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ON SUMMATION PROCESSES OF FOURIER EXPANSIONS IN BANACH SPACES. III: JACKSON- AND ZAMANSKY-TYPE INEQUALITIES FOR ABEL-BOUNDED EXPANSIONS

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This paper is a sequel to the preceding ones with the same title published in this Journal (Note I, Vol. 24, pp. 127-140; II, 24, pp. 551-569). The contents of these notes are assumed to be known. References are in alphabetical order in each paper; they as well as the sections are numbered consecutively throughout this series.

9. Jackson- and Zamansky-type inequalities. In this paper we continue the study of multipliers with respect to orthogonal projections in Banach spaces satisfying certain summability conditions. Whereas in Notes I, II the (C, j)-boundedness of the expansion was assumed (for treatment in the fractional case see [42]), we here commence with the (weaker) Abel-boundedness which leads to (more restrictive) multiplier classes. The usefulness of this multiplier concept may again be illustrated by a treatment of certain fundamental approximation-theoretical problems. Whereas Notes I and II dealt with the comparison and saturation problem, respectively, we here choose the problem of establishing Jackson- and Zamansky-type inequalities (for a first treatment see [33], [42]).

As was examplified in Sec. 2, 6, the translation of the above approximation-theoretical problems to uniform multiplier conditions is straight forward and indicated briefly in this section for the present topics. The essential point is to be seen in deriving convenient multiplier criteria to be carried out in Sec. 10 where we will also discuss the connection with the multiplier sets bv_{j+1} and BV_{j+1} of Sec. 7. Finally Sec. 11 gives applications to Hermite and Laguerre expansions in weighted L^p-spaces.

Motivated by the applications, for which the projections $\{P_k\}$ need not necessarily be bounded with respect to the topology of the Banach spaces X, let us modify slightly the multiplier concept (cf. [33]). Thus we commence with some Hilbert space H and suppose the Banach space X to be represented as the X-closure of H in the following sense. Let $\{P_k\}_{k=0}^{\infty}$ be a sequence of mutually orthogonal projections, which are

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continuous and complete on H, i.e., $\{P_k\} \subset [H]$ and $f = \sum_{k=0}^{\infty} P_k f$ for any $f \in H$. It is assumed that $P_k(H) \subset X$ for any $k \in \boldsymbol{P}$, and that the set of all polynomials Π, i.e. the set of all finite linear combinations $\sum_{k=0}^{n} f_k$, $f_k \in P_k(H)$, is dense in X (with norm $\|\cdot\|$). A sequence $\alpha \in s$ is called a multiplier on X if for each $f = \sum_{k=0}^{n} f_k$ the polynomial $f^\alpha = \sum_{k=0}^{n} \alpha_k f_k$ satisfies

$$\|f^\alpha\| \leqq A\|f\| \qquad (f \in \Pi)\,, \tag{9.1}$$

the constant A being independent of $f \in \Pi$. Then the operator, generated by $\alpha \in s$ on Π, can be uniquely extended to all of X so that we again write $\alpha \in M(X; \{P_k\})$ (cf. Sec. 2). Concerning the closed operator B^ψ in Sec. 6, $\psi \in s$ is said to generate an X-closed B^ψ if there exists a closed linear operator B^ψ with domain $D(B^\psi) \equiv X^\psi \subset X$ and range in X such that $\Pi \subset X^\psi$ and $B^\psi f = \sum_{k=0}^{\infty} \psi_k P_k f$ for any $f \in \Pi$. Note that the existence of B^ψ in case $\{P_k\} \subset [X]$ is obvious (cf. Sec. 6). With these modifications the following result holds.

THEOREM 9.1. *Let $\{T(\rho)\}_{\rho>0} \subset [X]_M$ be a strong approximation process with associated multiplier family $\{\tau(\rho)\}_{\rho>0}$, let $\chi(\rho)$ be some positive, monotone function with $\lim_{\rho\to\infty} \chi(\rho) = \infty$, and let $\psi \in s$ generate an X-closed operator B^ψ.*

a) *If there exists a uniformly bounded multiplier family $\{\eta(\rho)\}$ such that*

$$\chi(\rho)\{\tau_k(\rho) - 1\} = \psi_k \eta_k(\rho) \qquad (\rho > 0,\ k \in \boldsymbol{P})\,, \tag{9.2}$$

