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ON THE EXISTENCE OF A CONFORMAL MARTINGALE

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1. In a previous paper [2] we showed the existence of a conformal martingale by assuming that (F_t) has no time of discontinuity. The purpose of this note is to prove it without using this assumption, Lemma 1 and Lemma 2 in [2]. Roughly speaking we prove that for any L^2-bounded martingale X there exists a "conjugate" Y such that $X + iY$ is conformal.

2. The reader is assumed to be familiar with the basic notions of the theory of stochastic integrals relative to martingales as given in [3]. By a system (Ω, F, F_t, P) is meant a complete probability space (Ω, F, P) with an increasing right continuous family $(F_t)_{t\geqq 0}$ of sub σ-fields of F. We assume as usual that F_0 contains all P-null sets. Denote by $M(F_t)$ the class of all right continuous L^2-bounded martingales X over (F_t) such that $X_0 = 0$. For each $X \in M(F_t)$ we define:

$$||X||_{BMO}^2 = \sup_t \text{ess}\cdot\sup_\omega E[\langle X, X\rangle_\infty - \langle X, X\rangle_t | F_t] .$$

DEFINITION 1. Let X and Y belong to $M(F_t)$. Then a complex-valued martingale $X + iY$ is called conformal if $\langle X, Y\rangle = 0$ and $\langle X, X\rangle = \langle Y, Y\rangle$.

Originally the concept of a conformal martingale was introduced by R. K. Getoor and M. J. Sharpe in [1].

DEFINITION 2. A system $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ is said to be a lifting of (Ω, F, F_t, P) under the surjection $\pi: \tilde{\Omega} \to \Omega$ if

(1) $\pi^{-1}(F_t) \subset \tilde{F}_t$ for each t and $\pi^{-1}(F) \subset \tilde{F}$

(2) $P = \tilde{P}\circ\pi^{-1}$ on F

(3) If X is a uniformly integrable martingale over (F_t), then $X\circ\pi$ is a martingale over $(\tilde{F}_t)$.

It follows from (1) that if T is an F_t-stopping time, then $T\circ\pi$ is an $\tilde{F}_t$-stopping time. Then it is easy to see that if $H = (H_t, F_t)$ is previsible, $H\circ\pi = (H_t\circ\pi)$ is also a previsible process over $(\tilde{F}_t)$. Therefore we get $\langle X\circ\pi, X\circ\pi\rangle = \langle X, X\rangle\circ\pi$ for every $X \in M(F_t)$.

3. In what follows we denote by $H\cdot X$ the stochastic integral $\left(\int_0^t H_s dX_s\right)$. We do not assume that (F_t) has no time of discontinuity.

THEOREM. *Suppose that* (Ω, F, P) *is separable. Then there exists a lifting* $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ *of* (Ω, F, F_t, P) *under* $\pi: \tilde{\Omega} \to \Omega$ *which satisfies the following conditions*:

1°. *There exists a linear mapping* $\alpha: M(F_t) \to M(\tilde{F}_t)$ *such that*

(1) *for every* $X \in M(F_t)$, $X\circ\pi + i\alpha(X)$ *is conformal*

(2) *for every* $X \in M(F_t)$ *and* $C \in L^2(X)$, $\alpha(C\cdot X) = (C\circ\pi)\cdot\alpha(X)$

2° *There exists a linear mapping* $\bar{\alpha}: M(\tilde{F}_t) \to M(F_t)$ *such that*

(1) $\bar{\alpha}\circ\alpha$ *is the identity on* $M(F_t)$

(2) *if* $X \in M(F_t)$ *and* $\tilde{X} \in M(\tilde{F}_t)$, *then* $\tilde{E}[\alpha(X)_\infty \tilde{X}_\infty] = E[X_\infty \bar{\alpha}(\tilde{X})_\infty]$

(3) *for every* $\tilde{X} \in M(\tilde{F}_t)$, $\|\bar{\alpha}(\tilde{X})\|_{BMO} \leqq \|\tilde{X}\|_{BMO}$.

PROOF. We shall use essentially the method given in [1]. Let $X^0 \in M(F_t)$ be fundamental for $M(F_t)$; the existence of such an element X^0 is guaranteed by the separability of (Ω, F, P). Put $A_t = \langle X^0, X^0\rangle_t$ and $\tau_t = \inf\{s > 0; A_s > t\}$. Denote by (G_t) the right continuous family (F_{τ_t}). Then each A_t is a G_t-stopping time. Let (K_t) be the right continuous family (G_{A_t}). We have in general $\tau_{A_t} \geqq t$ a.s.

