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ON THE DIVERGENCE OF REARRANGED TRIGONOMETRIC SERIES

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K. Tandori studied the rearrangement of trigonometric series in [6], and that of Walsh series in [7]. The result of [6] was sharpened by F. Móricz ([1], [2]), then further by the author [3]. That is

THEOREM A. *If* $\{\rho(n)\}$ *is a sequence of positive numbers with* $\rho(n) = o(\sqrt[4]{\log n})$, *then there exists a sequence of real numbers* $\{a_1, b_1, \cdots, a_n, b_n, \cdots\}$ *for which*

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\rho^2(n) < \infty$$

and such that the trigonometric series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be rearranged into an everywhere divergent series.

The result of [7] was also sharpened in [4] and [5]. Writing $L_1(n) = \log n$ and $L_s(n) = L_1(L_{s-1}(n))$ $(s = 2, 3, \cdots)$, [5] reads as follows.

THEOREM B. *For any natural number* s, *there exists a sequence of real numbers* $\{a_1, \cdots, a_n, \cdots\}$ *for which*

$$\sum_{n=N+1}^{\infty} a_n^2 \sqrt{L_1(n)} L_2(n) L_3(n) \cdots L_s(n) < \infty^{1)}$$

and such that the Walsh series

$$\sum_{n=1}^{\infty} a_n w_n(x)$$

can be rearranged into an almost everywhere divergent series.

In the present paper, we are going to prove the following theorem by using the methods of [3] and [5].

THEOREM. *For any natural number* s, *there exists a sequence of real numbers* $\{a_1, b_1, \cdots, a_n, b_n, \cdots\}$ *for which*

1) N is a natural number depending s such that $L_s(N) > 0$.

$$\sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)\sqrt{L_1(n)}L_2(n)L_3(n) \cdots L_s(n) < \infty$$

and such that the trigonometric series (1) *can be rearranged into an almost everywhere divergent series.*

COROLLARY. *For any natural number s, there exists a sequence of real numbers* $\{a_1, b_1, \cdots, a_n, b_n, \cdots\} \in l_2$ *such that the trigonometric series* (1) *can be rearranged to satisfy*

$$\limsup_{N\to\infty} \frac{\left|\sum_{j=1}^{N} (a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x)\right|}{\{L_1(N)\}^{1/4}\cdot\{L_2(N) \cdots L_s(N)\}^{1/2}} > 0$$

almost everywhere.

1. Lemmas. Let us modify the original lemmas in [3] or [1].

LEMMA 1. *Let* $E = \bigcup_i J_i \subset [-\pi/8, \pi/8]$ *be a generalized simple set,* $0 < \varepsilon < \max_i |J_i|/2$ *and* $0 < \eta \leqq 1$ *real numbers, and n a natural number such that* $n \geqq (5/4)\pi/\varepsilon\eta - 1$. *Then for any natural number* $N(\geqq 4n + 3)$ *with* $N \equiv 3 \pmod 4$, *there exist trigonometric polynomials* $Q_l(x)$ $(l = 1, 2, 3)$ *with the following properties*:

(i) *denote by* ν_l *the frequencies occurring in* $Q_l(x)$ $(l = 1, 2, 3)$, *then*

$$\begin{aligned} &\nu_1 \equiv 3 \pmod 4\ , && N - 4n \leqq \nu_1 \leqq N + 4n\ , \\ &\nu_2 \equiv 1 \pmod 4\ , && N - 4n - 2 \leqq \nu_2 \leqq N + 4n + 2\ , \\ &\nu_3 \equiv 2 \pmod 4\ , && 2N - 4n \leqq \nu_3 \leqq 2N + 4n\ ; \end{aligned}$$

(ii) $\left|\sum_{l=1}^{3} Q_l(x)\right| < C_1\eta$ *if* $x \in [-\pi/8, \pi/8] - E$ $\quad (C_1 = 4 + 6\sqrt{2})$;

