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ON KAEHLER METRICS ON A COMPACT HOMOGENEOUS COMPLEX MANIFOLD

MITSUHIRO ITOH

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1. **Introduction.** It was shown by M. Berger [1] that an arbitrary Einstein Kaehler metric on a complex projective space is equivalent to the Fubini-Study metric. Moreover, Y. Matsushima proved in [3] that on a Kaehler C-space (i.e., a simply connected compact homogeneous complex manifold which admits a Kaehler metric), Einstein Kaehler metrics are mutually equivalent. Here the equivalency of Kaehler metrics g_1 and g_2 on a Kaehler manifold denotes that there exist a holomorphic transformation ϕ of the manifold and a positive constant c such that $cg_1 = \phi^* g_2$.

On a compact Kaehler manifold, the scalar multiple of the Ricci form by $1/(2\pi)$ represents the first Chern class of this manifold and the constancy of the scalar curvature means that the Ricci form is harmonic. On a Kaehler C-space M, we have a G_0-invariant Einstein Kaehler metric $\tilde{g}$ which is called the canonical Einstein metric, where G_0 is a compact group of holomorphic transformations of M ([3]). Then, Matsushima's theorem "any Einstein Kaehler metric g on a Kaehler C-space M is equivalent to the canonical metric $\tilde{g}$" is interpreted as the following "if any Kaehler metric on M satisfies that its Kaehler form is cohomologous to that of $\tilde{g}$ and its scalar curvature is equal to that of $\tilde{g}$, then it is equivalent to the canonical metric $\tilde{g}$".

The purpose of this paper is a generalization of Matsushima's theorem. In fact, we shall prove in Theorem B and Corollary C in §2 that any Kaehler metric on a Kaehler C-space satisfying a certain condition on curvature is equivalent to the canonical Einstein metric $\tilde{g}$.

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2. **Results.** Let g be a Kaehler metric on a Kaehler manifold of complex dimension n. Let S be the Ricci tensor of the metric g. The metric g is called an Einstein Kaehler metric if S is given by the scalar multiple of g. With respect to a local coordinate system $z^1, \cdots, z^n$, g and S can be expressed as

$$g = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha}\cdot dz^{\bar{\beta}}\,, \tag{2.1}$$

$$S = 2\sum_{\alpha,\beta} S_{\alpha\bar{\beta}} dz^{\alpha}\cdot dz^{\bar{\beta}}\,. \tag{2.2}$$

We define 2-forms ω and σ, called the Kaehler form and the Ricci form by

$$\omega = i\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^{\alpha}\wedge dz^{\bar{\beta}}\,, \tag{2.3}$$

$$\sigma = i\sum_{\alpha,\beta} S_{\alpha\bar{\beta}} dz^{\alpha}\wedge dz^{\bar{\beta}}\,. \tag{2.4}$$

The scalar curvature ρ of g is given by

$$\rho = 2\sum_{\alpha,\beta} g^{\alpha\bar{\beta}} S_{\alpha\bar{\beta}}\,, \tag{2.5}$$

where the matrix $(g^{\alpha\bar{\beta}})$ is the inverse of $(g_{\alpha\bar{\beta}})$.

LEMMA. *Let g be a Kaehler metric on a Kaehler manifold of complex dimension n. Then we have*

$$\sigma\wedge\omega^{n-1} = \frac{1}{2n}\rho\cdot\omega^{n}\,. \tag{2.6}$$

PROOF. We may check (2.6) pointwise. For an arbitrary point p, we can choose a suitable local coordinate system around p such that $g_{\alpha\bar{\beta}}(p) = \delta_{\alpha\beta}$, $S_{\alpha\bar{\beta}}(p) = S_{\alpha}\cdot\delta_{\alpha\beta}$ $\alpha, \beta = 1, \cdots, n$, that is, at p

$$\omega = i\sum_{\alpha} dz^{\alpha}\wedge dz^{\bar{\alpha}}\,,\quad \sigma = i\sum_{\alpha} S_{\alpha} dz^{\alpha}\wedge dz^{\bar{\alpha}}\,.$$

