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GEODESICS OF O_n^2 AND AN ANALYSIS ON A RELATED RIEMANN SURFACE

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0. Introduction. As is shown in [6] and [8], the following nonlinear differential equation:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + (1-x^2)(nx^2-1) = 0\,,$$

where $n(>1)$ is a real constant, is the equation for the support function $x(t)$ of a geodesic in the 2-dimensional Riemannian manifold O_n^2 with the metric:

$$\text{(0.1)} \qquad ds^2 = (1-u^2-v^2)^{n-2}\{(1-v^2)du^2 + 2uvdudv + (1-u^2)dv^2\}$$

in the unit disk: $u^2+v^2<1$. O_n^2 can be regarded as a surface of revolution in the 4-dimensional Lorentzian space punctured at a point from a closed one [10].

Any non constant solution $x(t)$ of (E) such that

$$x^2 + \left(\frac{dx}{dt}\right)^2 < 1$$

is periodic and its period T is given by the improper integral:

$$\text{(0.2)} \qquad T = 2\int_{a_0}^{a_1} \frac{dx}{\sqrt{1-x^2-C\left(\dfrac{1}{x^2}-1\right)^{\alpha}}}\,,$$

where

$$\text{(0.3)} \qquad C = (a_0^2)^{\alpha}(1-a_0^2)^{1-\alpha} = (a_1^2)^{\alpha}(1-a_1^2)^{1-\alpha}$$
$$(0 < a_0 < \sqrt{\alpha} < a_1 < 1,\ \alpha = 1/n)$$

is the integral constant of (E) and $0 < C < A = \alpha^{\alpha}(1-\alpha)^{1-\alpha}$.

By means of the above mentioned geometrical meaning of $x(t)$, T represents the angular period of a geodesic of O_n^2 in the unit disk. The following was proved in [4]:

(i) T is differentiable with respect to C,

(ii) $T > \pi$,

(iii) $\lim_{C\to 0} T = \pi$ and $\lim_{C\to A} T = \sqrt{2}\,\pi$;

and then the following inequality:

$$\text{(U)} \qquad T < \sqrt{2}\pi$$

was conjectured in [5] and [11] by means of a numerical analysis of (E) done by M. Urabe [11]. This inequality has been proved recently in [8] and [9] in cases of $n \geqq 3$ and $1 < n < 3$ respectively.

In [6], the author conjectured also that *T is a monotone increasing function of C* which will imply (U). He will prove this conjecture by means of an analysis on a related Riemann surface with O_n^2.

1. Preliminaries. The differential equation of geodesics of O_n^2 is

$$(1 - u^2 - v^2)\frac{d^2v}{du^2} = n\left(-v + u\frac{dv}{du}\right)\left\{1 - v^2 + 2uv\frac{dv}{du} + (1 - u^2)\left(\frac{dv}{du}\right)^2\right\}$$

in the coordinates (u, v), which can be written as

$$\text{(E}'\text{)} \qquad r(1 - r^2)\frac{d^2r}{d\theta^2} + \{(n + 2)r^2 - 2\}\left(\frac{dr}{d\theta}\right)^2 + r^2(1 - r^2)(nr^2 - 1) = 0$$

in the polar coordinates (r, θ) in the (u, v)-plane, i.e. $u = r\cos\theta$, $v = r\sin\theta$.

The differential equation (E′) has the following first integral:

$$\left(\frac{dr}{d\theta}\right)^2 = C_1r^4(1 - r^2)^n - r^2(1 - r^2)\,,$$

where C_1 is a positive integral constant. Any solution $r(\theta)$ of (E′) such that $r \neq 0$, $0 < r < 1$, is periodic and its period Θ is given by the improper integral:

$$\text{(1.1)} \qquad \Theta = 2\int_{r_0}^{r_1}[C_1r^4(1 - r^2)^n - r^2(1 - r^2)]^{-1/2}dr\,,$$

where

$$r_0^2(1 - r_0^2)^{n-1} = r_1^2(1 - r_1^2)^{n-1} = 1/C_1\,,$$
$$0 < r_0 < \sqrt{\alpha} < r_1 < 1\,.$$

If we put $C_1 = 1/C^n$, then we get $r_0 = a_0$ and $r_1 = a_1$, and we can prove the equality:

$$\Theta = T$$

by making use of the properties of the solution $x(t)$ and its geometrical meaning. Furthermore, if we change the integral variable in (1.1) from r to x by $nr^2 = x$, then we obtain easily

$$\text{(1.2)} \qquad T = T(c) = \sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n - x)\{x(n - x)^{n-1} - c\}}}\,,$$

where

$$(1.3) \qquad c = (nC)^n = x_0(n - x_0)^{n-1} = x_1(n - x_1)^{n-1} ,$$

$$(1.4) \qquad 0 < x_0 < 1 < x_1 < n .$$

Now, we try to express $T(c)$ by means of complex analysis. If we take a piecewise smooth, oriented, simple close curve γ in the complex z-plane such that x_0 and x_1 and 1 are inside of γ and the zero and n and the other solutions than x_0 and x_1 of the equation:

