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ON THE WEAKLY CONTINUOUS CONSTANT FIELD OF HILBERT SPACE AND ITS APPLICATION TO THE REDUCTION THEORY OF VON NEUMANN ALGEBRA

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1. Introduction. In the present paper, we shall introduce a notion of the weakly continuous constant fields of a Hilbert space over a hyperstonean space. We have defined in [10] the continuous fields of Hilbert spaces which give a characterization of AW^*-modules, where the following property has an important role in our definition: Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces introduced in [10] and $\xi = \{\xi(\omega)\}$, $\eta = \{\eta(\omega)\}$ be two elements of H, then the function $\omega \longrightarrow (\xi(\omega) | \eta(\omega))$ is continuous. But we can not expect, in general, the existence of the continuous constant fields of an infinite dimensional Hilbert space satisfying the conditions in [10; Definition 3.1]. Thus, replacing the condition (4) of [10; Definition 3.1], we define another continuous constant fields of a Hilbert space. Namely, we shall introduce a new notion of the weakly continuous constant field of a Hilbert space.

Let K be a Hilbert space and Ω a hyperstonean space. We set $F = C(\Omega, K)$ and consider the set $H = W_F(\Omega, K)$ of all weakly continuous vector fields with respect to F, where $C(\Omega, K)$ is the space of all K-valued continuous functions on Ω. Then, even if ξ and η are two elements of $H = W_F(\Omega, K)$, the function $\omega \longrightarrow (\xi(\omega) | \eta(\omega))$ is not necessarily continuous, and so H does not necessarily become an AW^*-module introduced by Kaplansky [4]. However, the space $H = W_F(\Omega, K)$ turns out to be a $C(\Omega)$-moduled Banach space with respect to the norm $\|\xi\| = \sup\{\|\xi(\omega)\| : \omega \in \Omega\}$ and we can show that the set $B(H)$ of all bounded $C(\Omega)$-module homomorphisms of H into H becomes a von Neumann algebra of type I with the center *-isomorphic to $C(\Omega)$. As an application of the above result, if Ω is a hyperstonean space, we can show that, if $\mathfrak{A}$ is a von Neumann algebra acting on K and $\mathscr{A} = C(\Omega)$, then the tensor product $\mathfrak{A} \otimes \mathscr{A}$ of $\mathfrak{A}$ and $\mathscr{A}$ is *-isomorphic to the algebra $W(\Omega, K, \mathfrak{A})$ where $W(\Omega, K, \mathfrak{A})$ means the set of all elements $A = \{A(\omega)\}$ of $B(H)$ such that $A(\omega) \in \mathfrak{A}$ for all $\omega \in \Omega$ (Definition 17). In general, let $C(\Omega, \mathfrak{A})$ be the set of all $\mathfrak{A}$-valued continuous functions on Ω, then $C(\Omega, \mathfrak{A})$ becomes a C^*-algebra under the pointwise multiplication defined by $AB = \{A(\omega)B(\omega)\}$ for every A and $B \in C(\Omega, \mathfrak{A})$. But, since our algebra $W(\Omega, K, \mathfrak{A})$ is a von Neumann algebra, we can not define the same multiplication in $W(\Omega, K, \mathfrak{A})$ by our result [8] and [9]. Nevertheless, we can fortunately define the multiplication in $W(\Omega, K, \mathfrak{A})$ as a subalgebra of $B(H)$.

The above mentioned second result is closely related to the Sakai's theorem [6] and [7; Theorem 1.22.13] which is given as follows: Let $\mathscr{A} = L^\infty(\Omega_0, \mu)$ be an abelian von Neumann algebra and $\mathfrak{A}$ a von Neumann algebra acting on a separable Hilbert space, then the tensor product $\mathfrak{A} \otimes \mathscr{A}$ of $\mathfrak{A}$ and $\mathscr{A}$ is represented as the algebra $L^\infty(\Omega_0, \mu, \mathfrak{A})$ of all essentially bounded weak *-measurable $\mathfrak{A}$-valued functions on Ω_0. The Sakai's theorem contains naturally the assumption of separability for $\mathfrak{A}$ as a measure theoretic result, whereas our result is free from the separability assumption.

2. The weakly continuous constant fields of Hilbert spaces. In this paper, we use the following notations: Ω is a hyperstonean space and $F = C(\Omega, K)$ is the set of all K-valued continuous functions on Ω where K is a Hilbert space. Then, if Ω is a hyperstonean space; any first category subset of Ω is a nowhere dense set in Ω [2; Corollary of Proposition 5].

We introduce first the following notion which says the weakly continuous constant field of Hilbert space over Ω.

DEFINITION 1. Let Ω be a hyperstonean space, K a Hilbert space and $H(\omega) = K$ for all $\omega \in \Omega$; then a vector field $\xi = \{\xi(\omega)\} \in \prod_{\omega \in \Omega} H(\omega)$ is called a weakly continuous vector field with respect to F if, for every $\eta \in F$, the function $\omega \longrightarrow (\xi(\omega) | \eta(\omega))$ is continuous on Ω. We denote the set of all weakly continuous vector fields with respect to F by $H = W_F(\Omega, K)$ or simply $W(\Omega, K)$, and call the weakly continuous constant field of Hilbert space K over Ω.

Under Definition 1, we give the norm by the form; for every $\xi \in H = W(\Omega, K)$, $||\xi|| = \sup\{||\xi(\omega)||: \omega \in \Omega\}$, then we can show by an elementary computation that $H = W(\Omega, K)$ becomes a $C(\Omega)$-moduled Banach space with respect to the above defined norm $||\cdot||$.

We have, in [10], introduced the notion of continuous vector field in which we have the following fact: If $\xi = \{\xi(\omega)\}$ is a continuous vector field, then the function $\omega \longrightarrow ||\xi(\omega)||$ is continuous on Ω. But, under Definition 1, even if ξ is an element of $W(\Omega, K)$, the function $\omega \longrightarrow ||\xi(\omega)||$ is not necessarily continuous. If ξ is an element of F, then the function

$\omega \to \|\xi(\omega)\|$ is continuous on Ω. Conversely, if ξ is an element of $H = W(\Omega, K)$ such that the function $\omega \to \|\xi(\omega)\|$ is continuous on Ω, then we can show that ξ is an element of F. This fact is useful for the later part in this paper and so we give the proof.

For the above fact, we have the following considerations. Let $\mathscr{M}$ be a closed submodule of $H = W(\Omega, K)$, then we can show by a similar way to [11; Proposition 1.3] that $(\mathscr{M} \cap F)(\omega)$ is a closed subspace of K for every $\omega \in \Omega$ where $(\mathscr{M} \cap F)(\omega) = \{\xi(\omega): \xi \in \mathscr{M} \cap F\}$. Then, we have the following result.

PROPOSITION 2. *Let* $\mathscr{M}$ *be a closed submodule of* $H = W(\Omega, K)$ *and* $\mathscr{M}_0 = \mathscr{M} \cap F$. *If* $\xi = \{\xi(\omega)\}$ *is an element of* $\prod_{\omega \in \Omega} \mathscr{M}_0(\omega)$ *such that the function* $\omega \to \|\xi(\omega)\|$ *is continuous and moreover, for every* $\eta \in \mathscr{M}_0$, *the function* $\omega \to (\xi(\omega) | \eta(\omega))$ *is continuous, then* ξ *is an element of* $\mathscr{M}_0$.

PROOF. For an arbitrary fixed element $\omega_0 \in \Omega$, there exists an element $\eta \in \mathscr{M}_0$ with $\xi(\omega_0) = \eta(\omega_0)$. Let $M = \max\{\|\xi(\omega)\|: \omega \in \Omega\}$, then, for every positive number ε, there exists a neighborhood $V(\omega_0)$ of ω_0 such that, for every $\omega \in V(\omega_0)$,

$$\big|\|\xi(\omega)\|^2 - \|\xi(\omega_0)\|^2\big| < \varepsilon/8\,, \quad |(\xi(\omega)|\eta(\omega)) - (\xi(\omega_0)|\eta(\omega_0))| < \varepsilon/8$$

and

$$\|\eta(\omega) - \eta(\omega_0)\| < \varepsilon/8M\,.$$

Hence, for every $\omega \in V(\omega_0)$, the following relations hold;

$$\begin{aligned} &|(\xi(\omega)|\eta(\omega_0)) - (\xi(\omega_0)|\eta(\omega_0))| \\ &\quad = |(\xi(\omega)|\eta(\omega_0)) - (\xi(\omega)|\eta(\omega)) + (\xi(\omega)|\eta(\omega)) - (\xi(\omega_0)|\eta(\omega_0))| \\ &\quad \leqq \|\xi(\omega)\|\cdot\|\eta(\omega) - \eta(\omega_0)\| + |(\xi(\omega)|\eta(\omega)) - (\xi(\omega_0)|\eta(\omega_0))| \\ &\quad < \varepsilon/8 + \varepsilon/8 = \varepsilon/4 \end{aligned}$$

and so

$$\begin{aligned} &\big|\|\xi(\omega)\|^2 - (\xi(\omega)|\eta(\omega_0))\big| \\ &\quad \leqq \big|\|\xi(\omega)\|^2 - (\xi(\omega_0)|\eta(\omega_0))\big| + |(\xi(\omega_0)|\eta(\omega_0)) - (\xi(\omega)|\eta(\omega_0))| \\ &\quad < \varepsilon/8 + \varepsilon/4 < \varepsilon/2\,. \end{aligned}$$

Thus, for every $\omega \in V(\omega_0)$, we have the following relation;

$$\begin{aligned} &\|\xi(\omega) - \xi(\omega_0)\|^2 \\ &\quad \leqq \big|\|\xi(\omega)\|^2 - (\xi(\omega)|\eta(\omega_0))\big| + \big|\|\xi(\omega_0)\|^2 - (\eta(\omega_0)|\xi(\omega))\big| \\ &\quad < \varepsilon/2 + \varepsilon/4 < \varepsilon\,. \end{aligned}$$

Therefore, ξ becomes an element of $F = C(\Omega, K)$.