We shall assume firstly that A is strictly increasing. One should be aware of $\tau_{A_t} = t$ a.s in this case. This implies that $F_t = K_t$. Now let (Ω', F', F'_t, P') be a separable system which carries a sequence $(B^n)_{n\geqq 1}$ of independent real Brownian motions with $B^n_0 = 0$ and $\langle B^n, B^n\rangle_t = t$ for all n. Denote by $(\tilde{\Omega}, \tilde{F}, \tilde{G}_t, \tilde{P})$ the product of the systems (Ω, F, G_t, P) and (Ω', F', F'_t, P') with π, π' the projections of $\tilde{\Omega} = \Omega \times \Omega'$ onto Ω and Ω' respectively. Then each $A_t\circ\pi$ is a $\tilde{G}_t$-stopping time. Let $\tilde{F}_t = \tilde{G}_{A_t\circ\pi}$ and consider now the system $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$. Clearly $P = \tilde{P}\circ\pi^{-1}$ on F and $\pi^{-1}(F) \subset \tilde{F}$. If $\Lambda \in F_t$, then

$$\pi^{-1}(\Lambda) \cap \{A_t\circ\pi < s\} = [\Lambda \cap \{A_t < s\}] \times \Omega' = [\Lambda \cap \{\tau_{A_t} < \tau_s\}] \times \Omega'$$

which belongs to $\tilde{G}_s$. Therefore $\pi^{-1}(\Lambda) \in \tilde{F}_t$. Next, if X is a uniformly integrable martingale over (F_t), then by Doob's optional sampling theorem X_{τ_t} is a G_t-martingale so $X_{\tau_t}\circ\pi$ is a $\tilde{G}_t$-martingale. Thus $X_t\circ\pi = X_{\tau_{A_t}}\circ\pi$ is an $\tilde{F}_t$-martingale. That is to say, $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ is a lifting of (Ω, F, F_t, P) under π.

Now we are going to construct $\alpha(X)$. Since $B^n\circ\pi'$ is a $\tilde{G}_t$-martingale, $\tilde{N}^n_t(\omega, \omega') = B^n_{A_t(\omega)}(\omega')$ is an $\tilde{F}_t$-martingale. Obviously $\langle \tilde{N}^j, \tilde{N}^k\rangle_t = \delta_{jk} A_t\circ\pi$ and $\langle X\circ\pi, \tilde{N}^n\rangle = 0$ for all $X \in M(F_t)$. Let $(X^n)_{n\geqq 1}$ be an integral basis for $M(F_t)$ whose existence is guaranteed by the separability of the space (Ω, F, P). Denote by D^n a previsible version of $d\langle X^n, X^n\rangle/dA$ and put:

$$C^n = (D^n)^{1/2}\ , \quad \tilde{M}^n = (C^n\circ\pi)\cdot\tilde{N}^n\ .$$

The process $\tilde{M}^n$ belongs to $M(\tilde{F}_t)$. If $\hat{D}^n$ is another previsible version of $d\langle X^n, X^n\rangle/dA$, it follows from the uniqueness of the density that

$$E\left[\int_0^\infty I_{\{D^n \neq \hat{D}^n\}}(t, \cdot)dA_t\right] = 0 .$$

Therefore $\tilde{M}^n$ does not depend on the choice of D^n. It is clear that $\langle \tilde{M}^n, X\circ\pi\rangle = 0$ for all n and that $\langle \tilde{M}^j, \tilde{M}^k\rangle_t = \delta_{jk}\langle X^j, X^k\rangle_t\circ\pi$. On the other hand, for each $X \in M(F_t)$

$$X = \sum_n H^n \cdot X^n$$

convergent in $M(F_t)$ with $H^n = d\langle X, X^n\rangle/d\langle X^n, X^n\rangle$. The sum