$$(1.5) \qquad z(n - z)^{n-1} - c = 0$$

are all outside of γ, and the orientation of γ is coherent to the canonical

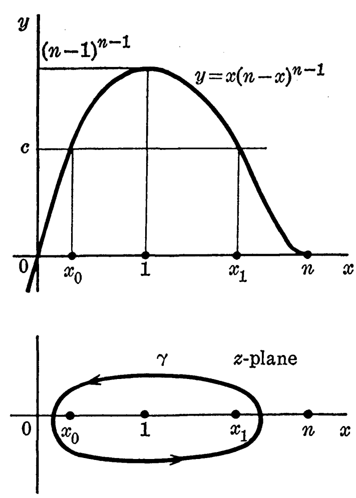

FIGURE 1.

one of the z-plane, then $T(c)$ can be written by the integral along γ as follows:

$$(1.6) \qquad T(c) = -\frac{\sqrt{nc}}{2}\int_\gamma \frac{dz}{z\sqrt{(n - z)\{z(n - z)^{n-1} - c\}}} .$$

This expression of $T(c)$ sets the integral (1.2) free from the improper property based on the interval (x_0, x_1) of integration and shows that $T(c)$ is analytic in c for $0 < c < (n - 1)^{n-1}$.

Differentiating (1.6) with respect to c, we obtain

$$T'(c) = -\frac{1}{4}\sqrt{\frac{n}{c}}\int_\gamma \left\{\frac{1}{z\sqrt{(n - z)\{z(n - z)^{n-1} - c\}}} + \frac{c}{z\sqrt{(n - z)\{z(n - z)^{n-1} - c\}^3}}\right\}dz ,$$

i.e.

$$T'(c) = -\frac{1}{4}\sqrt{\frac{n}{c}}\int_{\gamma}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\,. \tag{1.7}$$

Now, we set

$$I_n(c) := \int_{\gamma}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\,. \tag{1.8}$$

If we can prove the following inequality:

$$I_n(c) < 0 \quad \text{for} \quad 0 < c < (n-1)^{n-1}\,,$$

then the period T given by (0.2) is monotone increasing as a function of C for $0 < C < A$.

2. A Riemann surface related with the integral $I_n(c)$. Now, we define a Riemann surface $\mathscr{F} = \mathscr{F}_n(c)$ in C^2 with the coordinates (z, w) by the equation:

$$z(n-z)^{n-1} - w^2 = c\,, \tag{2.1}$$

which is an algebraic curve when n is an integer. The closed curve γ in (1.8) can be considered as an oriented closed curve on the surface and the integral (1.8) as an integral along γ on $\mathscr{F}$. Therefore the value of $I_n(c)$ does not change even if we replace γ by another piecewise smooth closed curve through a piecewise smooth homotopy on $\mathscr{F}$ whose projection on the z-plane avoids the roots of the equation (1.5) and $z = n$.

Let $b > 0$ be a real constant such that

$$b = \sqrt{(n-1)^{n-1} - c}\,, \tag{2.2}$$

then the projection γ_z and γ_w of the curve γ on $\mathscr{F}$ onto the z-plane and the w-plane respectively may be illustrated as in Fig. 2, taking into consideration of the transition of integrals from (1.2) to (1.6).

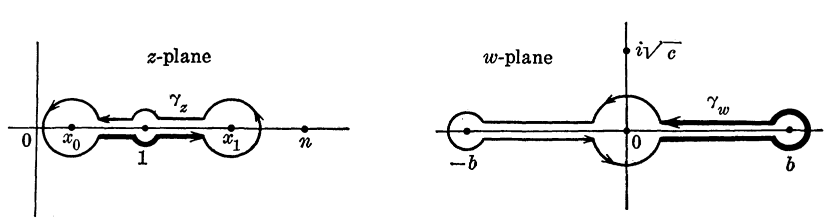

FIGURE 2.

In fact, since we have

$$\{z(n-z)^{n-1}\}' = n(1-z)(n-z)^{n-2} \tag{2.3}$$

and

$$\{z(n-z)^{n-1}\}'' = -n(n-1)(2-z)(n-z)^{n-3}\,, \tag{2.4}$$

we obtain easily from (2.1) around $z = x_i$ $(i = 0, 1)$

$$n(1-x_i)(n-x_i)^{n-2}(z-x_i) + O((z-x_i)^2) = w^2\,,$$

from which we get the relation

$$z - x_i = \frac{w^2}{n(1-x_i)(n-x_i)^{n-2}} + O(w^4); \tag{2.5}$$

and we obtain around $z = 1$

$$b^2 - \frac{n(n-1)^{n-2}}{2}(z-1)^2 + O((z-1)^3) = w^2\,,$$

or

$$\begin{aligned}\frac{n(n-1)^{n-2}}{2}(z-1)^2 + O((z-1)^3) &= b^2 - w^2 \\ &= \mp 2b(w \mp b) - (w \mp b)^2\,,\end{aligned}$$

from which we obtain the relation:

$$w \mp b = \mp \frac{n(n-1)^{n-2}}{4b}(z-1)^2 + O((z-1)^3)\,. \tag{2.6}$$

These relations implies the correspondence between γ_z and γ_w as is shown in Fig. 2.