then there holds the Jackson-type inequality

$$\chi(\rho)\,\|T(\rho)f - f\| \leqq \sup_{\rho>0} \|\eta(\rho)\|_M \, \|B^\psi f\| \qquad (f \in X^\psi)\,.$$

b) *With (9.2) replaced by*

$$\psi_k \tau_k(\rho) = \chi(\rho)\eta_k(\rho)\{\tau_k(\rho) - 1\} \qquad (\rho > 0,\ k \in \boldsymbol{P})\,, \tag{9.3}$$

one has the Zamansky-type inequality

$$\|B^\psi T(\rho) f\| \leqq \chi(\rho)\Big(\sup_{\rho>0} \|\eta(\rho)\|_M\Big)\, \|T(\rho)f - f\| \qquad (f \in X)\,.$$

Since the projections $\{P_k\}$ are total on H and Π is dense in X, the assertions are obvious (cf. Sec. 2, 6, and [33]).

Concerning applications, the actual problem is to be seen in the verification of the multiplier conditions (9.2) and (9.3). It hardly seems possible to develop a satisfactory multiplier theory without assuming further properties upon X and $\{P_k\}$. Here we suppose that the Abel means are uniformly bounded on Π. This clearly weakens the hypothesis of

Note II since Abel-boundedness follows from (C, j)-boundedness (cf. Note II, p. 560).

10. Multipliers for Abel-bounded expansions. We recall the definition of the Abel means (cf. (4.2) for $\kappa = 1$)

$$A(t)f = \sum_{k=0}^{\infty} e^{-kt} P_k f \qquad (f \in \Pi)\,, \tag{10.1}$$

which reduces to the more familiar form $\sum r^k P_k f$ by setting $-t = \log r$, $0 < r < 1$. Now assume the Abel means to be uniformly bounded, i.e.,

$$\|A(t)f\| \leqq C\|f\| \qquad (f \in \Pi)\,, \tag{10.2}$$

the constant C being independent of $t > 0$ and f. It is clear by the technique of Theorem 7.1 that we have to introduce the following moment sequence space ($\lim_{k\to\infty} \alpha_k = \alpha_\infty$):

$$cbv = \left\{\alpha \in l^\infty;\ \alpha_k - \alpha_\infty = \int_0^\infty e^{-kt} db(t) \text{ for some } b \in BV[0, \infty]\right\}, \tag{10.3}$$

$$\|\alpha\|_{cbv} = \int_0^\infty |db(t)| + |\alpha_\infty|\,.$$

THEOREM 10.1. *If $\{P_k\}$ as above satisfies* (10.2), *then every $\alpha \in cbv$ is a multiplier and*

$$\|\alpha\|_M \leqq C\|\alpha\|_{cbv}\,.$$

PROOF. For each $f \in \Pi$ set

$$f^\alpha = \int_0^\infty \sum_{k=0}^{\infty} e^{-kt} P_k f\, db(t) + \alpha_\infty f\,.$$

Since the sum is finite one may interchange the order of integration and summation to obtain $f^\alpha = \sum \alpha_k P_k f$. Furthermore, f^α satisfies the inequality (9.1) since by (10.2)

$$\|f^\alpha\| \leqq \sup_{t>0} \|A(t)f\| \int_0^\infty |db(t)| + |\alpha_\infty|\,\|f\| \leqq C\|\alpha\|_{cbv}\|f\|\,.$$

Hence all is proved.

The class cbv may be characterized via completely monotone sequences, $\alpha \in l^\infty$ being called completely monotone if $\Delta^m \alpha_k \geqq 0$, $k, m \in \boldsymbol{P}$ (note that here $\Delta\alpha_k = \alpha_k - \alpha_{k+1}$ in contrast to [44; p. 108]). Indeed,

PROPOSITION 10.2 (Hausdorff). *$\alpha \in cbv$ if and only if α is the difference of two completely monotone sequences.*

As already remarked in Sec. 3 and 7, it is quite useful to extend the sequence $\alpha \in s$ to a function $a(x)$ defined for $x \geqq 0$, particularly, if one has to deal with families of sequences. This is obvious in the present instance by introducing the class ($\lim_{t\to\infty} a(t) = a(\infty)$)

$$\begin{aligned}(10.4)\qquad CBV = \Big\{a \in C[0, \infty];\ a(x) - a(\infty) \\ = \int_0^\infty e^{-xt}db(t) \text{ for some } b \in BV[0, \infty]\Big\}, \\ \| a \|_{CBV} = \int_0^\infty | db(t) | + | a(\infty) | .\end{aligned}$$

Since the CBV-norm (also the cbv-norm) is invariant with respect to dilations, $\| \alpha_\rho \|_{cbv} = \| a \|_{CBV}$ (independent of ρ) provided the family $\{\alpha(\rho)\} \subset s$ is of Fejér's type: $\alpha_k(\rho) = a(k/\rho)$.

