Tôhoku Math. Journ.
28 (1976), 381-387.

THREE NOTES ON REAL ANALYTIC GROUPS

DRAGOMIR Ž. DJOKOVIĆ

(Received April 7, 1975)

Introduction. This paper consists of three independent parts. In the first part the main result is Theorem 3 which gives a thorough description of the closure of an analytic subgroup of a real Lie group. In the second part we determine the Lie algebra of the "commutator group" (A, B) of two subgroups A and B of a real Lie group, where A or B is an analytic subgroup. In the third part we show that the exponential map of a simply connected solvable real Lie group G is an isomorphism of analytic manifolds if G is the topological component of identity of a real algebraic group.

By L we denote the Lie algebra functor. Thus if $f: G_1 \to G_2$ is a morphism of Lie groups then $L(f): L(G_1) \to L(G_2)$ is the differential of f. If G and H are Lie groups and a continuous homomorphism $H \to \mathrm{Aut}(G)$ is given then we denote by $G \rtimes H$ the corresponding semi-direct product.

1. Closure of an analytic subgroup. Let G be a real Lie group and A an analytic subgroup of G. Our objective is to describe the closure $B = \bar{A}$ of A in G. The basic results are due to Malcev [6]. For a concise exposition of Malcev's results we refer to Hochschild's book [5], p. 190-193. The case when $G = GL_n(\boldsymbol{R})$ has been considered recently by M. Goto [4].

For any group G we denote by G' its commutator subgroup, i.e., the subgroup of G generated by all the commutators $(x, y) = xyx^{-1}y^{-1}$ with $x, y \in G$. A *vector group* is by definition a simply connected abelian analytic group. If V is such a group then $\exp: L(V) \to V$ is an isomorphism of analytic groups. This enables us to consider V as a real vector space.

THEOREM 1. *Let G be a real Lie group and A an analytic subgroup of G. We shall use* $^-$ *for the closure operator in G. Let* Ad *be the adjoint representation of $B = \bar{A}$ in $L(B)$. Then*

(i) *If a subspace of $L(B)$ is stable under* $\mathrm{Ad}(A)$ *then it is also stable under* $\mathrm{Ad}(B)$. *Hence, every analytic subgroup of B which is normalized by A is in fact normal in B;*

(ii) $B' = A'$;

(iii) *There exists a vector subgroup* $V \subset A$ *such that* $\bar{V}$ *is a torus,* $B = A\bar{V}$ *and* $A \cap \bar{V} = V$.

PROOF. (i) This follows from the fact that $\mathrm{Ad}(A)$ is dense in $\mathrm{Ad}(B)$.

(ii) We present here the proof from [5] because we need the details of it for the proof of (iii).

Let $f: B^* \to B$ be the universal group covering and identify $L(B^*)$ with $L(B)$ via $L(f)$. Let A^* be the analytic subgroup of B^* such that $L(A^*) = L(A)$. By (i) A^* and $(A^*)'$ are normal analytic subgroups of B^* and consequently they are closed in B^* by [5], Theorem 1.2, p. 135. From now on we shall use $^-$ to denote closure of subsets in both G and B^*. Since $\bar{A} = B$ we must have $\overline{A^*F} = B^*$ where $F = \mathrm{Ker}\, f$. Thus $(B^*)' = (\overline{A^*F})' \subset \overline{(A^*F)'} = \overline{(A^*)'} = (A^*)'$. Hence $(B^*)' = (A^*)'$ and consequently (ii) holds.

(iii) By [5], Theorem 1.2, p. 135 B^*/A^* is simply connected and since $(B^*)' = (A^*)'$ it follows that B^*/A^* is a vector group. By [5], Theorem 1.2, p. 189 there is an abelian analytic subgroup X of B^* containing the center Z of B^*. Since $F \subset Z \subset X$ and F is discrete it follows that F is finitely generated. $F/(A^* \cap F)$ is torsion-free because it is isomorphic to a subgroup of B^*/A^*. Hence we have a direct product decomposition $F = (A^* \cap F) \times F_1$ where F_1 is a free abelian subgroup of F of rank n, say. Let Y be an analytic subgroup of X of minimal dimension such that $Y \supset F_1$. Since F_1 is free abelian it is clear that Y must be a vector group and $\dim Y = n$. We have $\bar{Y} = Y$ because $\bar{F}_1 = F_1$ and Y/F_1 is a compact subgroup of B^*/F_1.