Then we have

$$\omega^{n} = i^{n}\cdot n!\cdot dz^{1}\wedge dz^{\bar{1}}\wedge\cdots\wedge dz^{n}\wedge dz^{\bar{n}}\,, \tag{2.7}$$

$$\omega^{n-1} = i^{n-1}\cdot(n-1)!\sum_{\alpha=1}^{n} dz^{1}\wedge dz^{\bar{1}}\wedge\cdots\wedge\widehat{dz^{\alpha}\wedge dz^{\bar{\alpha}}}\wedge\cdots\wedge dz^{n}\wedge dz^{\bar{n}} \tag{2.8}$$

which, together with $\rho = 2\sum_{\alpha,\beta} g^{\alpha\bar{\beta}} S_{\alpha\bar{\beta}} = 2\sum_{\alpha} S_{\alpha}$, imply

$$\begin{aligned}\sigma\wedge\omega^{n-1} &= i^{n}(n-1)!\Big(\sum_{\alpha} S_{\alpha} dz^{\alpha}\wedge dz^{\bar{\alpha}}\Big)\\ &\quad\wedge\Big(\sum_{\beta} dz^{1}\wedge dz^{\bar{1}}\wedge\cdots\wedge\widehat{dz^{\beta}\wedge dz^{\bar{\beta}}}\wedge\cdots\wedge dz^{n}\wedge dz^{\bar{n}}\Big)\\ &= i^{n}(n-1)!\Big(\sum_{\alpha} S_{\alpha}\Big) dz^{1}\wedge dz^{\bar{1}}\wedge\cdots\wedge dz^{n}\wedge dz^{\bar{n}}\\ &= \frac{1}{n}\Big(\sum_{\alpha} S_{\alpha}\Big)\omega^{n} = \frac{1}{2n}\rho\cdot\omega^{n}\,.\end{aligned}$$

q.e.d.

Making use of Lemma, we have the following.

THEOREM A. *Let g and $\tilde{g}$ be two Kaehler metrics on a compact Kaehler manifold M such that their Kaehler forms are mutually cohomologous. If the scalar curvature $\tilde{\rho}$ of $\tilde{g}$ is constant and the scalar curvature ρ of g satisfies $\rho \leqq \tilde{\rho}$ (or $\rho \geqq \tilde{\rho}$) everywhere on M, then ρ must be constant and equal to $\tilde{\rho}$.*

PROOF. Let ω and $\tilde{\omega}$ be the Kaehler forms of g and $\tilde{g}$. We shall write $\phi \sim \psi$ symbolically if ϕ is cohomologous to ψ. Then we have $\omega \sim \tilde{\omega}$ from the condition. If we denote by σ and $\tilde{\sigma}$ the Ricci forms of g and $\tilde{g}$ respectively, then we obtain $\sigma \wedge \omega^{n-1} \sim \tilde{\sigma} \wedge \tilde{\omega}^{n-1}$ where $n = \dim_c M$, since both $(1/2\pi)\sigma$ and $(1/2\pi)\tilde{\sigma}$ represent the first Chern class of M([2]). On the other hand $\tilde{\rho}\cdot\omega^n \sim \tilde{\rho}\cdot\tilde{\omega}^n$ since $\tilde{\rho}$ is constant. By the aid of (2.6) in Lemma, $\rho\cdot\omega^n - \tilde{\rho}\cdot\omega^n \sim \rho\cdot\omega^n - \tilde{\rho}\cdot\tilde{\omega}^n = 2n(\sigma \wedge \omega^{n-1} - \tilde{\sigma} \wedge \tilde{\omega}^{n-1}) \sim 0$. Then we have $\int_M (\rho - \tilde{\rho})\omega^n = 0$. By the condition on ρ, we can conclude that ρ is constant and equal to $\tilde{\rho}$. q.e.d.

Now we are in a position to prove the following theorem.

THEOREM B. *Let M be a Kaehler C-space with the canonical Einstein Kaehler metric $\tilde{g}$. Let g be another Kaehler metric on M whose Kaehler form is cohomologous to that of $\tilde{g}$. If the scalar curvatures ρ and $\tilde{\rho}$ of g and $\tilde{g}$ satisfy $\rho \leqq \tilde{\rho}$ or else $\rho \geqq \tilde{\rho}$ everywhere on M, then there exists a holomorphic transformation ϕ of M such that $g = \phi^*\tilde{g}$, that is, g is equivalent to $\tilde{g}$.*