Now, if we consider the facts such that $\xi(\omega) \in \mathscr{M}_0(\omega)$ for every

$\omega \in \Omega$ and $\eta \in F$, then, for any positive number ε, there exists a family $\{U(\omega), \eta_\omega\}_{\omega \in \Omega}$ of pairs consisting of closed and open neighborhood $U(\omega)$ of ω and element η_ω of $\mathscr{M}_0$ such that $\|\xi(\omega') - \eta_\omega(\omega')\| < \varepsilon$ for every $\omega' \in U(\omega)$. Since Ω is compact, there exists a finite subcovering $\{U(\omega_i); i = 1, 2, \cdots, n\}$ of Ω. We can suppose that $\{U(\omega_i); i = 1, 2, \cdots, n\}$ are mutually disjoint. Let z_i be the projection in $C(\Omega)$ corresponding to $U(\omega_i)$ and $\eta = \sum z_i \eta_{\omega_i} \in \mathscr{M}_0$, then $\|\xi(\omega) - \eta(\omega)\| < \varepsilon$ for all $\omega \in \Omega$ and so $\|\xi - \eta\| < \varepsilon$. Since $\mathscr{M}_0$ is a closed submodule, ξ is an element of $\mathscr{M}_0$.

In Proposition 2, put $\mathscr{M} = H$, then $\mathscr{M} \cap F = F$ and, by Definition 1, for every $\xi \in \mathscr{M} = H$ and $\eta \in \mathscr{M} \cap F = F$, the function $\omega \rightarrow (\xi(\omega) | \eta(\omega))$ is continuous. Hence, if ξ is an element of $H = W(\Omega, K)$ such that the function $\omega \rightarrow \|\xi(\omega)\|$ is continuous on Ω; then ξ is an element of F. Furthermore, we have the following result that is useful in this paper.

LEMMA 3. *Let $H = W(\Omega, K)$ be a weakly continuous constant field of Hilbert space K. Then if ξ is an element of H, there exists a maximal family $\{e_\iota\}$ of orthogonal projections in $C(\Omega)$ such that $e_\iota \xi$ is an element of F for every ι.*

PROOF. Let $\{\bar{\eta}_\alpha\}$ be a completely normalized orthogonal system for K. Then, for every $\omega \in \Omega$, $\xi(\omega)$ can be represented with $\xi(\omega) = \sum f_\alpha(\omega) \bar{\eta}_\alpha$. Now, since, for every $\eta \in F$, the function $\omega \rightarrow (\xi(\omega) | \eta(\omega))$ is continuous, each f_α is an element of $C(\Omega)$. Thus, since Ω is a hyperstonean space and each f_α is an element of $C(\Omega)$, the relation $\|\xi(\omega)\|^2 = \sum |f_\alpha(\omega)|^2$ induces the existence of a maximal family $\{e_\iota\}$ of orthogonal projections in $C(\Omega)$ such that the function $\omega \rightarrow \|(e_\iota \xi)(\omega)\|$ is continuous for every ι. Therefore, by the remark after Proposition 2, $e_\iota \xi$ is an element of F for every ι. The proof is completed.

From the representation shown above, we can show the following elementary fact: Let ξ be an element of H and $\{e_\iota\}$ be the maximal family of orthogonal projections in $C(\Omega)$ with the above property, then $\|\xi\| = \sup \|e_\iota \xi\|$. Furthermore, from Lemma 3, we have the following corollary which will be used in the proof of Theorem D. We can show this assertion by using the proof of Lemma 3, so we omit its proof.

COROLLARY 4. *Let ξ be an element of $F = C(\Omega, K)$ and $\{\bar{\eta}_\alpha\}_{\alpha \in A}$ a completely normalized orthogonal system for K. Then there exist a sequence $\{f_n\}$ in $C(\Omega)$ and a sequence $\{\bar{\eta}_n\}$ in $\{\bar{\eta}_\alpha\}_{\alpha \in A}$ such that $\xi = \sum_{n=1}^\infty f_n \eta_n$ in the sense of $\lim_{k \to \infty} \|\xi - \sum_{n=1}^k f_n \eta_n\| = 0$ where each η_n is a constant field with $\eta_n(\omega) = \bar{\eta}_n$ for all $\omega \in \Omega$.*

From the elementary property of $C(\Omega, K)$, let F_0 be the submodule of H generated by the set $\{f\xi: f \in C(\Omega)$ and ξ are constant vector field$\}$, then $\bar{F}_0 = F$ where $\bar{F}_0$ is the norm closure of F_0 in $H = W(\Omega, K)$.

Furthermore, let $B(H)$ be the set of all bounded $C(\Omega)$-module homomorphisms of H into H. Then, for $A, B \in B(H)$, if $A|F = B|F$; then, by Lemma 3, $A = B$ where $A|F$ is the restriction of A to F.

Now, since even if ξ and η are elements of H, (ξ, η) is not necessarily an element of $C(\Omega)$ where (ξ, η) is the function $\omega \rightarrow (\xi(\omega)|\eta(\omega))$ on Ω, $H = W(\Omega, K)$ does not become an AW^*-module which was introduced by Kaplansky [4]. An important property of AW^*-modules is the following; let $\{\xi_\alpha\}$ be a bounded set of an AW^*-module M over $C(\Omega)$ and $\{e_\alpha\}$ a family of orthogonal projections in $C(\Omega)$, then there exists uniquely an element ξ of M such that $e_\alpha\xi = e_\alpha\xi_\alpha$ for all α. Although, a weakly continuous constant field of Hilbert space is not necessarily an AW^*-module, we can give a property similar to the one in AW^*-modules. To prove this property, we shall show a result similar to the Riesz's theorem in Hilbert spaces (see [10; Theorem 3.6]). It can be shown by using Lemma 3, a similar way to the proof in [10; Theorem 3.6] and by the remark after Lemma 3.

LEMMA 5. *Let ϕ be a bounded $C(\Omega)$-module homomorphism of $H = W(\Omega, K)$ into $C(\Omega)$, then there exists uniquely in H an element ξ_0 such that $\phi(\xi) = (\xi, \xi_0)$ for every $\xi \in F$ and $||\phi|| = ||\xi_0||$.*

In Lemma 5, if we suppose that ϕ is a bounded $C(\Omega)$-module homomorphism of F into $C(\Omega)$, we can get the same conclusion in Lemma 5. Furthermore, we can show by an elementary computation that if ϕ is a bounded $C(\Omega)$-module homomorphism of H into $C(\Omega)$ then $||\phi|| = ||\phi|F||$. Furthermore, we can show that, if A is an element of $B(H)$, then $||A|| = ||A|F||$. This assertion can be shown by Lemma 3 and the remark after Lemma 3.

By Lemma 5, we can show the assertion mentioned before Lemma 5 in the following.

PROPOSITION 6. *Let $\{\xi_\alpha\}$ be a bounded set of $H = W(\Omega, K)$ and $\{e_\alpha\}$ a maximal family of mutually orthogonal projections in $C(\Omega)$, then there exists uniquely an element ξ in H such that $e_\alpha\xi = e_\alpha\xi_\alpha$ for all α.*

PROOF. For every $\eta \in F$, the set $\{(\eta, \xi_\alpha)\}$ is a bounded subset of $C(\Omega)$, thus $\sum e_\alpha(\eta, \xi_\alpha)$ in $C(\Omega)$ exists. Define $\phi(\eta) = \sum e_\alpha(\eta, \xi_\alpha)$, then ϕ is a bounded $C(\Omega)$-module homomorphism of F into $C(\Omega)$. Then, by the remark after Lemma 5, there exists an element ξ in H such that $\phi(\eta) = (\eta, \xi)$ for every $\eta \in F$. Therefore, $e_\alpha\xi = e_\alpha\xi_\alpha$ for all α. The unicity of ξ is evident.

Considering Proposition 6, we have the following definition.

DEFINITION 7. Let $\{\xi_\alpha\}$ be a bounded set in $H = W(\Omega, K)$ and $\{e_\alpha\}$ a maximal family of mutually orthogonal projections in $C(\Omega)$: Let ξ be the element determined in Proposition 6 as the form $e_\alpha\xi = e_\alpha\xi_\alpha$ for all α. We denote $\xi = \sum e_\alpha\xi_\alpha$.

By Definition 7, we can give a reformulation of Lemma 3 which will be useful in our later discussions.