$$\sum_n (H^n\circ\pi)\cdot \tilde{M}^n$$

converges in $M(\tilde{F}_t)$ because $\langle (H^n\circ\pi)\cdot\tilde{M}^n, (H^n\circ\pi)\cdot\tilde{M}^n\rangle = \langle H^n\cdot X^n, H^n\cdot X^n\rangle\circ\pi$ for each n; $(H^n\circ\pi)\cdot\tilde{M}^n$ does not depend on the choice of H^n. Then the mapping $\alpha\colon M(F_t) \to M(\tilde{F}_t)$ given by

$$\alpha(X) = \sum_n (H^n\circ\pi)\cdot\tilde{M}^n$$

is well defined and linear. From the above relation we get

$$\langle \alpha(X), \alpha(X)\rangle = \langle X\circ\pi, X\circ\pi\rangle , \qquad \langle X\circ\pi , \ \alpha(X)\rangle = 0 .$$

Consequently $X\circ\pi + i\alpha(X)$ is an $\tilde{F}_t$-conformal martingale. It is immediate that $\alpha(C\cdot X) = (C\circ\pi)\cdot\alpha(X)$ if $X \in M(F_t)$ and $C \in L^2(X)$.

Next, we shall explain briefly the definition of the adjoint mapping $\bar{\alpha}$. This part is an adaptation of the proof due to Getoor and Sharpe (see [1]). Denote by $\tilde{N}$ the stable subspace of $M(\tilde{F}_t)$ generated by the $X^n\circ\pi$ and $\tilde{M}^n$, by L_1 the projection of $M(\tilde{F}_t)$ onto $\tilde{N}$ and let $L_2\colon \tilde{N} \to \tilde{N}$ be defined as follows: if $\tilde{X} \in \tilde{N}$ has an expansion of the form $\sum_n C^n\cdot(X^n\circ\pi) + \sum_n D^n\cdot\tilde{M}^n$, then $L_2(\tilde{X}) = \sum_n D^n\cdot(X^n\circ\pi) + \sum_n C^n\cdot\tilde{M}^n$; $L_2(\tilde{X})$ does not depend on the previsible versions of C^n and D^n. Then it is clear that for every $X \in M(F_t)$, $L_2(\alpha(X)) = X\circ\pi$. Define $L_3\colon \tilde{N} \to M(F_t)$ by letting $L_3\tilde{X}$ for $\tilde{X} \in \tilde{N}$ be the unique right continuous martingale over (F_t) such that $(L_3\tilde{X})_t\circ\pi = \tilde{E}[\tilde{X}_\infty | \pi^{-1}(F_t)]$. Then the mapping $\bar{\alpha} = L_3L_2L_1\colon M(\tilde{F}_t) \to M(F_t)$ satisfies all the properties necessarily for the theorem.

Finally, we are going to consider the general case. Construct a system $(\Omega^*, F^*, F_t^*, P^*)$ by taking the product of the system (Ω, F, F_t, P) with another separable system $(\hat{\Omega}, \hat{F}, \hat{F}_t, \hat{P})$ which carries a real Brownian motion $(\hat{B}_t)$ with $\hat{B}_0 = 0$ and $\langle \hat{B}, \hat{B}\rangle_t = t$. As (Ω^*, F^*, P^*) is separable, $M(F_t^*)$ has a fundamental element Y^0. Let γ, $\hat{\gamma}$ the projections of Ω^* onto Ω and $\hat{\Omega}$ respectively. Then $\hat{B}\circ\hat{\gamma}$ is a continuous F_t^*-martingale. Since $\langle \hat{B}, \hat{B}\rangle$ is $\langle Y^0, Y^0\rangle$-absolutely continuous, $\langle Y^0, Y^0\rangle$ is strictly increasing.

Therefore, on some lifting $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ of $(\Omega^*, F^*, F_t^*, P^*)$ under π^*, there exist linear mappings $\alpha^*: M(F_t^*) \to M(\tilde{F}_t)$ and $\bar{\alpha}^*: M(\tilde{F}_t) \to M(F_t^*)$ which satisfy all the properties of the theorem; namely, for each $X^* \in M(F_t^*)$, $X^* \circ \pi^* + i\alpha^*(X^*)$ is a conformal martingale over $(\tilde{F}_t)$. Then $(\tilde{\Omega}, \tilde{F}, \tilde{F}_t, \tilde{P})$ is also a lifting of (Ω, F, F_t, P) under $\pi = \gamma \circ \pi^*$. As $X \circ \gamma \in M(F_t^*)$ for every $X \in M(F_t)$,