Apart from verifying the Laplace representation $\int e^{-xt}db(t)$ explicitly in concrete examples, one may use a characterization via completely monotone functions. Here a function $e(x)$ is called completely monotone on $0 \leqq x < \infty$ (on $0 < x < \infty$) if $e(x)$ is continuous for $x \geqq 0$ (for $x > 0$) and $(-1)^k e^{(k)}(x) \geqq 0$, $k \in \boldsymbol{P}$, for $x > 0$ (see e.g. [45; p. 154-155]).

PROPOSITION 10.3 (Bernstein). *$a \in CBV$ if and only if a is the difference of two completely monotone functions on $0 \leqq x < \infty$.*

Now Schoenberg [38] has proved that if $e(x)$ is completely monotone on $0 \leqq x < \infty$ so is $e(\phi(x))$ provided ϕ satisfies

$$\begin{aligned}(10.5)\qquad \phi(x) \geqq 0 \text{ is continuous for } x \geqq 0,\ \phi(0) = 0,\ \phi'(x) \text{ is} \\ \text{completely monotone on } x > 0 \text{ with } \int_{0+} \phi'(x)dx < \infty .\end{aligned}$$

Thus we arrive at the analogue of Hardy's "Second Theorem of Consistency" (cf. [42; p. 30, 50]).

THEOREM 10.4. *If $a \in CBV$, $\chi(\rho)$ is positive for $\rho > 0$, and ϕ satisfies* (10.5), *then $a(\phi(x)/\chi(\rho)) \in CBV$ uniformly in $\rho > 0$.*

Some examples of admissible functions ϕ are x^κ, $0 < \kappa \leqq 1$, $\log^\omega (1 + x)$, $0 < \omega \leqq 1$, but also $\log^\omega (1 + x^\kappa)$ since $\phi(x) = \phi_1(\phi_2(x))$ satisfies (10.5) if ϕ_1, ϕ_2 do by a result due to Schoenberg [38]. Clearly Theorem 10.4 will simplify the verification of (9.2) or (9.3).

Now it is interesting to ask for relations between the multiplier classes bv_{j+1} and cbv as well as between BV_{j+1} and CBV, where BV_{j+1} is normed by

$$\| a \|_{BV_{j+1}} = \frac{1}{\Gamma(j + 1)} \int_0^\infty x^j \,| da^{(j)}(x) | + | a(\infty) | .$$

Note that $BV_{j+1} \subset BV_{k+1}$, $0 \leqq k \leqq j$, in the sense of continuous embedding (cf. [42; p. 24]).

As a first result we have

PROPOSITION 10.5. *For each $j \in \boldsymbol{P}$, $cbv \subset bv_{j+1}$ in the sense of continuous embedding.*

PROOF. By Prop. 10.2 any $\alpha \in cbv$ may be written as $\alpha - \{\alpha_\infty\} = \alpha^1 - \alpha^2$ with completely monotone α^i. Thus (cf. [7])

$$\begin{aligned} \|\alpha\|_{bv_{j+1}} &\leqq \sum_{k=0}^{\infty} \binom{k+j}{j} \Delta^{j+1}\alpha_k^1 + \sum_{k=0}^{\infty} \binom{k+j}{j} \Delta^{j+1}\alpha_k^2 + |\alpha_\infty| \\ &= (\alpha_0^1 - \alpha_\infty^1) + (\alpha_0^2 - \alpha_\infty^2) + |\alpha_\infty| = \|\alpha\|_{cbv} . \end{aligned}$$

The relation between cbv and bv_{j+1} is, with respect to Prop. 10.2, nicely illuminated by the following result (cf. [7]).