PROPOSITION 8. *Let ξ be an element of H; then there exist a maximal family $\{e_\alpha\}$ of orthogonal projections in $C(\Omega)$ and a family $\{\xi_\alpha\}$ in F such that $\xi = \sum e_\alpha\xi_\alpha$ with $e_\alpha\xi_\alpha = \xi_\alpha$ for all α.*

Furthermore, by Lemma 5, we can define the adjoint operator of an arbitrary element $A \in B(H)$ as follows: Let A be a bounded $C(\Omega)$-module homomorphism on H, then by Lemma 5, there exists a bounded $C(\Omega)$-module homomorphism $A^\#$ of F into H such that, for every $\xi, \eta \in F$, $(A\xi, \eta) = (\xi, A^\#\eta)$. Now, we shall extend this $A^\#$ to a bounded $C(\Omega)$-module homomorphism A^* on H. If this is possible, by the remark after Corollary 4, we call A^* as the adjoint operator for A. Thus, we have the following consideration.

LEMMA 9. *Let A' be a bounded $C(\Omega)$-module homomorphism of F into H; then there exists uniquely a bounded $C(\Omega)$-module homomorphism A of H into H with $A|F = A'$.*

PROOF. If ξ is an arbitrary element of H, then ξ is expressed as $\xi = \sum e_\alpha\xi_\alpha$ with the condition in Proposition 8. Since A' is a bounded $C(\Omega)$-module homomorphism of F into H, $\{A'\xi_\alpha\}$ is a bounded set in H, so $\sum e_\alpha A'\xi_\alpha$ is an element of H by Proposition 6. Now, define an operator A as follows; $A\xi = \sum e_\alpha A'\xi_\alpha$. Then A is well-defined. In fact, if ξ is represented by two forms $\xi = \sum e_\alpha\xi_\alpha = \sum f_\iota\eta_\iota$ under the condition in Proposition 8, then, since $e_\alpha f_\iota\xi_\alpha = e_\alpha f_\iota\eta_\iota$ for all α and ι, we have $e_\alpha f_\iota A'\xi_\alpha = e_\alpha f_\iota A'\eta_\iota$ for all α and ι. Thus, $\sum e_\alpha A'\xi_\alpha = \sum f_\iota A'\eta_\iota$ and so A is well-defined. It is evident that A is $C(\Omega)$-module and bounded, and $A|F = A'$. We shall show the linearity of A. Let $\xi = \sum e_\alpha\xi_\alpha$ and $\eta = \sum f_\iota\eta_\iota$ in H with the expressions in Proposition 8, then $\xi + \eta = \sum e_\alpha f_\iota(\xi_\alpha + \eta_\iota)$. Thus, we have $e_\alpha f_\iota(\sum e_\alpha f_\iota(A'\xi_\alpha + A'\eta_\iota)) = e_\alpha f_\iota(A'\xi_\alpha + A'\eta_\iota) = e_\alpha f_\iota(A\xi + A\eta)$ for all α and ι. Thus $A\xi + A\eta = A(\xi + \eta)$. We can show by the remark after Corollary 4 that A is uniquely determined.

From Lemma 9, for any element $A \in B(H)$, there exists uniquely determined operator A^* such that $(A\xi, \eta) = (\xi, A^*\eta)$ for every $\xi, \eta \in F$.

Furthermore, one can show the equalities $||A|| = ||A^*||$ and $||A^*A|| = ||A||^2$ by the remark after Corollary 4 and the definition of A^*. Thus, $B(H)$ becomes a C^*-algebra. In particular, $B(H)$ becomes a von Neumann algebra of type I and its center is *-isomorphic to $C(\Omega)$. We shall show this assertion in the remainder of this section. The greater part of our proof is due to the proof of the Kaplansky's theorem [4; Theorem 7] and the author's result [11; Theorem 1.6]. Before the proof of this assertion, we have some considerations. We first determine the relation between the projections and the closed submodules of H. We noticed before Proposition 2 that if $\mathscr{M}$ is a closed submodule of $H = W(\Omega, K)$, then $(\mathscr{M} \cap F)(\omega)$ is a closed subspace of K. Thus, we can introduce the following notion.

DEFINITION 10. Let $\mathscr{M}$ be a closed submodule of $H = W(\Omega, K)$ and $\mathscr{M}_0 = \mathscr{M} \cap F$. Then $\mathscr{M}$ is called a continuous submodule of H if the following condition (*) is satisfied:

(*) If ξ is an element of $\prod_{\omega \in \Omega} \mathscr{M}_0(\omega)$ such that the function $\omega \to ||\xi(\omega)||$ is bounded and, for every $\eta \in \mathscr{M}_0$, the function $\omega \to (\xi(\omega) | \eta(\omega))$ is continuous, then ξ is an element of $\mathscr{M}$.

By Definition 10, we can determine the projection P with $PH = \mathscr{M}$. Before proving this assertion, we show the following.

LEMMA 11. *If A is an element of $B(H)$, then there exists a field $\{A(\omega)\}$ of bounded operators on K such that $(A\xi)(\omega) = A(\omega)\xi(\omega)$ for every $\xi \in F = C(\Omega, K)$ and $\omega \in \Omega$.*

Conversely, if $\{A(\omega)\}$ is a field of bounded operators on K such that the function $\omega \to ||A(\omega)||$ is bounded and, for every $\xi \in F$, the vector field $\{A(\omega)\xi(\omega)\}$ is an element of H (this operator field $\{A(\omega)\}$ is called a weakly continuous field), then there exists uniquely an element A of $B(H)$ such that $(A\xi)(\omega) = A(\omega)\xi(\omega)$ for every $\xi \in F$ and $\omega \in \Omega$.

PROOF. Let A be an element of $B(H)$. Then for every $\xi \in F$ such that $A\xi \in F$, $||(A\xi)(\omega)|| \leqq ||A|| \cdot ||\xi(\omega)||$ for every $\omega \in \Omega$, because both functions $\omega \to ||(A\xi)(\omega)||$ and $\omega \to ||\xi(\omega)||$ are continuous. For any elements $\xi \in F$, put $A\xi = \sum f_\iota \eta_\iota$ under the condition in Proposition 8. Let G_ι be the closed and open set in Ω corresponding to f_ι and let η be an element of F. Then since $f_\iota \xi \in F$ and $Af_\iota \xi = f_\iota A\xi = \eta_\iota \in F$ for all ι, for every $\omega \in G_\iota$, we have

$$|((A\xi)(\omega) | \eta(\omega))| = |((Af_\iota\xi)(\omega) | \eta(\omega))| \leqq ||A|| \cdot ||(f_\iota\xi)(\omega)|| \cdot ||\eta(\omega)||$$
$$= ||A|| \cdot ||\xi(\omega)|| \cdot ||\eta(\omega)|| .$$

Thus, $|((A\xi)(\omega) | \eta(\omega))| \leqq ||A|| \cdot ||\xi(\omega)|| \cdot ||\eta(\omega)||$ for all $\omega \in \cup G_\iota$. Since both functions $\omega \longrightarrow ((A\xi)(\omega)|\eta(\omega))$ and $\omega \longrightarrow \|A\|\cdot\|\xi(\omega)\|\cdot\|\eta(\omega)\|$ are continuous and the set $\cup G_\iota$ is dense in Ω, $|((A\xi)(\omega)|\eta(\omega))| \leqq \|A\|\cdot\|\xi(\omega)\|\cdot\|\eta(\omega))\|$ for every $\xi, \eta \in F$ and $\omega \in \Omega$. Thus, put $F_\omega((\xi(\omega)|\eta(\omega)) = ((A\xi)(\omega)|\eta(\omega))$ for $\xi, \eta \in F$, then, for every $\omega \in \Omega$, F_ω is a bounded bilinear form on K. Hence, there exists a field $\{A(\omega)\}$ of bounded operators on K such that $((A\xi)(\omega)|\eta(\omega)) = (A(\omega)\xi(\omega)|\eta(\omega))$ for every $\omega \in \Omega$ and $\xi, \eta \in F$. Furthermore, for every $\xi \in F$, the vector field $\{A(\omega)\xi(\omega)\}$ is an element of H; and $(A\xi)(\omega) = A(\omega)\xi(\omega)$ for every $\xi \in F$ and $\omega \in \Omega$.

Conversely, let $\{A(\omega)\}$ be a field of bounded operators on K such that the function $\omega \longrightarrow \|A(\omega)\|$ is bounded and, for every $\xi \in F$, the vector field $\{A(\omega)\xi(\omega)\}$ is an element of H. Then, define an operator A' as follows: $A'\xi = \{A(\omega)\xi(\omega)\}$ for every $\xi \in F$. Then A' is a bounded $C(\Omega)$-module homomorphism of F into H. By Lemma 9, A' is uniquely extended an operator A defined on H. Now, since, for every $\xi \in F$, $(A\xi)(\omega) = (A'\xi)(\omega) = A(\omega)\xi(\omega)$ for every $\omega \in \Omega$, we have the conclusion of the Lemma.

By Proposition 8 and Lemma 11, we have the following result.

PROPOSITION 12. *For each $A, B \in B(H)$ and $\xi \in F$, there exists a nowhere dense set N in Ω such that $(AB)(\omega)\xi(\omega) = A(\omega)B(\omega)\xi(\omega)$ for all $\omega \in \Omega\backslash N$.*

Furthermore, for every $A \in B(H)$, $A(\omega)^* = A^*(\omega)$ for all $\omega \in \Omega$.