Now, differentiating (2.1) we have $n(1-z)(n-z)^{n-2}dz = 2wdw$ and using this the integrand of (1.8) can be written as

$$\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = \frac{(n-z)^{n-3/2}}{w^3}\cdot\frac{2wdw}{n(1-z)(n-z)^{n-2}}\,,$$

hence we get the expression of I_n by

$$I_n(c) = \frac{2}{n}\int_\Gamma \frac{(n-z)^{1/2}dw}{(1-z)w^2}\,. \tag{2.7}$$

Next, we need the following lemmas with regard to the integrals (1.8) and (1.6).

LEMMA 1. *If $n > -1$, then*

$$\lim_{r\to\infty}\int_{|z|=r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = 0\,.$$

PROOF. Setting $z = re^{i\theta}$, for sufficiently large r there exists a positive constant K_1 such that

$$\left|\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\right| \leqq K_1 r^{-(n+1)/2}|d\theta| .$$

Hence we have

$$\left|\int_{|z|=r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\right| \leqq 2\pi K_1 r^{-(n+1)/2} ,$$

which implies this lemma. q.e.d.

REMARK 1. Let A_r be a set of subarcs on the circle $|z| = r$ with a bounded angular measure from 0, then we have also

$$\lim_{r\to\infty}\int_{A_r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = 0 .$$

LEMMA 2. *If $n > 1/2$, then*

$$\lim_{r\to 0}\int_{|z-n|=r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = 0 .$$

PROOF. Setting $z = n + re^{i\theta}$, for sufficiently small r there exists a positive constant K_2 such that

$$\left|\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\right| \leqq K_2 c^{-3/2} r^{n-1/2}|d\theta| .$$

Hence we have

$$\left|\int_{|z-n|=r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}}\right| \leqq 2\pi K_2 c^{-3/2} r^{n-1/2} ,$$

which implies this lemma. q.e.d.

REMARK 2. Let B_r be a set of subarcs on the circle $|z - n| = r$ with a bounded angular measure from n, then we have also

$$\lim_{r\to 0}\int_{B_r}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = 0 .$$

Finally, let ζ be a solution of the equation (1.5) other than x_0 and x_1.

LEMMA 3.

$$\lim_{r\to 0}\int_{|z-\zeta|=r}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}} = 0 .$$

PROOF. It is clear that $\zeta \neq 0$, 1 and n and ζ is simple by (2.3). Setting $z = \zeta + re^{i\theta}$, for sufficiently small r there exists a positive constant K_3 such that

$$\left|\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\right| \leqq \frac{K_3 n^{-1/2} r^{1/2}|d\theta|}{|\zeta\sqrt{(1-\zeta)(n-\zeta)^{n-1}}|} .$$

Hence we have

$$\left|\int_{|z-\zeta|=r}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\right| \leqq \frac{2\pi K_3 n^{-1/2} r^{1/2}}{|\zeta\sqrt{(1-\zeta)(n-\zeta)^{n-1}}|},$$

which imply this lemma. q.e.d.

REMARK 3. Let C_r be a set of subarcs on the circle $|z-\zeta|=r$ with a bounded angular measure from ζ, then we have also

$$\lim_{r\to 0}\int_{|z-\zeta|=r}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}=0 .$$

3. Changes of the curve of integration. In order to find a suitable change of the curve γ of integration on the Riemann surface $\mathscr{F}_n(c)$, we shall investigate the correspondence between z and w on this surface.

By the argument in § 2, we see that the singular points on $\mathscr{F}_n(c)\subset \boldsymbol{C}^2$ are

(i) $(x_0, 0)$, $(x_1, 0)$ and $(\zeta, 0)$, where ζ are the solutions of the equation (1.5) other than x_0 and x_1;

(ii) $(1, b)$ and $(1, -b)$;

(iii) $(n, i\sqrt{c})$ and $(n, -i\sqrt{c})$, if $n>2$.

The reasons of singularity are that $w=\sqrt{z(n-z)^{n-1}-c}$ vanishes at $z=x_0, x_1$ and ζ, and $\{z(n-z)^{n-1}-c\}'$ vanishes at $z=1$ and n (when $n>2$).

(2.5) and (2.6) show the state of $\mathscr{F}_n(c)$ around the points $(x_0, 0)$, $(x_1, 0)$, $(1, b)$ and $(1, -b)$.