PROPOSITION 10.6. *$\alpha \in bv_{j+1}$ if and only if $\alpha - \{\alpha_\infty\} = \alpha^1 - \alpha^2$ where $\alpha^i \in s$, $i = 1, 2$, are $(j+1)$-times monotone, i.e., $\Delta^m \alpha_k^i \geqq 0$, $k \in \boldsymbol{P}$, $0 \leqq m \leqq j+1$.*

The analogue to this proposition for the BV_{j+1}-classes reads

PROPOSITION 10.7. *$a \in BV_{j+1}$ if and only if $a(x) - a(\infty) = a^1(x) - a^2(x)$, where $a^i(x)$, $i = 1, 2$, are $(j+1)$-times monotone on $0 \leqq x < \infty$, i.e., $a^i(x)$ is defined for $x \geqq 0$ (continuous if $j \geqq 1$), $(-d/dx)^k a^i(x) \geqq 0$ for $x > 0$, $0 \leqq k \leqq j-1$, and $(-d/dx)^j a^i(x) \geqq 0$ almost everywhere and non-increasing.*

PROOF. If $a \in BV_{j+1}$ then one has the representation (cf. [42; p. 25])

$$a(x) = \frac{(-1)^{j+1}}{\Gamma(j+1)} \int_x^\infty (t-x)^j da^{(j)}(t) + a(\infty) \qquad (x \geqq 0) .$$

Since $(-1)^j a^{(j)}(x)$ is of bounded variation on $[\varepsilon, \infty)$ for each $\varepsilon > 0$, one may apply the Jordan decomposition to obtain $(-1)^j a^{(j)}(x) = c^1(x) - c^2(x)$ with $c^i(x) \geqq 0$ and decreasing for $x > 0$, thus

$$a^i(x) = \frac{-1}{\Gamma(j+1)} \int_x^\infty (t-x)^j dc^i(t) . \tag{10.6}$$

$a^i(x)$ exists at the origin since

$$a^i(0) = \frac{-1}{\Gamma(j+1)} \int_0^\infty t^j dc^i(t) \leqq \frac{1}{\Gamma(j+1)} \int_0^\infty t^j \, |da^{(j)}(t)| ;$$

the further properties required of a^i are readily read from the representation (10.6).

Now assume conversely that $a(x)$ is $(j+1)$-times monotone, where one may suppose $a(\infty)=0$ without loss of generality. Then

$$a(\varepsilon)-a(N)=-\int_\varepsilon^N a'(u)du=-ua'(u)\,|_\varepsilon^N+\int_\varepsilon^N ua''(u)du$$
$$=\sum_{k=1}^{j}\frac{(-1)^k}{\Gamma(k+1)}u^k a^{(k)}(u)\,|_\varepsilon^N+\frac{(-1)^{j+1}}{\Gamma(j+1)}\int_\varepsilon^N u^j da^{(j)}(u)\,.$$

Now Lévy [36] and Williamson [46] have shown that $a(x)$ being $(j+1)$-times monotone implies $u^k a^{(k)}(u)=o(1)$ for $u\to 0+$ and for $u\to\infty$, $1\leqq k\leqq j$. Thus, one has

$$\|a\|_{BV_{j+1}}=\frac{(-1)^{j+1}}{\Gamma(j+1)}\int_0^\infty u^j da^{(j)}(u)=a(0)<\infty$$

by letting $\varepsilon\to 0+$, $N\to\infty$, i.e., the converse holds.

Now carrying over the proof of Prop. 10.5 we immediately arrive at

PROPOSITION 10.8. *For each $j\in \boldsymbol{P}$, $CBV\subset BV_{j+1}$ in the sense of continuous embedding.*

Let us conclude this section with the observation that everything has turned out to be as expected: the stronger the summability method the narrower the multiplier class. For we have seen in Note II or in [42; p. 19, 54] that the Abel means are stronger than the (C,β)-means, which in turn are stronger than the (C,γ)-means for $0\leqq\gamma\leqq\beta$. Correspondingly (for the last inclusions with arbitrary $\beta\geqq\gamma\geqq 0$ see [42; p. 20, 37])

$$cbv\subset bv_{\beta+1}\subset bv_{\gamma+1}\,,\qquad CBV\subset BV_{\beta+1}\subset BV_{\gamma+1}\,.$$