PROOF. Since we can show the second assertion by an elementary computation and Lemma 11, we give the proof of the first assertion. Put $\zeta = \{B(\omega)\xi(\omega)\} \in H$, then ζ is written by the form $\zeta = \sum e_\alpha\zeta_\alpha$ under the condition in Proposition 8. Let G_α be the closed and open set corresponding to e_α: Then, since $e_\alpha B\xi = e_\alpha\zeta = \zeta_\alpha \in F$, we have, for every $\omega \in G_\alpha$, $(AB)(\omega)\xi(\omega) = (Ae_\alpha B\xi)(\omega) = A(\omega)(e_\alpha B\xi)(\omega) = A(\omega)B(\omega)\xi(\omega)$. Hence, $(AB)(\omega)\xi(\omega) = A(\omega)B(\omega)\xi(\omega)$ for all $\omega \in \cup G_\alpha$.

Next, from Proposition 12, we shall give a condition which is equivalent to the notion of continuous submodule.

LEMMA 13. *Let $\mathscr{M}$ be a continuous submodule of H; then the following properties are shown:*

(1) *Let $\{\xi_\alpha\}$ be a bounded subset of $\mathscr{M}$ and $\{e_\alpha\}$ a maximal family of orthogonal projections in $C(\Omega)$; then $\xi = \sum e_\alpha\xi_\alpha$ is also an element of $\mathscr{M}$.*

(2) *For every $\omega \in \Omega$, $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$ where $\mathscr{M}_0 = \mathscr{M} \cap F$.*

PROOF. Since $\mathscr{M}_0(\omega)$ is a closed subspace of K by the remark before Proposition 2, we can put the projection $P(\omega)$ of K onto $\mathscr{M}_0(\omega)$ for every $\omega \in \Omega$. Then, by the definition of the continuous submodule, the field $\{P(\omega)\}$ becomes a weakly continuous field. Furthermore, put $P = \{P(\omega)\}$, then $PH = \mathscr{M}$. Hence, the proof of the assertion for (1) is evident because $P\xi = \sum e_\alpha P\xi_\alpha = \sum e_\alpha \xi_\alpha$.

Next, we shall show the proof of (2). For every $\xi \in H$ and $\eta \in F$, the both functions $\omega \to (P(\omega)\xi(\omega) | \eta(\omega))$ and $\omega \to ((P\xi)(\omega) | \eta(\omega))$ are continuous, and there exists a nowhere dense set N such that $(P\xi)(\omega) = P(\omega)\xi(\omega)$ for every $\omega \in \Omega \backslash N$. Hence $(P\xi)(\omega) = P(\omega)\xi(\omega)$ for every $\xi \in H$ and $\omega \in \Omega$. Therefore, $(PH)(\omega) \subset \mathscr{M}_0(\omega)$, and so $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$ for every $\omega \in \Omega$.

We shall show a sufficient condition under which a closed submodule becomes a continuous submodule. Before going to show the result, we introduce the following condition $(**)$ similar to the notion [11; Theorem 1.5 (3)]:

$(**)$ $\mathscr{M}$ is a closed submodule of $H = W(\Omega, K)$ which satisfies; for every bounded subset $\{\xi_\alpha\}$ in $\mathscr{M}$ and a maximal family $\{e_\alpha\}$ of orthogonal projections in $C(\Omega)$, the element $\sum e_\alpha \xi_\alpha$ is in $\mathscr{M}$.

Let P be a projection in $B(H)$ and $\mathscr{M} = PH$, then it is evident that $\mathscr{M}$ satisfies the condition $(**)$. Furthermore, we have the converse by the following Proposition 14.

PROPOSITION 14. *Let $\mathscr{M}$ be a closed submodule of $H = W(\Omega, K)$ and $\mathscr{M}_0 = \mathscr{M} \cap F$; then the following conditions are equivalent*:

(1) *$\mathscr{M}$ satisfies the condition $(**)$.*

(2) *There exists a projection P in $B(H)$ with $PH = \mathscr{M}$.*

PROOF. $(2) \Rightarrow (1)$: This assertion has been already mentioned in the remark before this proposition and in the proof of Lemma 11.

$(1) \Rightarrow (2)$: Let $P'(\omega)$ be the projection of K onto $\mathscr{M}_0(\omega)$ for every $\omega \in \Omega$, then, every $\xi \in H$ and $\eta \in \mathscr{M}_0$, the function $\omega \to (P'(\omega)\xi(\omega) | \eta(\omega)) = (\xi(\omega) | P'(\omega)\eta(\omega)) = (\xi(\omega) | \eta(\omega))$ is continuous. Furthermore,

$$\| P'(\omega)\xi(\omega) \| = \sup \{ | (P'(\omega)\xi(\omega) | \eta(\omega)) | : \eta \in \mathscr{M}_0 \text{ with } \|\eta\| \leqq 1 \},$$

thus the function $\omega \to \|P'(\omega)\xi(\omega)\|$ is lower semi-continuous. Hence, there exists a maximal family $\{e_\alpha\}$ of orthogonal projections in $C(\Omega)$ such that the function $\omega \to \|P'(\omega)\xi(\omega)\|$ is continuous on $\cup G_\alpha$ where G_α is the closed and open set in Ω corresponding to e_α. Then, since the function $\omega \to \|P'(\omega)\xi(\omega)\|$ is continuous on $\cup G_\alpha$, if we define an element ξ_α such that $\xi_\alpha(\omega) = P'(\omega)\xi(\omega)$ for $\omega \in G_\alpha$ and $\xi_\alpha(\omega) = 0$ for $\omega \notin G_\alpha$, then ξ_α is an element of $\mathscr{M}_0$ by Proposition 2. Thus $\zeta = \sum e_\alpha \xi_\alpha$ is an element of $\mathscr{M}$, because $\mathscr{M}$ satisfies the condition $(**)$. Put an operator P determined as the form; $P\xi = \zeta$. Then, P is well-defined and it is evident that P is a bounded $C(\Omega)$-module homomorphism on F. We shall show that P is a projection. At first, we shall show that P is linear. For any ξ and ξ' in H, put $\zeta = \sum e_\alpha \zeta_\alpha$ and $\zeta' = \sum f_\iota \zeta'_\iota$ where $\zeta_\alpha(\omega) = P'(\omega)\xi(\omega)$ for every $\omega \in G_\alpha$ and $\zeta'_\iota(\omega) = P'(\omega)\xi'(\omega)$ for every $\omega \in F_\iota$ where G_α and F_ι are closed and open set in Ω corresponding to e_α and f_ι respectively; then, for every $\omega \in F_\iota \cap G_\alpha$, $(\zeta + \zeta')(\omega) = P'(\omega)(\xi(\omega) + \xi'(\omega))$ and so $\zeta + \zeta' = \sum e_\alpha f_\iota(\zeta_\alpha + \zeta'_\iota)$. Thus, $P(\xi + \xi') = \zeta + \zeta'$. Therefore, P is a bounded $C(\Omega)$-module homomorphism of H into H. Conversely, if ξ is an element of $\mathscr{M}$, we can represent ξ as the form $\xi = \sum e_\alpha \xi_\alpha$ such that $\{\xi_\alpha\} \subset \mathscr{M}_0$ and $\{e_\alpha\}$ is a maximal family of orthogonal projections in $C(\Omega)$. Now, $P\eta = \eta$ for every $\eta \in \mathscr{M}_0$, $P\xi = \sum e_\alpha P\xi_\alpha = \sum e_\alpha \xi_\alpha = \xi$, and so $PH = \mathscr{M}$. Furthermore, for every $\xi \in H$, since $P\xi \in \mathscr{M}$, $P^2\xi = P(P\xi) = P\xi$, thus $P^2 = P$.

It is evident by Lemma 11 and Proposition 12 that $P^* = P$.

We have shown the condition for a closed submodule to satisfy the condition $(**)$. But, we can not show the relation $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$ for every $\omega \in \Omega$. Now, if we suppose that $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$ for every $\omega \in \Omega$, then we can show that $\mathscr{M}$ is a continuous submodule.

PROPOSITION 15. *Let $\mathscr{M}$ be a closed submodule of H and $\mathscr{M}_0 = \mathscr{M} \cap F$. If $\mathscr{M}$ satisfies the condition $(**)$ and $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$ for every $\omega \in \Omega$, then $\mathscr{M}$ is a continuous submodule of H.*

PROOF. Let $\xi = \{\xi(\omega)\}$ be an element of $\prod_{\omega \in \Omega} \mathscr{M}_0(\omega)$ such that the function $\omega \to \|\xi(\omega)\|$ is bounded and, for any $\eta \in \mathscr{M}_0$, the function $\omega \to (\xi(\omega)|\eta(\omega))$ is continuous. Then, we must show that ξ is an element of H and $P\xi = \xi$ where $P = \{P(\omega)\}$ is the projection of H onto $\mathscr{M}$ determined by Proposition 14. Now, since, for any $\eta \in \mathscr{M}_0$, the function $\omega \to (\xi(\omega)|\eta(\omega))$ is continuous and $\xi(\omega) \in \mathscr{M}_0(\omega)$ for every $\omega \in \Omega$, the function $\omega \to \|\xi(\omega)\|$ is lower semi-continuous on Ω. Hence, there exists a maximal family $\{e_\alpha\}$ of orthogonal projections in $C(\Omega)$ such that the function $\omega \to \|(e_\alpha\xi)(\omega)\|$ is continuous for every α. Then, by Proposition 2, $e_\alpha\xi$ is an element of $\mathscr{M}_0$ for all α. Thus, put $\xi' = \sum e_\alpha(e_\alpha\xi)$, then ξ' is an element of $\mathscr{M}$ because $\mathscr{M}$ satisfies the condition $(**)$. Now, for every $\eta \in \mathscr{M}_0$, both functions $\omega \to (\xi(\omega)|\eta(\omega))$ and $\omega \to (\xi'(\omega)|\eta(\omega))$ are continuous. Furthermore, the relation $\xi' = \sum e_\alpha(e_\alpha\xi)$ leads us to the fact that $\xi(\omega) = \xi'(\omega)$ for every $\omega \in \cup G_\alpha$ where G_α is the closed and open set in Ω corresponding to e_α. Thus, since $\cup G_\alpha$ is dense, $(\xi(\omega)|\eta(\omega)) = (\xi'(\omega)|\eta(\omega))$ for every $\eta \in \mathscr{M}_0$. Therefore $\xi(\omega) = \xi'(\omega)$ for all $\omega \in \Omega$, and $\xi = \xi' \in \mathscr{M}$.