(iii) $\int_{-\pi}^{\pi} \left|\sum_{l=1}^{3} Q_l(x)\right|^2 dx \leqq C_2 |E|$ $\quad (C_2 = 9(25 + 12\sqrt{2})\pi^4)$;

(iv) *there exists a decomposition* $E^{(\varepsilon)} = E_1 + E_2 + E_3$ *such that*

$$\sum_{k=1}^{l} Q_k(x) \geqq 1/2 \quad \textit{if} \quad x \in E_l \qquad (l = 1, 2, 3)\ .$$

PROOF. Setting

$$E^{(\varepsilon)} = \bigcup_{i=1}^{m} [\alpha_i + \varepsilon, \beta_i - \varepsilon]\ ,$$

$a = \pi/4(n + 1)$ and $b_k = 2ak$ $(k = 0, \pm 1, \pm 2, \cdots)$, we determine the integers ρ_i and σ_i $(i = 1, \cdots, m)$ such that

$$b_{\rho_i} - a \leqq \alpha_i + \varepsilon < b_{\rho_i} \quad \text{or} \quad b_{\rho_i} \leqq \alpha_i + \varepsilon < b_{\rho_i} + a$$

and

$$b_{\sigma_i} - a < \beta_i - \varepsilon \leqq b_{\sigma_i} \quad \text{or} \quad b_{\sigma_i} < \beta_i - \varepsilon \leqq b_{\sigma_i} + a$$

respectively. Then the trigonometric polynomial

$$\begin{aligned} P(x) &= 2\pi a \sum_{i=1}^{m} \sum_{r=\rho_i}^{\sigma_i} K_n(4(x - b_r))^{2)} \\ &= \sum_{\nu=0}^{n} (a_{4\nu} \cos 4\nu x + b_{4\nu} \sin 4\nu x) \end{aligned}$$

has the following properties:

$$\begin{aligned} &P(x) \geqq 1 \quad \text{if} \quad x \in E^{(\varepsilon)} ; \\ &P(x) < \eta \quad \text{if} \quad x \in [-\pi/8, \pi/8] - E ; \\ &\int_{-\pi}^{\pi} P^2(x)dx \leqq (9/2)\pi^4 \, |E| \, . \end{aligned}$$

Now set

$$\begin{aligned} Q_1(x) &= (\cos Nx)P(x) , \\ Q_2(x) &= -2\sqrt{2}(\cos 2x)(\cos Nx)P(x) , \\ Q_3(x) &= -(3 + 4\sqrt{2})(\cos 2Nx)P(x) \end{aligned}$$

and

$$\begin{aligned} E_1 &= E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi - \frac{1}{3}\pi\right), \ \frac{1}{N}\left(2k\pi + \frac{1}{3}\pi\right)\right], \\ E_2 &= E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi + \frac{2}{3}\pi\right), \ \frac{1}{N}\left(2k\pi + \frac{4}{3}\pi\right)\right], \\ E_3 &= E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left(\frac{1}{N}\left(k\pi + \frac{1}{3}\pi\right), \ \frac{1}{N}\left(k\pi + \frac{2}{3}\pi\right)\right) . \end{aligned}$$

Then we can see that (i)-(iv) hold (cf. [3; Lemma 2]).

LEMMA 2. *For every natural number m, there exist generalized simple sets $E(m, i, k) \subset [-\pi/8, \pi/8]$ and mutually disjoint trigonometric polynomials $R(m, i, k; x)$ $(k = 1, \cdots, 3^i;\ i = 0, 1, \cdots)$ with the following properties*:

(i) $E(m, i, k) \cap E(m, i, k') = \emptyset \quad (1 \leqq k < k' \leqq 3^i)$;

(ii) $E(m, i, k) \supset \bigcup_{l=1}^{3} E(m, i + 1, 3(k - 1) + l)$;