PROOF. Let ω and $\tilde{\omega}$ be the Kaehler forms, σ and $\tilde{\sigma}$ the Ricci forms of g and $\tilde{g}$. Since $\omega \sim \tilde{\omega}$, $\sigma \sim \tilde{\sigma}$ and $\tilde{\sigma} = c\tilde{\omega}$ for a positive constant c, we have $\sigma \sim c\omega$. The constancy of $\tilde{\rho}$ means that ρ is constant from Theorem A, hence σ is harmonic (see [2]). We can conclude that $\sigma = c\omega$, i.e., g is an Einstein Kaehler metric. Then from Matsushima's theorem ([3]), there exist $\phi \in G$ and a positive constant α such that $g = \alpha\phi^*\tilde{g}$ where G is the identity component of the group of all holomorphic transformations of M. Since $\omega \sim \tilde{\omega}$ and ϕ is a transformation homotopic to the identity transformation, we have $\alpha = 1$, that is, $g = \phi^*\tilde{g}$. q.e.d.

Let ρ, S and K be the scalar curvature, the Ricci tensor and the sectional curvature of a Kaehler metric g on a manifold of complex dimension n. By the definition of ρ, S and K, we obtain the following formulas (see [1]):

$$\rho = 2\sum_{i=1}^{n} S(V_i, V_i)\,, \tag{2.9}$$

$$(2.10)\qquad \rho = 2\left[\sum_{i=1}^{n} K(\{V_i, JV_i\}) + 2\sum_{i<j}(K(\{V_i, V_j\}) + K(\{V_i, JV_j\}))\right],$$

where $\{V_i, JV_i\}_{i=1,\cdots,n}$ is an orthonormal frame at a point p and $\{X, Y\}$ is the plane spanned by tangent vectors X and Y at p and

$$(2.11)\qquad \rho = \frac{n(n+1)}{\mathrm{vol}\,(S^{2n-1})}\int_{X\in U_p} K(\{X, JX\})dX\,,$$

where U_p, which denotes the set of all unit tangent vectors at p, is identified with S^{2n-1}, the volume element dX and the volume of S^{2n-1} are canonical. From Theorem B, the following is easily obtained.

COROLLARY C. *Let M, $\tilde{g}$ and $\tilde{\rho}$ be as in Theorem B. Let g be another Kaehler metric on M whose Kaehler form is cohomologous to that of $\tilde{g}$. If the metric g satisfies one of the following conditions, then the metric g is equivalent to the canonical metric $\tilde{g}$.*

I) *The Ricci tensor S of g satisfies either $S(V, V) \leqq (1/2n)\tilde{\rho}$ or $S(V, V) \geqq (1/2n)\tilde{\rho}$ for any unit vector V.*

II) *The sectional curvature of any plane with respect to g is not greater (or not smaller) than $(1/2n^2)\tilde{\rho}$.*

III) *The sectional curvature of any holomorphic plane with respect to g is not greater (or not smaller) than $(1/n(n+1))\tilde{\rho}$.*

An n-dimensional complex projective space $P_n(C)$ admits the Fubini-Study metric of positive constant holomorphic sectional curvature c. It is well known that $P_n(C)$ is a Kaehler C-space and the scalar curvature of the metric is equal to $n(n+1)c$. Therefore we have:

COROLLARY D. *Let $\tilde{g}$ be Fubini-Study metric on $P_n(C)$ of constant holomorphic curvature c and g be a Kaehler metric on $P_n(C)$ whose Kaehler form is cohomologous to that of $\tilde{g}$. If g satisfies one of the following, then g is equivalent to $\tilde{g}$.*

I) *The scalar curvature of g is not greater (or not smaller) than $n(n+1)c$ everywhere on $P_n(C)$.*

II) *The Ricci tensor S of g satisfies either $S(V, V) \leqq ((n+1)/2)c$ or else $S(V, V) \geqq ((n+1)/2)c$ for any unit vector V.*

III) *The sectional curvature of any plane with respect to g is not greater (or not smaller) than $(1/2 + 1/2n)c$.*

IV) *The sectional curvature of any holomorphic plane with respect to g is not greater (or not smaller) than c.*

REMARK. If the second Betti number of a Kaehler C-space M is equal to one (for example, an irreducible Hermitian symmetric space of compact type), there can not exist any Kaehler metric on M such that the volumes of M take the same value with respect to it and the canonical metric, and its scalar curvature is not equal to the scalar curvature of the canonical metric anywhere on M.

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DEPARTMENT OF THE FOUNDATIONS
OF MATHEMATICAL SCIENCES
TOKYO UNIVERSITY OF EDUCATION
TOKYO, JAPAN