11. Applications.

11.1. Abel-Cartwright and Picard means. Let $X, H, \{P_k\}$ be as in Sec. 9 such that the sequence of projections $\{P_k\}$ satisfies (10.2). Then the Abel-Cartwright means (4.2) of order $0<\kappa\leqq 1$ are uniformly bounded operators on X since $\exp\{-x^\kappa\}$, $0<\kappa\leqq 1$, is completely monotone on $0\leqq x<\infty$ (cf. Theorem 10.4) and thus belongs to CBV by Prop. 10.3. The Picard means of order $\kappa>0$, defined by

$$P_\kappa(\rho)f=\sum_{k=0}^{\infty}p_\kappa(k/\rho)P_kf\,,\qquad p_\kappa(x)=(1+x)^{-\kappa}\qquad (f\in\Pi)\,,$$

are also uniformly bounded on X since $(1+x)^{-\kappa}$ is completely monotone. Since $\lim_{\rho\to\infty}w_\kappa(k/\rho)=\lim_{\rho\to\infty}p_\kappa(k/\rho)=1$ for each $k\in\boldsymbol{P}$, both means are approximation processes.

Replacing the discrete parameter $(n+1)$ of the Abel-Cartwright means (4.2) by the continuous one $\rho=\varepsilon^{-1/\kappa}$,

$$W_\kappa^*(\varepsilon)f = \sum_{k=0}^{\infty} e^{-\varepsilon k^\kappa} P_k f \qquad (0 < \kappa \leqq 1,\ f \in \Pi)\ ,$$

it is immediately clear that $\{W_\kappa^*(\varepsilon)\}_{\varepsilon>0}$ is a uniformly bounded semi-group of class (C_0) on all of X with (closed linear) infinitesimal generator $A_\kappa =$ s-$\lim_{\varepsilon\to 0+} \varepsilon^{-1}\{W_\kappa^*(\varepsilon) - I\}$. The restriction of A_κ to Π gives the representation

$$A_\kappa f = \sum_{k=0}^{\infty} - k^\kappa P_k f \qquad (f \in \Pi)\ .$$

Since with A_κ also all integral powers of A_κ are closed linear operators (cf. [26; p. 12]) we have proved that $\{k^\kappa\} \in s$ generates an X-closed operator B^ψ for any $\kappa > 0$.

THEOREM 11.1. *Let X, H, $\{P_k\}$ be as in Sec. 9 with $\{P_k\}$ satisfying (10.2), and let B^β be a closed linear operator generated by $\psi = \{k^\beta\}$, $0 < \beta \leqq \kappa \leqq 1$. Then*

$$\| W_\kappa(\rho)f - f \| \leqq C'\rho^{-\beta} \| B^\beta f \| \qquad (f \in X^\psi)\ .$$

In view of Theorem 9.1 a) we have only to examine the sequence $(\gamma = \beta/\kappa)$

$$\eta_k(\rho) = e(k^\kappa/\rho^\kappa)\ , \qquad e(x) = x^{-\gamma}(e^{-x} - 1) \qquad (0 < \gamma \leqq 1)\ .$$

By Theorems 10.1 and 10.4 one can restrict the matter to a discussion of $e(x)$. Since (cf. [43])

$$e(x) = -\int_0^\infty e^{-xt}c(t)dt\ , \qquad c(u) = \frac{1}{\Gamma(\gamma)}\begin{cases} u^{\gamma-1} & , \quad 0 < u < 1 \\ u^{\gamma-1} - (u-1)^{\gamma-1}\ , & \quad u > 1\ , \end{cases}$$

$c(t)$ being clearly integrable on $(0, \infty)$, everything is proved.

Note that for $\beta = \kappa$ we have examined the saturation quotient of Theorem 8.1 so that we could reformulate Theorem 8.1 for $0 < \kappa \leqq 1$ under the weaker hypothesis (10.2) (after an obvious modification of statement iii) in Theorem 8.1).

It is not possible to derive a Zamansky-type inequality for the Abel-Cartwright means since one would have to examine functions of type $x^\gamma e^{-x}/(1 - e^{-x})$, $\gamma \geqq 1$, which, after analytic continuation, are not bounded on vertical lines, thus cannot be Laplace-Stieltjes transforms of some $b \in BV[0, \infty]$.