In the following, we suppose $n\geqq 2$ and investigate the state of $\mathscr{F}_n(c)$ around the points $(n, i\sqrt{c})$ and $(n, -i\sqrt{c})$. Setting

$$z=n+re^{i\theta}\ (r>0) \quad\text{and}\quad w=\pm i\sqrt{c}+te^{i\varphi}\ (t>0)$$

and substituting these into (2.1), we have

$$(n+re^{i\theta})\{re^{i(\theta+m\pi)}\}^{n-1}-c=-c\pm 2i\sqrt{c}\,te^{i\varphi}+t^2e^{2i\varphi},$$

where m is an odd integer, hence

$$nr^{n-1}e^{i(n-1)(\theta+m\pi)}+r^ne^{i\{n\theta+(n-1)m\pi\}}=2\sqrt{c}te^{i(\varphi\pm\pi/2)}+t^2e^{2i\varphi},$$

which implies

$$t=\frac{n}{2\sqrt{c}}r^{n-1}+O(r^n) \tag{3.1}$$

and

$$\varphi=(n-1)(\theta+m\pi)\mp\frac{\pi}{2}+O(r).$$

From the above relation between the arguments θ and φ, we get especially for $\theta = \pi$

$$\varphi = (n-1)(m+1)\pi \mp \frac{\pi}{2} + O(r)\,.$$

Considering the correspondence between γ_z and γ_w as is shown in Fig. 2, we may put $m = -1$ for our purpose. Therefore we have the relations

(3.2) $$\varphi = (n-1)(\theta-\pi) - \frac{\pi}{2} + O(r) \quad \text{around } (n, i\sqrt{c})$$

and

(3.3) $$\varphi = (n-1)(\theta-\pi) + \frac{\pi}{2} + O(r) \quad \text{around } (n, -i\sqrt{c})\,.$$

Now, for sufficiently small $r > 0$ we can choose two angles $\theta_1 = \theta_1(r)$ and $\theta_2 = \theta_2(r)$ around $(n, i\sqrt{c})$ such that

(i) $\theta_1 < \pi$ and the value φ_1 of φ in (3.2) for $\theta = \theta_1$ is $-3\pi/2$ and

(ii) $\theta_2 > \pi$ and the value φ_2 of φ in (3.2) for $\theta = \theta_2$ is $\pi/2$.

Since we have for $\theta = \pi$ the equality $\varphi = -\pi/2$, we may put

(3.4) $$\pi - \theta_1 = \frac{\pi}{n-1} + O(r)$$

and

(3.5) $$\theta_2 - \pi = \frac{\pi}{n-1} + O(r)\,.$$

If $n > 2$, we may consider for sufficiently small r as

(3.6) $$0 < \theta_1 < \pi \quad \text{and} \quad \pi < \theta_2 < 2\pi\,.$$

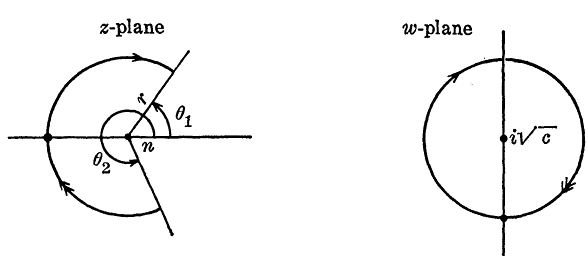

FIGURE 3.

Using the above argument, we shall firstly deform the original curve γ in the integral (1.8) to a curve γ_1 on $\mathscr{F}_n(c)$ as is shown in Fig. 4_i, $i = 1, 2$, without through the singular points described in this section.

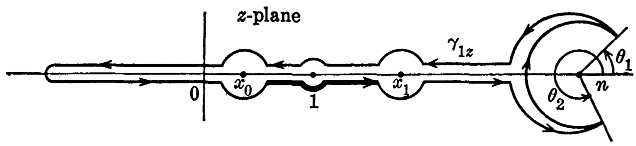

FIGURE 4_1.

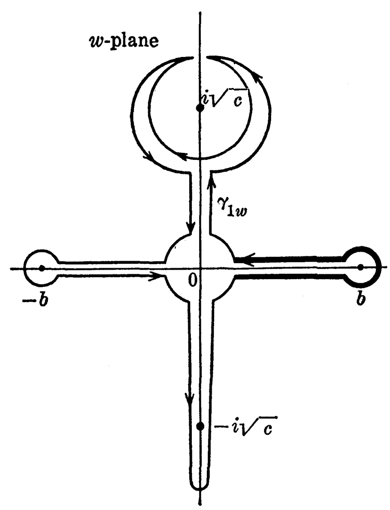

FIGURE 4_2.

