z}_d)$, but $\tilde{Z}_w$ and $\tilde{Z}_d$ are different.

However, we have a following proposition;

PROPOSITION 3.1. *Let G_1 be a stronger topological group weaker than that of a stronger locally compact group G_0. If $M(G_0) = M(G_1)$ and there exists a nonzero positive measure μ in $M(G_1)$ such that the mapping $x \longrightarrow F(\mu * \delta_x)$ on G_1 is continuous for each fixed $F \in M(G_1)^*$, where δ_x is denoted the measure with point mass 1 at x and $M(G_1)^*$ is the Banach dual of $M(G_1)$, then $G_1 = G_0$.*

PROOF. To prove the Proposition, we will now repeat the argument of [8].

The set $L = \{\mu \in M(G_1): x \longrightarrow F(|\mu| * \delta_x)$ is continuous on G_1 for each $F \in M(G_1)^*\}$ has not L-ideals in itself and is an L-ideal in $M(G_1)$. Hence, $L = L^1(G_0)$, where $L^1(G_0)$ is the group algebra on G_0. Let $\{x_\alpha\}$ be a generalized sequence converging to x_0 in G_1 then

$$F(\lambda * \delta_{x_\alpha}) \longrightarrow F(\lambda * \delta_{x_0})$$

for every $F \in M(G_0)^*$ and every $\lambda \in L^1(G_0)$. The mapping $\mu \longrightarrow \int f d\mu$ for each $f \in C_0(G_0)$ is a bounded linear functional on $M(G_0)$, so that

$$\int f d(\lambda * \delta_{x_\alpha}) \longrightarrow \int f d(\lambda * \delta_{x_0})$$

for each $f \in C_0(G_0)$, the set of all elements in $C(G_0)$ vanishing at infinity. Since every $f \in C_0(G_0)$ is uniformly continuous, for $\varepsilon > 0$, there exists $\lambda \in L^1(G_0)$ such that $\|\lambda\| = 1$ and

$$\left| f(x) - \int f(x+y) d\lambda(y) \right| < \varepsilon/3 \qquad \text{for every } x \in G_0 .$$

Hence, there exists an index α_0 such that

$$\begin{aligned} &|f(x_\alpha) - f(x_0)| \\ &\leqq \left| f(x_\alpha) - \int f(x_\alpha + y)d\lambda(y) \right| + \left| f(x_0) - \int f(x_0 + y)d\lambda(y) \right| \\ &\quad + \left| \int f(y)d(\lambda * \delta_{x_\alpha})(y) - \int f(y)d(\lambda * \delta_{x_0})(y) \right| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon , \end{aligned}$$

for $\alpha \geqq \alpha_0$. Therefore, $f(x_\alpha) \to f(x_0)$ for every $f \in C_0(G_0)$, so that $x_\alpha \to x_0$ in G_0. This proves that the topology of G_1 is stronger than that of G_0, and therefore, $G_1 = G_0$.

COROLLARY (cf. [11]). *Let G_0, G_1 be two locally compact groups stronger than the original topology of G. If $M(G_0) = M(G_1)$ then $G_0 = G_1$.*

PROOF. Let G' be the inductive limit group of G_0 and G_1. By Proposition 2.4, $M(G_0) = M(G_1) = M(G')$. By Proposition 3.1, $G_0 = G' = G_1$.

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