To derive a Zamansky-type inequality for the Picard means we only consider the case $\kappa = 2$ for the sake of simplicity. Thus we have to examine the function

$$e(x) = x^\beta(1 + x)^{-2}/(1 - (1 + x)^{-2}) = x^{\beta-1}(2 + x)^{-1} \qquad (1 \leqq \beta \leqq 2)$$

($\beta \geqq 1$ being necessary to remove the singularity at $x = 0$, $\beta \leqq 2$ being necessary for the boundedness of $e(x)$). In case $\beta = 1$ ($\beta = 2$) the function $e(x)$ is completely monotone for $x \geqq 0$ (difference of two completely monotone functions), thus a Laplace-Stieltjes transform; in case $1 < \beta < 2$ one has

$$e(x) = \int_0^\infty e^{-xt} c(t)dt ,$$

$$c(t) = \frac{t^{1-\beta}}{\Gamma(2-\beta)} - \frac{2}{\Gamma(2-\beta)} \int_0^t (t-y)^{1-\beta} e^{-2y} dy \in L^1(0, \infty) .$$

THEOREM 11.2. *Let X, H, $\{P_k\}$ be as in Sec. 9 with $\{P_k\}$ satisfying* (10.2), *and let B^β be a closed linear operator generated by $\psi = \{k^\beta\}$, $1 \leqq \beta \leqq 2$. Then*

$$\| B^\beta P_2(\rho) f \| \leqq C' \rho^\beta \| P_2(\rho) f - f \| \qquad (f \in X) .$$

11.2. Hermite series with weight. Choose $X = L_w^p(-\infty, \infty)$, $w(x) = e^{-x^2}$, $1 \leqq p < \infty$, $H = L_w^2(-\infty, \infty)$, where

$$L_w^p(-\infty, \infty) = \left\{ f ; \| f \|_{p,w} = \left(\int_{-\infty}^\infty | f(x) |^p e^{-x^2} dx \right)^{1/p} < \infty \right\} .$$

Define projections on H with the aid of the Hermite polynomials $H_k(x)$ (cf. Sec. 8.5) by

$$P_k f(x) = \{2^k k! \sqrt{\pi}\}^{-1} \left[\int_{-\infty}^\infty f(u) H_k(u) e^{-u^2} du \right] H_k(x) \qquad (f \in H) .$$

Then the projections $\{P_k\} \subset [H]$ are mutually orthogonal, and satisfy the representation $f = \sum_{k=0}^\infty P_k f$ for $f \in H$ since

$$\lim_{n \to \infty} \int_{-\infty}^\infty \left| \sum_{k=0}^n P_k f(x) - f(x) \right|^2 e^{-x^2} dx = 0 ,$$

this being not so for $p \neq 2$ as was shown by Pollard [16]. Furthermore, $P_k(H) \subset X$ for any $k \in \boldsymbol{P}$, and the finite linear combinations Π of $H_k(x)$ are dense in L_w^p, $1 \leqq p < \infty$. For, according to the Stone-Weierstrass theorem the (ordinary) polynomials are dense in L_w^p, and each (ordinary) polynomial can be uniquely represented by a finite linear combination of Hermite polynomials.

Muckenhoupt [37] has proved ($0 < r < 1$)

$$\left\| \sum_{k=0}^\infty r^k P_k f \right\| \leqq C \| f \| \qquad (f \in \Pi) ,$$

and this coincides with our hypothesis (10.2) for $-t = \log r$. Muckenhoupt's result is best possible in the sense that the (C, j)-means of $\sum P_k f$ are not uniformly bounded for any $j \in \boldsymbol{N}$ in case $p \neq 2$ as Askey-Hirschman

[2] have shown.

Hence all hypotheses are fulfilled; in particular, $W_\kappa(\rho)$, $0 < \kappa \leqq 1$, is an approximation process on L_w^p, $1 \leqq p < \infty$, for $\rho \to \infty$; note that $W_{1/2}(\rho)$ for $\rho^{-1} = 2t^2$ coincides with Muckenhoupt's "alternate Poisson integral" which satisfies a certain second order elliptic differential equation. Muckenhoupt also gives the concrete extension of $W_{1/2}(\rho)$ to all of L_w^p by an explicit representation. Further note that $e^{-t\log(2x+1)} \in CBV$ according to Theorem 10.4, so that

$$U^t f = \sum_{k=0}^{\infty} (2k+1)^{-t} P_k f \qquad (f \in \Pi)$$

is a uniformly bounded semi-group of class (C_0) on L_w^p, $1 \leqq p < \infty$. The operators U^t were essentially introduced in [35; p. 672] as a semi-group of class (C_0). Before restating Theorems 11.1 and 11.2 observe that B^β for $\beta = 1$ has the representation $B^1 = 1/2(d/dx)^2 - x(d/dx)$.