By the above considerations, we have the following theorem.

THEOREM A. *Let $\mathscr{M}$ be a closed submodule of $H = W(\Omega, K)$ and*

$\mathscr{M}_0 = \mathscr{M} \cap F$. *Then, the following two properties are shown.*

(1) *The following conditions* (α) *and* (β) *are equivalent:*

(α) *For every bounded subset* $\{\xi_\alpha\}$ *in* $\mathscr{M}$ *and every maximal family* $\{e_\alpha\}$ *of orthogonal projections in* $C(\Omega)$, *the element* $\sum e_\alpha \xi_\alpha$ *is in* $\mathscr{M}$.

(β) *There exists a projection* P *of* H *onto* $\mathscr{M}$.

(2) *The following conditions* (α') *and* (β') *are equivalent:*

(α') $\mathscr{M}$ *is a continuous submodule of* H.

(β') *For every bounded subset* $\{\xi_\alpha\}$ *in* $\mathscr{M}$ *and every maximal family* $\{e_\alpha\}$ *of orthogonal projections in* $C(\Omega)$, *the element* $\sum e_\alpha \xi_\alpha$ *is in* $\mathscr{M}$, *and for every* $\omega \in \Omega$, $\mathscr{M}(\omega) = \mathscr{M}_0(\omega)$.

We say that P is an abelian projection in $B(H)$ if $PB(H)P$ is abelian. If ξ is an element of H such that the function $\omega \longrightarrow \|\xi(\omega)\|$ is continuous, by the remark after Proposition 2, ξ becomes an element of $F = C(\Omega, K)$. Thus, if $\xi \in H$ and $|\xi|$ is a projection in $C(\Omega)$, then ξ is an element of F. With this thing, we can determine the abelian projections of $B(H)$ in the following form by considering Lemma 3, Theorem A, the remark after Proposition 2 and by the similar way to the proof of [4; Lemma 13]. We leave the proof to the readers.

LEMMA 16. *Let* $H = W(\Omega, K)$ *be a weakly continuous constant field of Hilbert space* K *over* Ω. *Let* η *be an element of* H *such that there exists a family* $\{e_\alpha\}$ *of mutually orthogonal projections in* $C(\Omega)$ *which satisfies the relations;* $e_\alpha|\eta| = e_\alpha$ *for all* α *and* $(I - e)|\eta| = 0$ *where* $e = \sum e_\alpha$. *Then the operator* P *determined by* $P\xi = (\xi, \eta)\eta$ *for every* $\xi \in F$ *is an abelian projection. Conversely every abelian projection arises in this form.*

Let $\mathfrak{Z}$ be the set of all elements T_f for $f \in C(\Omega)$ defined by $T_f\xi = \{f(\omega)\xi(\omega)\}$ for every $\xi \in H$. Then $\mathfrak{Z}$ is a C^*-subalgebra of $B(H)$ (in fact, $\mathfrak{Z}$ is a von Neumann algebra and *-isomorphic to $C(\Omega)$).

From the above considerations, we can show the main theorem.

THEOREM B. *Let* $H = W(\Omega, K)$ *be a weakly continuous constant field of Hilbert space* K *with respect to* $F = C(\Omega, K)$ *over a hyperstonean space* Ω; *then the algebra* $B(H)$ *of all bounded* $C(\Omega)$*-module homomorphisms of* H *into* H *is a von Neumann algebra of type I and its center is *-isomorphic to* $C(\Omega)$.

PROOF. If we can show that $B(H)$ is an AW^*-algebra, it is evident by Lemma 16 that $B(H)$ is an AW^*-algebra of type I. Furthermore, if we can show that the center of $B(H)$ is *-isomorphic to the von

Neumann algebra $C(\Omega)$, then $B(H)$ becomes a von Neumann algebra of type I by [3; Theorem 2].

Now, since we can show by an elementary computation that $\mathfrak{Z}$ is *-isomorphic to $C(\Omega)$ and $\mathfrak{Z}$ is the center of $B(H)$, we must show that $B(H)$ is an AW^*-algebra. For this assertion, it is sufficient for us to show the following: The left annihilator $\mathscr{L}(\mathfrak{S})$ of any subset $\mathfrak{S}$ of $B(H)$ is of the form $B(H)P$ with P a projection in $B(H)$ (see [4]). Let $\mathfrak{S}$ be a subset of $B(H)$, $\mathscr{L}(\mathfrak{S})$ the left annihilator of $\mathfrak{S}$ and $\mathfrak{N}=$ the linear span of the set $\{R(B): B\in\mathfrak{S}\}$, then $\mathscr{L}(\mathfrak{S})=\{A\in B(H): A\mathfrak{N}=\{0\}\}$ where $R(B)$ is the range of B. Now, put $\mathscr{M}=$ the norm closure of $\{\sum e_\alpha\xi_\alpha$: $\{\xi_\alpha\}$ is a bounded subset of $\mathfrak{N}$ and $\{e_\alpha\}$ is a maximal family of orthogonal projections in $C(\Omega)\}$, then by the proof in [11; Lemma 2.5] and Theorem A, $\mathscr{M}$ is a closed submodule satisfying the condition (**); furthermore, for every $A\in\mathscr{L}(\mathfrak{S})$, $A\mathscr{M}=\{0\}$. Thus, $\mathscr{L}(\mathfrak{S})=\{A\in B(H): A\mathscr{M}=\{0\}\}$. Since $\mathscr{M}$ satisfies the condition (**), there exists a projection Q of H onto $\mathscr{M}$. Put $P=I-Q$, then $\mathscr{L}(\mathfrak{S})=B(H)P$. This completes the proof.

For Theorem B, let V be the norm closure of linear subspace generated by $\{\phi\circ\omega_\xi: \xi\in F$ and $\phi\in\mathscr{A}_*\}$ in $B(H)^*$, then we can show by an elementary computation that $B(H)_*=V$ where $\omega_\xi(A)=(A\xi,\xi)$ in $\mathscr{A}$ for $A\in B(H)$ and $\mathscr{A}=C(\Omega)$ (for example, see [12]).

3. The weakly continuous constant field of von Neumann algebra. In this section, we shall consider the second problem in the introduction as an application of Theorem B and Theorem C which will be shown later.

In the previous section, we have shown that an arbitrary element $A\in B(H)$ has the following representation; there exists an operator field $\{A(\omega)\}$ satisfying the relation $(A\xi)(\omega)=A(\omega)\xi(\omega)$ for every $\xi\in F$ and $\omega\in\Omega$. Thus, let π_ω be the mapping of $B(H)$ onto $B(K)$ defined as follows; $\pi_\omega(A)=A(\omega)$ for $A=\{A(\omega)\}$. Then, by the author's theorem [8] and [9], π_ω is not a *-homomorphism in general. But, it is evident that π_ω is a positive linear mapping.

DEFINITION 17. Let $H=W(\Omega, K)$ be a weakly continuous constant field of Hilbert space K and $\mathfrak{A}$ a von Neumann algebra acting on K. Then we define $W(\Omega, K, \mathfrak{A})=\{A\in B(H): A(\omega)\in\mathfrak{A}$ for every $\omega\in\Omega\}$ and call $W(\Omega, K, \mathfrak{A})$ as a weakly continuous constant field of the von Neumann algebra $\mathfrak{A}$.

Under the above definition, if $\mathfrak{A}$ is a von Neumann algebra acting on K, then we must show that the weakly continuous constant field

$W(\Omega, K, \mathfrak{A})$ of $\mathfrak{A}$ becomes a von Neumann algebra. To prove this assertion, we shall show that $W(\Omega, K, \mathfrak{A}) = W(\Omega, K, \mathfrak{A})''$ where $W(\Omega, K, \mathfrak{A})''$ means the double commutant of $W(\Omega, K, \mathfrak{A})$ in the von Neumann algebra $B(H)$.