Now, we consider the point

$$w = i\sqrt{c+y} \quad (y > 0)$$

in the w-plane. If y is sufficiently small, then we can choose uniquely the points $\sigma_1(y) = n + r_1 e^{i\theta_1}$ and $\sigma_2(y) = n + r_2 e^{i\theta_2}$ around $z = n$ in the z-plane such that

$$(\sigma_1(y),\, i\sqrt{c+y}),\ (\sigma_2(y),\, i\sqrt{c+y}) \in \mathscr{F}_n(c) \quad \text{and} \quad \theta_1 = \theta_1(r_1),\ \theta_2 = \theta_2(r_2)\,.$$

Since we have from (2.3)

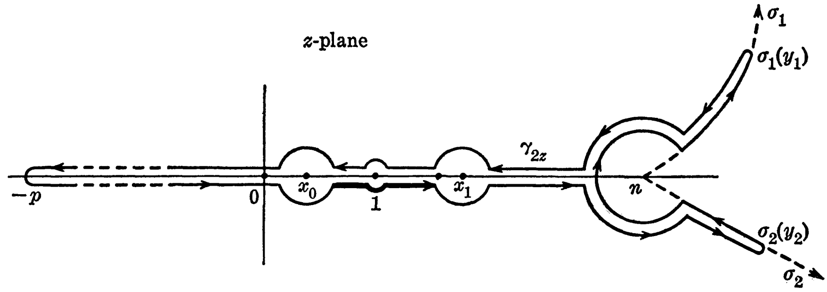

FIGURE 5_1.

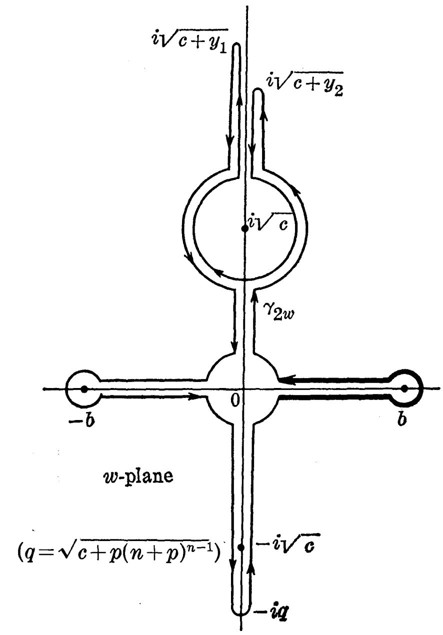

FIGURE 5_2.

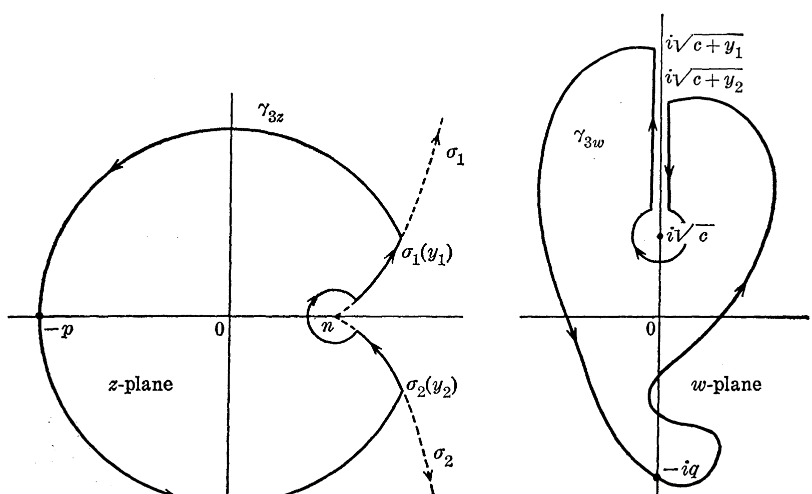

FIGURE 6.

$$n(1-z)(n-z)^{n-2}\frac{dz}{dw} = 2w$$

and so for $w = i\sqrt{c+y}$ $(y > 0)$ we have $dz/dw \neq 0$. Therefore, if we vary y in the interval $0 < y < \infty$, then we obtain two curves $\sigma_1(y)$ and

$\sigma_2(y)$ starting from the point $z = n$ and diverging to the infinity.

Then, we shall deform the curve γ_1 to a curve γ_2 on $\mathscr{F}_n(c)$ as is shown in Fig. 5_i, $i = 1, 2$, without through the singular points.

Finally, we take sufficiently large positive numbers y_1, y_2 and p such that

$$|\sigma_1(y_1)| = |\sigma_2(y_2)| = p > \sup\{|\zeta|\}$$

where ζ are the solutions of the equation (1.5). This choice of y_1 and y_2 is assured by means of the above consideration about $\mathscr{F}_n(c)$. Then, we deform the curve γ_2 to a curve γ_3 on $\mathscr{F}_n(c)$ as is shown in Fig. 6.

In this deformation from γ_2 to γ_3, we admitted that the intermediate piecewise smooth curve passes through the singular points $(\zeta, 0) \in \mathscr{F}_n(c)$, where ζ are solutions of (1.5).

LEMMA 4.

$$T(c) = -\pi - \frac{\sqrt{nc}}{2}\int_{\gamma_3}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\,.$$

PROOF. By the method of the construction of the curves γ_1 and γ_2 we get easily for sufficiently small $r > 0$

$$\int_{\gamma_1}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}} = \int_{\gamma}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}} + \oint_{|z|=r}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\,.$$

On the 2nd term of the right handside, $z = 0$ corresponds to $w = -i\sqrt{c}$ from the arguments in § 1 and this section. Hence we obtain easily

$$\oint_{|z|=r}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}} = -\frac{2\pi}{\sqrt{nc}}\,.$$

Furthermore, we have

$$\int_{\gamma_1}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}} = \int_{\gamma_2}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}$$

and in the right hand side we can also replace γ_2 with γ_3 by virtue of Lemma 3. Hence we obtain the formula expressed in this lemma.

q.e.d.