COROLLARY 11.3. *Let X be given as above.*

a) *For $0 < \beta \leqq \kappa \leqq 1$ one has the Jackson-type inequality*

$$\left(\int_{-\infty}^{\infty} | W_\kappa(\rho)f(x) - f(x) |^p e^{-x^2} dx\right)^{1/p} \leqq C'\rho^{-\beta}\left(\int_{-\infty}^{\infty} | B^\beta f(x) |^p e^{-x^2} dx\right)^{1/p}$$

for all $f \in X^\psi$, where $\psi = \{k^\beta\}$ generates the closed operator B^β.

b) *There holds the Zamansky-type inequality*

$$\left\| \left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right) P_2(\rho) f(x) \right\|_{p,w} \leqq C'\rho \, \| P_2(\rho) f - f \|_{p,w} \qquad (f \in X) .$$

REMARK. Let us return to Hermite expansions in usual $L^p(-\infty, \infty)$-spaces (cf. Sec. 8.5). Since Abel summability is then an old result we can also cover the material in [25] on saturation completely. Moreover, we recently noticed that Freud-Knapowski [34] have even proved $(C, 1)$-boundedness for $p = \infty$; observing that the multipliers on $L^1(-\infty, \infty)$ and $L^\infty(-\infty, \infty)$ coincide (cf. [35; p. 572–574] and the literature cited there), we may immediately apply all the general results of Sec. 4.1 and 8.1 for all $1 \leqq p \leqq \infty$. Finally let us mention that our Theorem 10.1 (in case of Hermite expansions in $L^p(-\infty, \infty)$) is derived in Hille-Phillips [35; p. 574] by the same technique from Abel-boundedness (indeed, the substitution $r = e^{-t}$ shows that cbv is identical with the moment sequence space used there).

11.3. Laguerre series with weight. Choose $X = L_w^p(0, \infty)$, $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, $1 \leqq p < \infty$, $H = L_w^2(0, \infty)$, where

$$L_w^p(0, \infty) = \left\{f; \|f\|_{p,w} = \left(\int_0^\infty |f(x)|^p x^\alpha e^{-x} dx\right)^{1/p} < \infty\right\}.$$

Define projections on H with the aid of the Laguerre polynomials $L_k^{(\alpha)}(x)$, $\alpha > -1$ (cf. Sec. 4.4) by

$$P_k^{(\alpha)} f(x) = \left\{\Gamma(\alpha + 1)\binom{k+\alpha}{k}\right\}^{-1} \left[\int_0^\infty f(u) L_k^{(\alpha)}(u) u^\alpha e^{-u} du\right] L_k^{(\alpha)}(x) \quad (\alpha > -1) .$$

Then the projections $\{P_k\} \subset [H]$ are mutually orthogonal, and satisfy the representation $f = \sum_{k=0}^\infty P_k^{(\alpha)} f$ for any $f \in H$ since

$$\lim_{n\to\infty} \int_0^\infty \left|\sum_{k=0}^n P_k^{(\alpha)} f(x) - f(x)\right|^2 x^\alpha e^{-x} dx = 0 \qquad (\alpha > -1) ,$$

this being not so for $p \neq 2$ as was indicated by Pollard [16]. Furthermore, $P_k^{(\alpha)}(H) \subset X$ for any $k \in \boldsymbol{P}$, and the finite linear combinations Π of $L_k^{(\alpha)}(x)$ are dense in L_w^p, $1 \leqq p < \infty$. Indeed, each $f \in L_w^p$ can be approximated by a continuous function g with compact support in the L_w^p-norm (cf. [39; p. 45]), and the procedure described in Stone [40; p. 76] may be applied to g so that the (ordinary) polynomials are dense in L_w^p, thus also Π is dense in L_w^p (see also [41; Sec. 5.7]).