THEOREM C. *Let $\mathfrak{A}$ be a von Neumann algebra acting on K; then $W(\Omega, K, \mathfrak{A})$ is a von Neumann subalgebra of $B(H)$ where $H = W(\Omega, K)$.*

PROOF. By Proposition 12, it is evident that $W(\Omega, K, \mathfrak{A})$ is a self-adjoint and linear subspace of $B(H)$. We shall show that, for every $A, B \in W(\Omega, K, \mathfrak{A})$, AB is also an element of $W(\Omega, K, \mathfrak{A})$, that is, $(AB)(\omega) \in \mathfrak{A}$ for every $\omega \in \Omega$. Thus, we must show that, for any $\bar{C}' \in \mathfrak{A}'$ and $\xi, \eta \in F$, $(\bar{C}'(AB)(\omega)\xi(\omega) \mid \eta(\omega)) = ((AB)(\omega)\bar{C}'\xi(\omega) \mid \eta(\omega))$ for all $\omega \in \Omega$. For $\xi \in F$, there exists a nowhere dense set N such that $(AB)(\omega)\xi(\omega) = A(\omega)B(\omega)\xi(\omega)$ for all $\omega \in \Omega \backslash N$. Furthermore, since the vector field $\{\bar{C}'\xi(\omega)\}$ is an element of F, by the same argument, there exists a nowhere dense set N' such that $(AB)(\omega)\bar{C}'\xi(\omega) = A(\omega)B(\omega)\bar{C}'\xi(\omega)$ for every $\omega \in \Omega \backslash N'$. Thus, put $N'' = N \cup N'$, then N'' is also a nowhere dense set, and, for every $\omega \in \Omega \backslash N''$, we have the following relation;

$$\begin{aligned}(\bar{C}'(AB)(\omega)\xi(\omega)|\eta(\omega)) &= (A(\omega)B(\omega)\xi(\omega) \mid \bar{C}'^{*}\eta(\omega)) = (A(\omega)B(\omega)\bar{C}'\xi(\omega) \mid \eta(\omega)) \\ &= ((AB)(\omega)\bar{C}'\xi(\omega) \mid \eta(\omega)) .\end{aligned}$$

Now, since both functions

$$\omega \longrightarrow (\bar{C}'(AB)(\omega)\xi(\omega) \mid \eta(\omega)) \quad \text{and} \quad \omega \longrightarrow ((AB)(\omega)\bar{C}'\xi(\omega) \mid \eta(\omega))$$

are continuous and N'' is a nowhere dense set, $(\bar{C}'(AB)(\omega)\xi(\omega) \mid \eta(\omega)) = ((AB)(\omega)\bar{C}'\xi(\omega) \mid \eta(\omega))$ for every $\xi, \eta \in F$ and $\omega \in \Omega$. Therefore AB is an element of $W(\Omega, K, \mathfrak{A})$ and so $W(\Omega, K, \mathfrak{A})$ becomes a C^*-subalgebra of $B(H)$. By the same argument as above, we can also show the relation $W(\Omega, K, \mathfrak{A}') = W(\Omega, K, \mathfrak{A})'$, and so $W(\Omega, K, \mathfrak{A}) = W(\Omega, K, \mathfrak{A})''$. Therefore $W(\Omega, K, \mathfrak{A})$ becomes a von Neumann algebra.

Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be a von Neumann algebra acting on a Hilbert space K (resp. L). Then the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is defined as the weak closure of the algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ on the Hilbert space $K \otimes L$, and this tensor product $\mathfrak{A} \otimes \mathfrak{B}$ does not depend on underlying Hilbert spaces in algebraic sense (see [5; Theorem 1]).

Let $\mathfrak{A}_*$ be the predual space of $\mathfrak{A}$ (that is, $\mathfrak{A}_*$ is the set of all σ-weakly continuous linear functionals on $\mathfrak{A}$), then we have the following Tomiyama's result [13; Theorem 1].

LEMMA 18. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be von Neumann algebras acting on K and L respectively. Let $\mathfrak{A} \otimes \mathfrak{B}$ be the tensor product of $\mathfrak{A}$ and $\mathfrak{B}$. Then* *for each* $\phi \in \mathfrak{A}_*$ *there exists a* σ*-weakly continuous mapping* $R_\phi: \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B}$ *satisfying the following*;

(1) $R_\phi(\sum_{i=1}^n A_i \otimes B_i) = \sum_{i=1}^n \langle A_i, \phi \rangle B_i$,

(2) $R_\phi((I \otimes A)X(I \otimes B)) = AR_\phi(X)B$ *for* $X \in \mathfrak{A} \otimes \mathfrak{B}$,

(3) $\langle X, \phi \otimes \psi \rangle = \langle R_\phi(X), \psi \rangle$ *for* $X \in \mathfrak{A} \otimes \mathfrak{B}$ *and* $\psi \in \mathfrak{B}_*$.

Next, let $\mathfrak{A}$ be a von Neumann algebra acting on K and $\mathscr{A}$ an abelian von Neumann algebra acting on L, then we can show that the tensor product $\mathfrak{A} \otimes \mathscr{A}$ of $\mathfrak{A}$ and $\mathscr{A}$ is *-isomorphic to the weakly continuous constant field $W(\Omega, K, \mathfrak{A})$ of $\mathfrak{A}$ where Ω is the spectrum space of $\mathscr{A}$. For this assertion, we can suppose that $\mathscr{A}$ is a maximal abelian von Neumann algebra of $B(L)$ by [5; Theorem 1].

For every $A = \{A(\omega)\} \in W(\Omega, K, \mathfrak{A})$ and $\bar{\xi}, \bar{\eta} \in K$ (if we consider an expression of each fibre $\xi(\omega)$ of a constant vector field $\xi = \{\xi(\omega)\} \in H$, we shall denote by $\xi(\omega) = \bar{\xi} \in K$), the function $\omega \to (A(\omega)\bar{\xi} | \bar{\eta})$ is continuous. Hence, if we let $\phi = \sum_{n=1}^\infty \omega_{\bar{\xi}_n, \bar{\eta}_n} | \mathfrak{A} \in \mathfrak{A}_*$ (in the later part of this paper, we denote $\sum_{n=1}^\infty \omega_{\bar{\xi}_n, \bar{\eta}_n} | \mathfrak{A}$ by $\sum_{n=1}^\infty \omega_{\bar{\xi}_n, \bar{\eta}_n}$), then the function $\omega \to \phi(A(\omega))$ is continuous on Ω. Thus, we define the continuous function $\phi \circ A$ on Ω such as $(\phi \circ A)(\omega) = \phi(A(\omega)) = \langle A(\omega), \phi \rangle$ for every $\omega \in \Omega$.

Let $A = \{A(\omega)\} \in W(\Omega, K, \mathfrak{A})$, then we define a bilinear form on $K \odot L$ as follows; for $\Xi = \sum_{i=1}^n \bar{\xi}_i \otimes \chi_i$, $\Xi' = \sum_{j=1}^m \bar{\xi}'_j \otimes \chi'_j$,

$$\langle \Xi | \Xi' \rangle = \sum_{i=1}^n \sum_{j=1}^m ((\omega_{\bar{\xi}_i, \bar{\xi}'_j} \circ A)\chi_i | \chi'_j):$$

Then, the above defined bilinear form is well-defined. In fact, if we suppose that $\Xi \sim 0 \otimes 0$ (or $\Xi \sim 0 \otimes 0$), then we can show that

$$\sum_{i=1}^n \sum_{j=1}^m ((\omega_{\bar{\xi}_i, \bar{\xi}'_j} \circ A)\chi_i | \chi'_j) = 0 .$$

To prove it, we can suppose by an elementary consideration that $\{\chi_i\}_{i=1}^n$ is linearly independent. Then, since $A(\omega) \in B(K)$ for every $\omega \in \Omega$ and so $A(\omega)^* \bar{\xi}'_j \in K$, the equation $\sum_{i=1}^n \bar{\xi}_i \otimes \chi_i \sim 0 \otimes 0$ induces $\sum_{i=1}^n (A(\omega)^* \bar{\xi}'_j | \bar{\xi}_j)\chi_i = 0$ for all j and $\omega \in \Omega$. Hence, since $\{\chi_i\}_{i=1}^n$ is linearly independent, $(A(\omega)\bar{\xi}_i | \bar{\xi}'_j) = 0$ for all i, j and $\omega \in \Omega$, and so $\omega_{\bar{\xi}_i, \bar{\xi}'_j} \circ A = 0$ for all i, j. Thus, $\Xi \sim 0 \otimes 0$ induces the identity $\sum_{i=1}^n \sum_{j=1}^m ((\omega_{\bar{\xi}_i, \bar{\xi}'_j} \circ A)\chi_i | \chi'_j) = 0$, and by the similar way, we have the same conclusion when $\Xi' \sim 0 \otimes 0$. Next, we shall show that $\langle \Xi | \Xi' \rangle$ becomes a bounded bilinear form on $K \odot L$. For this, we can assume that A is a positive element of $W(\Omega, K, \mathfrak{A})$. Let $\zeta_i = \{A(\omega)^{1/2}\bar{\xi}_i\}$, then $\langle \Xi | \Xi \rangle = \sum_{i,j=1}^n ((\omega_{\bar{\xi}_i, \bar{\xi}_j} \circ A)\chi_i | \chi_j) = \sum_{i,j=1}^n ((\zeta_i, \zeta_j)\chi_i | \chi_j) \geqq 0$ where $\Xi = \sum_{i=1}^n \bar{\xi}_i \otimes \chi_i$. Thus, by the Schwartz's inequality, for every $\Xi = \sum_{i=1}^n \bar{\xi}_i \otimes \chi_i$ and $\Xi' = \sum_{j=1}^m \bar{\xi}'_j \otimes \chi'_j$, $|\langle \Xi | \Xi' \rangle| \leqq \{\langle \Xi | \Xi \rangle \langle \Xi' | \Xi' \rangle\}^{1/2}$. Now, since we can suppose that $\{\bar{\xi}_i\}_{i=1}^n$ and $\{\bar{\xi}'_j\}_{j=1}^m$ are normalized orthogonal system, and $\|A\|I - A \geqq 0$; we have the following result:

$$0 \leqq \sum_{i,j=1}^{n}((\omega_{\bar{\xi}_i,\bar{\xi}_j}\circ A)\chi_i|\chi_j) \leqq \sum_{i,j=1}^{n}((\omega_{\bar{\xi}_i,\bar{\xi}_j}\circ\|A\|I)\chi_i|\chi_j)$$
$$= \sum_{i,j=1}^{n}(\|A\|(\xi_i, \xi_j)\chi_i|\chi_j) = \sum_{i=1}^{n}\|A\|\cdot\|\chi_i\|^2 = \|A\|\cdot\|\Xi\|^2$$

where $\xi_i(\omega) = \bar{\xi}_i$ for all i and $\omega \in \Omega$. Thus, $\langle \Xi|\Xi\rangle \leqq \|A\|\cdot\|\Xi\|^2$, and so, by the similar way, $\langle \Xi'|\Xi'\rangle \leqq \|A\|\cdot\|\Xi'\|^2$. Therefore, $|\langle \Xi|\Xi'\rangle| \leqq \|A\|\cdot\|\Xi\|\cdot\|\Xi'\|$, and so $\langle \Xi|\Xi'\rangle$ is bounded on $K \odot L$. Hence, there exists an element $\tilde{A}$ in $B(K\otimes L)$ satisfying

$$(\tilde{A}\Xi|\Xi') = \sum_{i=1}^{n}\sum_{j=1}^{m}((\omega_{\bar{\xi}_i,\bar{\xi}'_j}\circ A)\chi_i|\chi'_j)$$

and $\|\tilde{A}\| \leqq \|A\|$.

Next, we shall show that $\tilde{A}$ is an element of $\mathfrak{A}\otimes\mathscr{A}$. Since we suppose that $\mathscr{A}$ is a maximal abelian von Neumann algebra, $(\mathfrak{A}\otimes\mathscr{A})' = \mathfrak{A}'\otimes\mathscr{A}$. Hence, to prove this assertion, it is sufficient to show that $\tilde{A}$ commutes with $\bar{B}'\otimes z \in \mathfrak{A}'\odot\mathscr{A}$. For every $\bar{\xi}, \bar{\eta} \in K$ and $\bar{B}' \in \mathfrak{A}'$, we can show the equation $\omega_{\bar{B}'\bar{\xi},\bar{\eta}} = \omega_{\bar{\xi},\bar{B}'^*\bar{\eta}}$. Thus for every $\bar{B}'\otimes z \in \mathfrak{A}'\odot\mathscr{A}$ and $\Xi = \sum_{i=1}^{n}\bar{\xi}_i\otimes\chi_i$, $\Xi' = \sum_{j=1}^{m}\bar{\xi}'_j\otimes\chi'_j \in K\odot L$,

$$\begin{aligned}(\tilde{A}(\bar{B}'\otimes z)\Xi|\Xi') &= (\tilde{A}(\sum_{i=1}^{n}\bar{B}'\bar{\xi}_i\otimes z\chi_i)|\sum_{j=1}^{m}\bar{\xi}'_j\otimes\chi'_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}((\omega_{\bar{B}'\bar{\xi}_i,\bar{\xi}'_j}\circ A)z\chi_i|\chi'_j)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}((\omega_{\bar{\xi}_i,\bar{B}'^*\bar{\xi}'_j}\circ A)\chi_i|z^*\chi'_j)\\ &= (\tilde{A}(\sum_{i=1}^{n}\bar{\xi}_i\otimes\chi_i)|\sum_{j=1}^{m}\bar{B}'^*\bar{\xi}'_j\otimes z^*\chi'_j)\\ &= (\tilde{A}\Xi|(\bar{B}'\otimes z)^*\Xi') = ((\bar{B}'\otimes z)\tilde{A}\Xi|\Xi')\ .\end{aligned}$$

Therefore, $\tilde{A} \in (\mathfrak{A}'\otimes\mathscr{A})' = \mathfrak{A}\otimes\mathscr{A}$.

Furthermore, we can show that $R_\phi(\tilde{A}) = \phi\circ A$ for each $\phi \in \mathfrak{A}_*$ where R_ϕ is the σ-weakly continuous linear mapping for ϕ determined in Lemma 18. In fact, if we take a functional $\phi = \sum_{i=1}^{n}\omega_{\bar{\xi}_i,\bar{\eta}_i}$ then, for every $\chi \in L$, we have the following equation;

$$\langle R_\phi(\tilde{A}), \omega_\chi\rangle = \langle \tilde{A}, \phi\otimes\omega_\chi\rangle = \langle \tilde{A}, (\sum_{i=1}^{n}\omega_{\bar{\xi}_i,\bar{\eta}_i})\otimes\omega_\chi\rangle = \sum_{i=1}^{n}\langle \tilde{A}, \omega_{\bar{\xi}_i,\bar{\eta}_i}\otimes\omega_\chi\rangle$$
$$= \sum_{i=1}^{n}(\tilde{A}(\bar{\xi}_i\otimes\chi)|\bar{\eta}_i\otimes\chi) = \sum_{i=1}^{n}((\omega_{\bar{\xi}_i,\bar{\eta}_i}\circ A)\chi|\chi) = ((\phi\circ A)\chi|\chi)\ .$$

Hence, $R_\phi(\tilde{A}) = \phi\circ A$ for every $\phi \in \mathfrak{A}_*$. Furthermore, we can show the converse of this fact by similar way to the proof of the above fact.

Next, let $\tilde{A}$ be an element of $\mathfrak{A} \otimes \mathscr{A}$ and $\bar{\xi}, \bar{\eta} \in K$. Then, $R_{\bar{\xi},\bar{\eta}}(\tilde{A})$ is an element of $\mathscr{A} = C(\Omega)$ where $R_{\bar{\xi},\bar{\eta}}$ means the map $R_{\omega_{\bar{\xi},\bar{\eta}}}$ determined in Lemma 18. Now, define a bounded bilinear form on K for every $\omega \in \Omega$ as follows; $\langle \bar{\xi} | \bar{\eta} \rangle = R_{\bar{\xi},\bar{\eta}}(\tilde{A})(\omega)$ for any $\bar{\xi}, \bar{\eta} \in K$. Then, there exists an element $A(\omega) \in B(K)$ such that $(A(\omega)\bar{\xi} | \bar{\eta}) = R_{\bar{\xi},\bar{\eta}}(\tilde{A})(\omega)$ for every $\omega \in \Omega$. Put $A = \{A(\omega)\}$, then, by Lemma 9, A can be extended onto H as an element of $B(H)$. We assert that A belongs to $W(\Omega, K, \mathfrak{A})$. For any element $\bar{B}' \in \mathfrak{A}'$ and $\omega \in \Omega$, $(A(\omega)\bar{B}'\bar{\xi} | \bar{\eta}) = R_{\bar{B}'\bar{\xi},\bar{\eta}}(\tilde{A})(\omega) = R_{\bar{\xi},\bar{B}'^*\bar{\eta}}(\tilde{A})(\omega) = (A(\omega)\bar{\xi} | \bar{B}'^*\bar{\eta}) = (\bar{B}'A(\omega)\bar{\xi} | \bar{\eta})$ for every $\bar{\xi}, \bar{\eta} \in K$. Hence $A(\omega) \in \mathfrak{A}$ for every $\omega \in \Omega$, and so $A = \{A(\omega)\} \in W(\Omega, K, \mathfrak{A})$. Thus, if we consider the above relation and use an elementary computation, we have the equation; $R_\phi(\tilde{A}) = \phi \circ A$ for every $\phi \in \mathfrak{A}_*$.

With the above correspondence, we have the following theorem.