(iii) *set*

$$E(m, i, k) = \bigcup_{j=1}^{g(m,i,k)} J_j \quad \text{and} \quad v_i(m) = \sum_{k=1}^{3^i} g(m, i, k) ,$$

then

2) $K_n(x) = \dfrac{1}{2(n + 1)}\left(\dfrac{\sin (n + 1)x/2}{\sin x/2}\right)^2.$

$$v_0(m) = 1 \quad and \quad v_i(m) \leqq 5^{-1} f_i(m) \qquad (i \geqq 1)$$

where

$$f_i(m) = m^i (3\sqrt{3})^{i(i+C_3)} \qquad (C_3 = 4)\ ;$$

(iv) *set*

$$F(m, i) = \left[-\frac{\pi}{8}, \frac{\pi}{8}\right] - \bigcup_{k=1}^{3^i} E(m, i, k)$$

and

$$F_m = \lim_{i\to\infty} F(m, i)\ ,$$

then

$$|F_m| \leqq \frac{C_4}{m} \qquad \left(C_4 = \frac{3\pi}{16}\right)\ ;$$

(v) *denote by* ν *the frequencies occurring in* $R(m, i, k;\ x)$, *then*

$$\nu \not\equiv 0 \ (\mathrm{mod}\ 4) \quad and \quad \nu \leqq f_i(m)\ ;$$

(vi) $\displaystyle\int_{-\pi}^{\pi} \left\{\sum_{k=1}^{3^i} R(m, i, k;\ x)\right\}^2 dx \leqq C_5 \quad \left(C_5 = C_2 \frac{\pi}{4}\right)$;

(vii) $\displaystyle\sum_{j=1}^{k} R(m, i, j;\ x) \geqq \frac{1}{6}$ *if* $x \in E(m, i, k)$.

PROOF. We set

$$\varepsilon_i = \varepsilon_i(m) = \begin{cases} \pi/16m & (i = 0)\ , \\ 5\pi/16m3^i f_i(m) & (i \geqq 1)\ , \end{cases}$$

$$\eta_i = \begin{cases} 1 & (i = 0)\ , \\ 1/C_1 3^{i+1} & (i \geqq 1)\ , \end{cases}$$

$$n_i = n_i(m) = \left[\frac{(5/4)\pi}{\varepsilon_i \eta_i}\right] = \begin{cases} 20m & (i = 0)\ , \\ [4C_1 m 3^{2i+1} f_i(m)] & (i \geqq 1) \end{cases}$$

and

$$N(m, i, k) = 8(n_i + 1)k - 1 \qquad (k = 1, \cdots, 3^i;\ i = 0, 1, \cdots)\ .$$

Set $E(m, 0, 1) = [-\pi/8, \pi/8]$ and $R(m, 0, 1;\ x) = \cos x$. Supposing $E(m, i, k)$ and $R(m, i, k; x)$ $(k = 1, \cdots, 3^i)$ defined, we apply Lemma 1 to $(E(m, i, k),\ \varepsilon_i(m),\ \eta_i,\ n_i(m)$ and $N(m, i, k))$ and get the trigonometric polynomials $Q_l(x)$ $(l = 1, 2, 3)$ and the decomposition $E(m, i, k)^{(\varepsilon_i)} = E_1 + E_2 + E_3$. Then set $R(m, i+1, 3(k-1)+l;\ x) = Q_l(x)$ and $E(m, i+1, 3(k-1)+l) = E_l$ $(l = 1, 2, 3;\ k = 1, \cdots, 3^i)$. And we can see that (i)-(vii) hold (cf. [3; Lemma 3]).