$$X \circ \pi + i\alpha^*(X \circ \gamma) = (X \circ \gamma) \circ \pi^* + i\alpha^*(X \circ \gamma)$$

is a conformal martingale over $(\tilde{F}_t)$. The mapping $\alpha: M(F_t) \to M(\tilde{F}_t)$ defined by $\alpha(X) = \alpha^*(X \circ \gamma)$ is linear. If $C \in L^2(X)$, then $C \circ \gamma \in L^2(X \circ \gamma)$ and so we get

$$\begin{aligned} \alpha(C \circ X) &= \alpha^*((C \circ \gamma) \cdot (X \circ \gamma)) \\ &= (C \circ \gamma \circ \pi^*) \cdot \alpha^*(X \circ \gamma) \\ &= (C \circ \pi) \cdot \alpha(X) . \end{aligned}$$

We are now going to define the adjoint mapping $\bar{\alpha}$. Define L^*: $M(F_t^*) \to M(F_t)$ by letting $L^* X^*$ for $X^* \in M(F_t^*)$ be the unique right continuous martingale over (F_t) such that

$$(L^* X^*)_t \circ \gamma = E^*[X_\infty^* \mid \gamma^{-1}(F_t)] .$$

It is easy to see that $\{\gamma^{-1}(F_t)\}$ is a right continuous family.

Now we put

$$\bar{\alpha}(\tilde{X}) = L^*(\bar{\alpha}^*(\tilde{X})) , \qquad \tilde{X} \in M(\tilde{F}_t) .$$

Obviously $\bar{\alpha}$ is a linear mapping of $M(\tilde{F}_t)$ into $M(F_t)$. If $X \in M(F_t)$, then $X \circ \gamma \in M(F_t^*)$ and

$$\begin{aligned} E^*[X_\infty \circ \gamma \mid \gamma^{-1}(F_t)] &= E[X_\infty \mid F_t] \circ \gamma \\ &= X_t \circ \gamma \end{aligned}$$

from which $L^*(X \circ \gamma) = X$. Thus we get

$$\begin{aligned} \bar{\alpha}(\alpha(X)) &= L^* \bar{\alpha}^*(\alpha^*(X \circ \gamma)) \\ &= L^*(X \circ \gamma) \\ &= X . \end{aligned}$$

If $X \in M(F_t)$ and $\tilde{X} \in M(\tilde{F}_t)$, then

$$\begin{aligned} \tilde{E}[\alpha(X)_\infty \tilde{X}_\infty] &= \tilde{E}[\alpha^*(X \circ \gamma)_\infty \tilde{X}_\infty] \\ &= E^*[(X_\infty \circ \gamma)(\bar{\alpha}^*(\tilde{X})_\infty)] \\ &= E^*[X_\infty \circ \gamma (L^* \bar{\alpha}^*(\tilde{X}))_\infty \circ \gamma] \\ &= E[X_\infty (\bar{\alpha}(\tilde{X})_\infty)] . \end{aligned}$$

And if $X^* \in M(F_t^*)$,

$$E[(L^*X^*_\infty - L^*X^*_t)^2|F_t] = E^*[\{E^*[X^*_\infty - X^*_t \mid \gamma^{-1}(F)]\}^2|\gamma^{-1}(F_t)]$$
$$\leqq E^*[\{E^*(X^*_\infty - X^*_t)^2|F^*_t]\}|\gamma^{-1}(F_t)]$$
$$\leqq ||X^*||^2_{BMO}$$

meaning $||L^*X^*||_{BMO} \leqq ||X^*||_{BMO}$. Therefore for every $\tilde{X} \in M(\tilde{F}_t)$ we get

$$||\bar{\alpha}(\tilde{X})||_{BMO} = ||L^*(\bar{\alpha}^*(\tilde{X}))||_{BMO}$$
$$\leqq ||\bar{\alpha}^*(\tilde{X})||_{BMO}$$
$$\leqq ||\tilde{X}||_{BMO} .$$

This completes the proof.

REMARK. If X is a locally square integrable martingale over (F_t) and (T_n) reduces X to $M(F_t)$, then for every n $\alpha(X^{T_{n+1}}) = \alpha(X^{T_n})$ on $[0, T_n \circ \pi]$. Thus $\alpha(X)$ can be defined for locally square integrable martingales.

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