LEMMA 5.

$$I_n(c) = \int_{\gamma_3}\frac{(n-z)^{n-3/2}dz}{\sqrt{\{z(n-z)^{n-1}-c\}^3}} = \frac{2}{n}\int_{\gamma_3}\frac{(n-z)^{1/2}dw}{(1-z)w^2}\,.$$

PROOF. By (1.6), (1.7) and Lemma 4, we obtain

$$I_n(c) = -4\sqrt{\frac{c}{n}}T'(c)$$
$$= -4\sqrt{\frac{c}{n}}\frac{d}{dc}\left\{-\frac{\sqrt{nc}}{2}\int_{\gamma_3}\frac{dz}{z\sqrt{(n-z)\{z(n-z)^{n-1}-c\}}}\right\}.$$

By the analogous computation to that of (1.7), we obtain the formula expressed in this lemma. q.e.d.

4. Proof of $I_n(c) < 0$, when $n > 2$.

LEMMA 6. *When $n > 2$, the curve $\sigma_1(y)$ $(0 < y)$ lies in the upper half plane and the curve $\sigma_2(y)$ $(0 < y)$ in the lower half plane of the real axis of the z-plane.*

PROOF. For the real variable x, we have
(i) $x(n-x)^{n-1} - c \geqq 0$ for $x_0 \leqq x \leqq x_1$;
(ii) $-c \leqq x(n-x)^{n-1} - c < 0$ for $0 \leqq x < x_0$ and $x_1 < x \leqq n$;
(iii) $x(n-x)^{n-1} - c < -c$ for $x < 0$
and
(iv) $x(n-x)^{n-1} - c = e^{i(n-1)\pi}x(x-n)^{n-1} - c$ for $n < x$.

Now, for sufficiently small $y > 0$, the statement is true by (3.6). From the above property of the function $x(n-x)^{n-1} - c$, we see that if the curves $\sigma_1(y)$ or $\sigma_2(y)$ meet with the real axis of the z-plane, the coordinate x^* of the meeting point must be $x^* < 0$ or $n < x^*$.

If $x^* < 0$ and $\sigma_1(y^*) = x^*$, then the curve $\sigma_1(y)$ must lie on the real axis around y^* since $\{z(n-z)^{n-1}\}'_{z=x} \neq 0$ for $x < 0$, in another words, a small neighborhood of $z = x^*$ corresponds regularly to a small neighborhood of $w = -i\sqrt{c+y^*}$ through the Riemann surface $\mathscr{F}_n(c)$. Hence, continuing this process, we can take x^* sufficiently near 0, then we obtain a contradiction, because y^* becomes then sufficiently small. We shall obtain also a contradiction in case $\sigma_2(y^*) = x^*$.

Next, $n < x^*$, from (iv) of the above mentioned facts, it must be

$$e^{i(n-1)\pi} = -1 = e^{i\pi},$$

i.e. n = even. If n = even, then we have

$$x(n-x)^{n-1} - c = -\{c + x(x-n)^{n-1}\} \quad \text{for} \quad n < x$$

and

$$c + x(x-n)^{n-1} > c.$$

By an analogous argument to the case $x^* < 0$, we can show that the both curves $\sigma_1(y)$ and $\sigma_2(y)$ can not meet with the real axis on the interval $n < x < \infty$. q.e.d.

PROPOSITION 1. *When* $n > 2$, *we have* $I_n(c) < 0$ for $0 < c < (n-1)^{n-1}$.

PROOF. By Lemma 5, we have

$$I_n(c) = \int_{\gamma_3} \frac{(n-z)^{n-3/2} dz}{\sqrt{\{z(n-z)^{n-1} - c\}^3}} .$$

By means of Remark 1 on Lemma 1 and Remark 2 on Lemma 2, we obtain easily

$$\begin{aligned} I_n(c) &= \int_{\sigma_1} \frac{(n-z)^{n-3/2} dz}{\sqrt{\{z(n-z)^{n-1} - c\}^3}} - \int_{\sigma_2} \frac{(n-z)^{n-3/2} dz}{\sqrt{\{z(n-z)^{n-1} - c\}^3}} \\ &= \frac{2}{n}\left\{\int_{\sigma_1} \frac{(n-z)^{1/2} dw}{(1-z)w^2} - \int_{\sigma_2} \frac{(n-z)^{1/2} dw}{(1-z)w^2}\right\} . \end{aligned}$$