2. Proof of the theorem. Setting $D_n = \{nL_1(n)L_2(n)\cdots L_s(n)\}^{-1}$, we define a sequence of integers $\{p(n)\}$ such that

$$\sum_{i=n}^{p(n)} D_i \geqq 1 \qquad (n = N+1, N+2, \cdots),$$

and $\{t_m\}$ such that

$$L_s(t_m - 1) \geqq m^2 \qquad (m = 1, 2, \cdots).$$

Arrange the system of trigonometric polynomials

$$\{D_i R(m, i, k; x)\} \qquad (k = 1, \cdots, 3^i;\ i = t_m, \cdots, p(t_m))$$

into

$$\{U_j(m; x)\} \qquad \left(j = 1, \cdots, h(m);\ h(m) = \sum_{i=t_m}^{p(t_m)} 3^i\right)$$

so that

$$U_{j(i,k)}(m; x) = D_i R(m, i, k; x),$$

$$j(i, k) < j(i+1, 3k-2) < j(i+1, 3k-1) < j(i+1, 3k) < j(i, k+1)$$

hold. Then we see by Lemma 2 that

$$(2) \qquad \sum_{j=1}^{\mu_m(x)} U_j(m; x) \geqq \frac{1}{6}, \qquad 1 \leqq \mu_m(x) \leqq h(m)$$

for $x \in [-\pi/8, \pi/8] - F_m$. Since $|F_m| \to 0$ $(m \to \infty)$, (2) holds almost everywhere in $[-\pi/8, \pi/8]$ for infinitely many m.

Now let us define the rearranged trigonometric series

$$\sum_{j=1}^{\infty} (a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x)$$

by considering the series

$$\sum_{m=1}^{\infty} \sum_{j=1}^{h(m)} U_j(m; 8^{m-1}x).$$

It is obvious that these series diverge almost everywhere in $[-\pi, \pi]$ from the second Borel-Cantelli lemma.

Setting $\lambda(n) = \sqrt{L_1(n)}L_2(n) \cdots L_s(n)$, we get

$$\sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)\lambda(n)$$
$$\leqq \pi^{-1} \sum_{m=1}^{\infty} \sum_{i=t_m}^{p(t_m)} D_i^2 \lambda(8^{m-1} f_i(m)) \int_{-\pi}^{\pi} \left\{\sum_{k=1}^{3^i} R^2(m, i, k;\ 8^{m-1}x)\right\} dx$$
$$\leqq \pi^{-1} C_5 \sum_{m=1}^{\infty} \sum_{i=t_m}^{\infty} D_i^2 \lambda(e^{4i^2})$$

$$\leqq C_6 \sum_{m=1}^{\infty} \sum_{i=t_m}^{\infty} \{iL_1(i) \cdots L_{s-1}(i)\}^{-1}\{L_s(i)\}^{-2} \qquad (C_6 = 4\pi^{-1}C_5)$$

$$\leqq C_6 \sum_{m=1}^{\infty} \{L_s(t_m - 1)\}^{-1} \leqq C_6 \sum_{m=1}^{\infty} m^{-2} < \infty \ .$$

This completes the proof of the theorem.

REFERENCES

[1] F. MÓRICZ, On the order of magnitude of the partial sums of rearranged Fourier series of square integrable functions, Acta Sci. Math., 28 (1967), 155-167.

[2] F. MÓRICZ, On the convergence of Fourier series in every arrangement of the terms, Acta Sci. Math., 31 (1970), 33-41.

[3] S. NAKATA, On the divergence of rearranged Fourier series of square integrable functions, Acta Sci. Math., 32 (1971), 59-70.

[4] S. NAKATA, On the divergence of rearranged Walsh series, Tôhoku Math. J., 24 (1972), 275-280.

[5] S. NAKATA, On the divergence of rearranged Walsh series II, to appear in Tôhoku Math. J.

[6] K. TANDORI, Beispiel der Fourierreihe einer quadratisch integrierbaren Funktionen, die in gewisser Anordnung ihrer Glieder überall divergiert, Acta Math. Hung., 15 (1964), 165-173.

[7] K. TANDORI, Über die Divergenz der Walshschen Reihen, Acta Sci. Math., 27 (1966), 261-263.

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