THEOREM D. *Let $\mathfrak{A}$ be a von Neumann algebra acting on K, $\mathscr{A}$ an abelian von Neumann algebra acting on L and Ω the spectrum space of $\mathscr{A}$, then the tensor product $\mathfrak{A} \otimes \mathscr{A}$ of $\mathfrak{A}$ and $\mathscr{A}$ is *-isomorphic to the weakly continuous constant field $W(\Omega, K, \mathfrak{A})$ of $\mathfrak{A}$.*

PROOF. Without loss of generality we may assume that $\mathscr{A}$ is a maximal abelian von Neumann algebra. Then, from the considerations before this theorem, we can define the following mapping π of $W(\Omega, K, \mathfrak{A})$ onto $\mathfrak{A} \otimes \mathscr{A}$; π: $W(\Omega, K, \mathfrak{A}) \ni A = \{A(\omega)\} \to \tilde{A} \in \mathfrak{A} \otimes \mathscr{A}$ determined as $R_\phi(\tilde{A}) = \phi \circ A$ for every $\phi \in \mathfrak{A}_*$ (in fact, the above relation is determined by the form $R_{\bar{\xi},\bar{\eta}}(\tilde{A}) = (A\xi, \eta)$ for every $\bar{\xi}, \bar{\eta} \in K$ where $\xi(\omega) = \bar{\xi}$, $\eta(\omega) = \bar{\eta}$ for every $\omega \in \Omega$). Then, it is clear that π is a *-preserving linear, one-to-one mapping. Thus, we must show that π is multiplicative. To prove this assertion, we must show that $((AB)(\omega)\bar{\xi} | \bar{\eta}) = R_{\bar{\xi},\bar{\eta}}(\tilde{A}\tilde{B})(\omega)$ for every $\bar{\xi}, \bar{\eta} \in K$ and $\omega \in \Omega$. At first, for every $\tilde{A} \in \mathfrak{A} \otimes \mathscr{A}$ and $\tilde{B} \in \mathfrak{A} \odot \mathscr{A}$, we have that $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = R_{\bar{\xi},\bar{\eta}}(\widetilde{BA})$ and $R_{\bar{\xi},\bar{\eta}}(\tilde{A}\tilde{B}) = R_{\bar{\xi},\bar{\eta}}(\widetilde{AB})$. In fact, if $\tilde{B} = \bar{B} \otimes I$, then $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = R_{\bar{\xi},\bar{B}^*\bar{\eta}}(\tilde{A})$ and $R_{\bar{\xi},\bar{\eta}}(\tilde{A}\tilde{B}) = R_{\bar{B}\bar{\xi},\bar{\eta}}(\tilde{A})$. Thus, suppose $\tilde{B} = \sum_{i=1}^n \bar{B}_i \otimes z_i \in \mathfrak{A} \odot \mathscr{A}$, then, the following equation holds; $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = \sum_{i=1}^n R_{\bar{\xi},\bar{\eta}}((\bar{B}_i \otimes z_i)\tilde{A}) = \sum_{i=1}^n z_i R_{\bar{\xi},\bar{\eta}}((\bar{B}_i \otimes I)\tilde{A}) = \sum_{i=1}^n z_i R_{\bar{\xi},\bar{B}_i^*\bar{\eta}}(\tilde{A})$. Furthermore, for every $\omega \in \Omega$,

$$(B(\omega)A(\omega)\bar{\xi} | \bar{\eta}) = \sum_{i=1}^n z_i(\omega)\bar{B}_i A(\omega)\bar{\xi} | \bar{\eta}) = \sum_{i=1}^n z_i(\omega)(A(\omega)\bar{\xi} | \bar{B}_i^*\bar{\eta}) .$$

Thus, the function $\omega \to (B(\omega)A(\omega)\bar{\xi} | \bar{\eta})$ is continuous, and so, by Proposition 12, $(B(\omega)A(\omega)\bar{\xi} | \bar{\eta}) = ((BA)(\omega)\bar{\xi} | \bar{\eta})$ for all $\omega \in \Omega$ and $\bar{\xi}, \bar{\eta} \in K$. Therefore, by the above facts, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = R_{\bar{\xi},\bar{\eta}}(\widetilde{BA})$ for all $\bar{\xi}, \bar{\eta} \in K$. By a similar argument, we can show that $R_{\bar{\xi},\bar{\eta}}(\tilde{A}\tilde{B}) = R_{\bar{\xi},\bar{\eta}}(\widetilde{AB})$ for all $\bar{\xi}$, $\bar{\eta} \in K$. In the above argument we have shown that if $\tilde{A} \in \mathfrak{A} \otimes \mathscr{A}$ and $\tilde{B} \in \mathfrak{A} \odot \mathscr{A}$, $A(\omega)B(\omega) = (AB)(\omega)$ and $B(\omega)A(\omega) = (BA)(\omega)$ for every $\omega \in \Omega$.

At last, we shall show that if $\tilde{A}$, $\tilde{B} \in \mathfrak{A} \otimes \mathscr{A}$ and $\bar{\xi}$, $\bar{\eta} \in K$, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = R_{\bar{\xi},\bar{\eta}}(\widetilde{BA})$. Since $\tilde{A}$ is an element of $\mathfrak{A} \otimes \mathscr{A}$, there exists a net $\{\tilde{A}_\alpha\}$ in $\mathfrak{A} \odot \mathscr{A}$ such that $\tilde{A}_\alpha \to \tilde{A}$ in the σ-weak topology. Then, the σ-weak continuity of the mapping $R_{\bar{\xi},\bar{\eta}}$ and the fact such that $R_{\bar{\xi},\bar{\eta}}(\tilde{A}) = (A\xi, \eta)$ induce the following fact; $(A_\alpha\xi, \eta) \to (A\xi, \eta)$ in the σ-weak topology where $\xi(\omega) = \bar{\xi}$ and $\eta(\omega) = \bar{\eta}$ for all $\omega \in \Omega$. Thus, considering Corollary 4, for every ξ, $\eta \in F$, we can show that $(A_\alpha\xi, \eta) \to (A\xi, \eta)$ in the σ-weak topology. Now, since $\tilde{A}_\alpha \to \tilde{A}$ in the σ-weak topology, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}_\alpha) \to R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})$ in the σ-weak topology. Furthermore, we have by the previous proof; for every $\omega \in \Omega$,

$$\begin{aligned} R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}_\alpha)(\omega) &= ((BA_\alpha)(\omega)\bar{\xi} \mid \bar{\eta}) = (B(\omega)A_\alpha(\omega)\bar{\xi} \mid \bar{\eta}) \\ &= (A_\alpha(\omega)\bar{\xi} \mid B(\omega)^*\bar{\eta}) = (A_\alpha\xi, B^*\eta)(\omega) \end{aligned}$$

where $\xi(\omega) = \bar{\xi}$ and $\eta(\omega) = \bar{\eta}$ for all $\omega \in \Omega$. Write $B^*\eta = \sum e_\iota\eta_\iota$ under the condition in Proposition 8. Let G_ι be the closed and open set in Ω corresponding to e_ι. Then, for every ι, $e_\iota(A_\alpha\xi, B^*\eta) = (A_\alpha\xi, \eta_\iota) \to e_\iota(A\xi, \eta_\iota)$ in the σ-weak topology. Thus $e_\iota R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = e_\iota(A\xi, \eta_\iota) = e_\iota(A\xi, B^*\eta)$. Hence, we have; for every $\omega \in G_\iota$, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})(\omega) = (A\xi, \eta_\iota)(\omega) = (A\xi, B^*\eta)(\omega)$. Furthermore, since ξ and η are element of F,

$$(A\xi, B^*\eta)(\omega) = ((A\xi)(\omega) \mid (B^*\eta)(\omega)) = (A(\omega)\bar{\xi} \mid B(\omega)^*\bar{\eta}) = (B(\omega)A(\omega)\bar{\xi} \mid \bar{\eta})$$

for every $\omega \in \Omega$. Thus, for every $\omega \in \cup G_\iota$, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})(\omega) = (B(\omega)A(\omega)\bar{\xi} \mid \bar{\eta})$. By Proposition 12, there exists a nowhere dense set N such that $((BA)(\omega)\bar{\xi} \mid \bar{\eta}) = (B(\omega)A(\omega)\bar{\xi} \mid \bar{\eta})$ for all $\omega \in \Omega \backslash N$. Hence, for every $\omega \in (\cup G_\iota) \cap (\Omega \backslash N)$, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})(\omega) = ((BA)(\omega)\bar{\xi} \mid \bar{\eta})$. Now, since both functions $\omega \to R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})(\omega)$ and $\omega \to ((BA)(\omega)\bar{\xi} \mid \bar{\eta})$ are continuous on Ω and the set $(\cup G_\iota) \cap (\Omega \backslash N)$ is dense in Ω, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A})(\omega) = ((BA)(\omega)\bar{\xi} \mid \bar{\eta})$ for all $\omega \in \Omega$. Thus, $R_{\bar{\xi},\bar{\eta}}(\tilde{B}\tilde{A}) = (BA\xi, \eta) = R_{\bar{\xi},\bar{\eta}}(\widetilde{BA})$. Therefore, π is multiplicative, and so π is a $*$-isomorphism of $W(\Omega, K, \mathfrak{A})$ onto $\mathfrak{A} \otimes \mathscr{A}$. This completes the proof of Theorem D.

In Theorem D, we have successfully dealt with the effect of nowhere dense sets in Ω which may be considered as replacement of null sets in the measure theoretic arguments (for example, see [2. Proposition 1]).

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CONCLUDING REMARKS: After this work was finished, the author came to know the paper by B. B. Renshaw appeared in Trans. of the A.M.S., 194 (1974), 337-347, in which his main theorem is similar to our Theorem D in the present paper. In his theorem, Renshaw has proved the isometric property of the mapping π. On the other hand, although he uses the theory of module mappings in order to treat the multiplicativity of π his arguments there still leave the problem of nowhere dense subsets in nonseparable case, whereas the present work goes through this problem as a reduction theory and shows that the space $W(\Omega, K, \mathfrak{A})$ is an algebra and that the mapping π is actually a *-isomorphism between $\mathfrak{A} \otimes \mathscr{A}$ and $W(\Omega, K, \mathfrak{A})$.