Again, Muckenhoupt [37] has verified our hypothesis (10.2),

$$\left\|\sum_{k=0}^\infty e^{-kt} P_k^{(\alpha)} f\right\| \leqq C \|f\| \qquad (f \in \Pi) .$$

This result is again best possible in the sense that the (C, j)-means of $\sum P_k^{(\alpha)} f$ are not uniformly bounded for any $j \in \boldsymbol{N}$ unless $p = 2$. In fact, following a written communication of Professor Askey, one may use the method of Askey-Hirschman [2] for the function $e^{cx} \in L_w^p$, $1/2 < c < 1/p$, $1 \leqq p < 2$ (a duality argument then gives the assertion for $p > 2$). Thus with the aid of the generating function for Laguerre polynomials [8; p. 189, (17)] it is clear that for $-\infty < c < 1/2$

$$\left\{\Gamma(\alpha + 1)\binom{k+\alpha}{k}\right\}^{-1} \int_0^\infty e^{cx} L_k^{(\alpha)}(x) x^\alpha e^{-x} dx = \left(1 - \frac{c}{c-1}\right)^{\alpha+1} \left(\frac{c}{c-1}\right)^k .$$

Since both sides are analytic in c for $\operatorname{Re} c < 1$, this relation yields $\|P_k\| = O[c/(1-c)]^k$, $1/2 < c < 1$, in contradiction to (7.6) which is necessary for the (C, j)-means, $j \in \boldsymbol{P}$, to be uniformly bounded.

Before restating the results of Sec. 11.1 we note that a natural closed linear extension of B^β for $\beta = 1$, $\psi = \{-k\}$, is given by the differential operator $B^1 = x(d^2/dx^2) + (\alpha + 1 - x)(d/dx)$.

COROLLARY 11.4. *Let X be given as above.*

a) *For $0 < \beta \leqq \kappa \leqq 1$ one has the Jackson-type inequality*

$$\left(\int_0^\infty | W_\kappa(\rho)f(x) - f(x) |^p x^\alpha e^{-x} dx\right)^{1/p} \leqq C'\rho^{-\beta}\left(\int_0^\infty | B^\beta f(x) |^p x^\alpha e^{-x} dx\right)^{1/p}$$

for all $f \in X^\psi$, where $\psi = \{-k^\beta\}$ generates the closed operator B^β.

b) *There holds the Zamansky-type inequality*

$$\left\| \left(x\frac{d^2}{dx^2} + (\alpha + 1 - x)\frac{d}{dx}\right)^2 P_2(\rho)f \right\|_{p,w} \leqq C'\rho^2 \| P_2(\rho)f - f \|_{p,w} \qquad (f \in X)\,.$$

REFERENCES

[33] P. L. BUTZER, R. J. NESSEL AND W. TREBELS, Multipliers with respect to spectral measures in Banach spaces and approximation. I: Radial multipliers in connection with Riesz-bounded spectral measures, J. Approximation Theory 8 (1973), 335-356.

[34] G. FREUD AND S. KNAPOWSKI, On linear processes of approximation III, Studia Math. 25 (1965), 374-383.

[35] E. HILLE AND R. S. PHILLIPS, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ. 31, Providence, Rhode Island 1957.

[36] P. LÉVY, Extensions d'un théorème de D. Dugué et M. Girault, Z. Wahrscheinlichkeitstheorie 1 (1962), 159-173.

[37] B. MUCKENHOUPT, Poisson integrals for Hermite and Laguerre expansions, Trans. Amer. Math. Soc. 139 (1969), 231-242.

[38] I. J. SCHOENBERG, Metric spaces and completely monotone functions, Ann. of Math. 39 (1938), 811-841.

[39] A. SCHÖNHAGE, Approximationstheorie, de Gruyter, Berlin 1971.

[40] M. H. STONE, A generalized Weierstrass approximation theorem, in: Studies in Modern Analysis (R.C. Buck, Ed.) MAA, Prentice-Hall 1962, 30-87.

[41] G. SZEGÖ, Orthogonal Polynomials, rev. ed. Amer. Math. Soc. Colloq. Publ. 23, Providence, Rhode Island 1959.

[42] W. TREBELS, Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory, Lecture Notes in Math. Vol. 329, Springer, Berlin 1973.

[43] U. WESTPHAL, Ein Kalkül für gebrochene Potenzen infinitesimaler Erzeuger von Halbgruppen und Gruppen von Operatoren I: Halbgruppenerzeuger, Compositio Math. 22 (1970), 67-103.

[44] D. V. WIDDER, The Laplace Transform, Princeton Univ. Press, Princeton 1946.

[45] D. V. WIDDER, An Introduction to Transform Theory, Academic Press, New York 1971.

[46] R. E. WILLIAMSON, Multiply monotone functions and their Laplace transforms, Duke Math. J. 23 (1956), 189-207.

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