Now, along the curves $\sigma_1(y)$ and $\sigma_2(y)$ we can set

$$z = \sigma_j(y) = n + r_j(y) e^{i\theta_j(y)} , \quad j = 1, 2 ,$$

and

$$0 < \theta_1(y) < \pi \quad \text{and} \quad \pi < \theta_2(y) < 2\pi \tag{4.1}$$

by Lemma 6. Since we may put $w = i\sqrt{c+y}$, we have for the curve $\sigma_1(y)$

$$\begin{aligned} \frac{(n-z)^{1/2} dw}{(1-z)w^2} &= -\frac{i\sqrt{r_1} e^{i\theta_1/2}}{(n-1) + r_1 e^{i\theta_1}} \cdot \frac{d(i\sqrt{c+y})}{-(c+y)} \\ &= -\frac{\sqrt{r_1} e^{i\theta_1/2}\{(n-1) + r_1 e^{-i\theta_1}\}}{2\{(n-1)^2 + r_1^2 + 2(n-1) r_1 \cos\theta_1\}} \cdot \frac{dy}{\sqrt{(c+y)^3}} , \end{aligned} \tag{4.2}$$

here we have used the expression

$$n - z = r e^{i(\theta + m\pi)} , \quad m = -1 ,$$

in the argument of the beginning of § 3. Hence we obtain

$$\begin{aligned} \frac{(n-z)^{1/2} dw}{(1-z)w^2} = &-\frac{\{(n-1)\sqrt{r_1} + \sqrt{r_1^3}\}\cos\theta_1/2}{2\{(n-1)^2 + r_1^2 + 2(n-1) r_1 \cos\theta_1\}} \cdot \frac{dy}{\sqrt{(c+y)^3}} \\ &+ \{\text{imag. part}\} , \end{aligned} \tag{4.3}$$

in which the real part of the right hand side < 0 by (4.1).

Analogously, along the curve $\sigma_2(y)$ we have

$$\begin{aligned} \frac{(n-z)^{1/2} dw}{(1-z)w^2} = &-\frac{\{(n-1)\sqrt{r_2} + \sqrt{r_2^3}\}\cos\theta_2/2}{2\{(n-1)^2 + r_2^2 + 2(n-1) r_2 \cos\theta_2\}} \cdot \frac{dy}{\sqrt{(c+y)^3}} \\ &+ \{\text{imag. part}\} , \end{aligned}$$

in which the real part of the right hand side > 0 by (4.1).

Since $I_n(c)$ is real valued, we obtain

$$I_n(c) = \frac{2}{n}\left\{\mathscr{R}\int_{\sigma_1}\frac{(n-z)^{1/2}dw}{(1-z)w^2} - \mathscr{R}\int_{\sigma_2}\frac{(n-z)^{1/2}dw}{(1-z)w^2}\right\} < 0$$

by means of the above mentioned facts. q.e.d.

PROPOSITION 2. *When $n = 2$, we have $I_2(c) < 0$ for $0 < c < 1$.*

PROOF. In this case, $\mathscr{F}_2(c)$ is given by $z(2-z) - c = w^2$ and its singular points are $(x_0, 0)$, $(x_1, 0)$, $(1, b)$ and $(1, -b)$ in all, where $b = \sqrt{1-c}$. We have

$$I_2(c) = \int_r \frac{(2-z)^{1/2}dz}{\sqrt{\{z(2-z)-c\}^3}} = \int_r \frac{(2-z)^{1/2}dw}{(1-z)w^2}\ .$$

Since the equation (1.5) has the only real roots x_0 and x_1, we obtain the equality as in case $n > 2$:

$$I_2(c) = \int_{\sigma_1} \frac{(2-z)^{1/2}dz}{\sqrt{\{z(2-z)-c\}^3}} - \int_{\sigma_2} \frac{(2-z)^{1/2}dz}{\sqrt{\{z(2-z)-c\}^3}}\ .$$

On the other hand, since the point $(2, i\sqrt{c})$ is regular on $\mathscr{F}_2(c)$ and for real $x > 2$ we have

$$x(2-x) - c = -\{c + x(x-2)\}\ , \quad x(x-2) > 0\ ,$$

we obtain easily that

$$\theta_1(r) \equiv 0 \quad \text{and} \quad \theta_2(r) = 2\pi \ \text{for}\ 0 < r\ . \tag{4.4}$$

Hence the both curves $\sigma_1(y)$ and $\sigma_2(y)$ $(0 < y)$ coincides with the half line: $2 < x$ of the real axis, including their directions by $y = x(x-2)$.

Now, we shall compute the integrand of $I_2(c)$ along the curves $\sigma_1(y)$ and $\sigma_2(y)$. In this case, we can also use (4.2) and (4.3), then by means of (4.4) we obtain: (i) Along $\sigma_1(y)$

$$\frac{(2-z)^{1/2}dz}{\sqrt{\{z(2-z)-c\}^3}} = -\frac{\sqrt{r_1}}{2(1+r_1)}\cdot\frac{dy}{\sqrt{(c+y)^3}} = -\frac{\sqrt{x-2}}{2(x-1)}\cdot\frac{dy}{\sqrt{(c+y)^3}}\ ;$$

(ii) along $\sigma_2(y)$

$$\frac{(2-z)^{1/2}dz}{\sqrt{\{z(2-z)-c\}^3}} = \frac{\sqrt{r_2}}{2(1+r_2)}\cdot\frac{dy}{\sqrt{(c+y)^3}} = \frac{\sqrt{x-2}}{2(x-1)}\cdot\frac{dy}{\sqrt{(c+y)^3}}\ .$$

Hence we obtain

$$I_2(c) = -\int_0^\infty \frac{\sqrt{x-2}}{(x-1)}\cdot\frac{dy}{\sqrt{(c+y)^3}} = -2\int_2^\infty \frac{\sqrt{x-2}dx}{\sqrt{\{c+x(x-2)\}^3}} < 0\ .$$

q.e.d.

5. **Proof of $I_n(c) < 0$, when $1 < n < 2$.** In this section, we shall prove indirectly the inequality $I_n(c) < 0$ for $0 < c < (n-1)^{n-1}$, when $1 < n < 2$.

Replacing nx^2 and nC by x and C respectively, the period T given by (0.2) can be written as:

$$T_n(x_0) := \int_{x_0}^{x_1} \frac{dx}{\sqrt{x(n-x) - Cx^{1-\alpha}(n-x)^{\alpha}}}, \tag{5.1}$$

where

$$C = x_0^{\alpha}(n-x_0)^{1-\alpha} = x_1^{\alpha}(n-x_1)^{1-\alpha}, \quad \alpha = 1/n \tag{5.2}$$

and

$$0 < x_0 < 1 < x_1 < n. \tag{5.3}$$

We shall denote anew the period given by (1.2) as

$$\Theta_n(c) := \sqrt{nc}\int_{x_0}^{x_1} \frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1} - c\}}}, \tag{5.4}$$

where

$$c = C^n = x_0(n-x_0)^{n-1}. \tag{5.5}$$

As is stated in § 1, we have the equality [6]

$$T_n(x_0) = \Theta_n(c).$$

The following was proved in [9].

LEMMA 7. *$T_n(x_0) = T_m(y_0)$, where $m = n/(n-1)$, $y_0 = m - (m-1)x_1$.*

PROPOSITION 3. *When $1 < n < 2$, we have $I_n(c) < 0$ for $0 < c < (n-1)^{n-1}$.*

PROOF. By the above change of notation and (1.7), (1.8) and Lemma 7, we obtain

$$I_n(c) = -4\sqrt{\frac{c}{n}}\frac{d}{dc}\Theta_n(c) = -4\sqrt{\frac{c}{n}}\frac{d}{dc}T_n(x_0) = -4\sqrt{\frac{c}{n}}\frac{d}{dc}T_m(y_0)$$
$$= -4\sqrt{\frac{c}{n}}\frac{d}{dc}\Theta_m(h),$$

where $h = y_0(m-y_0)^{m-1}$ by (5.5) replaced n, x_0 by m, y_0. Hence, we have

$$I_n(c) = -4\sqrt{\frac{c}{n}} \cdot \frac{dh}{dc} \cdot \frac{d}{dh}\Theta_m(h). \tag{5.6}$$

Since $1 < n < 2$, we see easily that $2 < m$. Hence, by Proposition 1 and

(1.7) we have

$$\frac{d}{dh}\Theta_m(h) > 0 . \tag{5.7}$$

On the other hand, from the equalities:

$$c = x_0(n - x_0)^{n-1} = x_1(n - x_1)^{n-1} , \quad y_0 = m - (m - 1)x_1 ,$$

we get

$$\frac{dh}{dc} = \frac{dh}{dy_0} \cdot \frac{dy_0}{dx_1} \cdot \frac{dx_1}{dc} = -(m-1)\frac{dh}{dy_0}\Big/\frac{dc}{dx_1}$$
$$= -\frac{(m-1)m(1-y_0)(m-y_0)^{m-2}}{n(1-x_1)(n-x_1)^{n-2}} ,$$

hence by $1 < x_1 < n$ and $0 < y_0 < 1$ we obtain

$$dh/dc > 0 . \tag{5.8}$$

Thus (5.6), (5.7) and (5.8) imply $I_n(c) < 0$. q.e.d.

Finally, from (1.7), (1.8) and Propositions 1, 2 and 3, we obtain our main theorem.

THEOREM. *For any real constant $n > 1$, the period T of the non-linear differential equation (E) given by*

$$T = 2\int_{a_0}^{a_1}\left\{1 - x^2 - C\left(\frac{1}{x^2} - 1\right)^{1/n}\right\}^{-1/2} dx ,$$

where $C = (a_0^2)^{1/n}(1 - a_0^2)^{1-1/n} = (a_1^2)^{1/n}(1 - a_1^2)^{1-1/n}$ $(0 < a_0 < \sqrt{1/n} < a_1 < 1)$, is increasing as function of the integral constant C $(0 < C < A = (1/n)^{1/n}(1 - 1/n)^{1-1/n})$.

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