By (ii) A^*Y is a normal analytic subgroup of B^*. By [5], Theorem 1.2, p. 135 $\overline{A^*Y} = A^*Y$. Therefore, from $A^*Y \supset A^*F_1 = A^*F$ and $\overline{A^*F} = B^*$ it follows now that $B^* = A^*Y$. From here we find that $Y/(A^* \cap Y) \cong B^*/A^*$ is a vector group and consequently $A^* \cap Y$ is also a vector group. Let W be a vector space complement of $A^* \cap Y$ in Y. Since $\bar{Y} = Y$ we also have $\overline{A^* \cap Y} = A^* \cap Y$ and $\bar{W} = W$. We have $B^* = A^*Y = A^*W$ and $A^* \cap W = (A^* \cap Y) \cap W = 1$ so that

$$B^* = A^* \rtimes W. \tag{1}$$

Since $A^*F = A^*(A^*F \cap W)$ and $\overline{A^*F} = B^*$ it follows from (1) that

$$\overline{A^*F \cap W} = W. \tag{2}$$

From $F = (A^* \cap F) \times F_1$ and $Y \supset F_1$ it follows that we have a direct product decomposition $F \cap Y = F_1 \times (A^* \cap Y \cap F)$. Since $F \cap Y$ is a discrete sebgroup of Y of rank n we must have $A^* \cap Y \cap F = 1$ and $Y \cap F = F_1$. Therefore $S = f(Y) \cong Y/(Y \cap F) = Y/F_1$ is a torus and $V = f(A^* \cap Y) \cong$

$A^* \cap Y$ is a vector group. It follows from (1) that $B = f(B^*) = Af(W)$ and since $f(W) \subset S \subset B$ we have $B = AS$.

Using that $Y \supset F_1$ and $A^*F = A^*F_1$ we get

$$A^*F \cap YF = (A^*F_1 \cap Y)F = (A^* \cap Y)F . \tag{3}$$

By applying f to (3) we obtain $A \cap S = V$.

By (3) we have $F(A^* \cap Y) = A^*F \cap YF \supset A^*F \cap W$ and then using (2) we get $\overline{F(A^* \cap Y)} \supset W$. Therefore

$$\bar{V} = f(\overline{F(A^* \cap Y)}) \supset Vf(W) = f((A^* \cap Y)W) = f(Y) = S .$$

Since $\bar{V} \subset S$ is clear, we have $\bar{V} = S$ and the proof is completed.

THEOREM 2. *Let G, A and $B = \bar{A}$ be as in Theorem 1. Then there exists a 1-parameter group $\varphi: \boldsymbol{R} \to A$ such that $\overline{\varphi(\boldsymbol{R})} = S$ is a torus and $B = AS$.*

PROOF. This follows from Theorem 1 (iii) and the next Lemma.

LEMMA. *Let $S = \boldsymbol{R}^n/\boldsymbol{Z}^n$ be a torus and let V be a vector subgroup of S such that $\bar{V} = S$. Then V contains a generator of S, i.e., an element such that the cyclic group generated by it is dense in S.*

PROOF. Let $U = L(V) \subset \boldsymbol{R}^n$ where as usual we identify $L(S)$ with $\boldsymbol{R}^n$. Then by hypothesis $U + \boldsymbol{Z}^n$ is dense in $\boldsymbol{R}^n$ and we have to show that for suitable $a \in U$ the group $D = \boldsymbol{Z}a + \boldsymbol{Z}^n$ is dense in $\boldsymbol{R}^n$.

Let $a_1, \cdots, a_m$ be a basis of U and let $P \subset \boldsymbol{R}$ be the $\boldsymbol{Q}$-vector space generated by the coordinates of all a_i, $1 \leqq i \leqq m$. Since $\boldsymbol{Q}(P)$ is a finitely generated extension of $\boldsymbol{Q}$, it is clear that we can choose $\alpha_1, \cdots, \alpha_m$ in $\boldsymbol{R}$ so that the sum of $\boldsymbol{Q}$-vector subspaces $\boldsymbol{Q}, \alpha_1P, \alpha_2P, \cdots, \alpha_mP$ of $\boldsymbol{R}$ is direct. We claim that we can take $a = \alpha_1a_1 + \cdots + \alpha_ma_m$.

We shall use the technique of associated subgroups as presented in Bourbaki [1]. For any subgroup $X \subset \boldsymbol{R}^n$ one defines its associated subgroup $X^* \subset \boldsymbol{R}^n$ as follows: $y \in X^*$ if and only if the dot product $\langle x, y\rangle \in \boldsymbol{Z}$ for all $x \in X$. We need only to show that $D^* = 0$. Since $D^* = (\boldsymbol{Z}a)^* \cap (\boldsymbol{Z}^n)^* = (\boldsymbol{Z}a)^* \cap \boldsymbol{Z}^n$ we see that $y \in D^*$ if and only if $y \in \boldsymbol{Z}^n$ and $\langle a, y\rangle = 0$. By our choice of $\alpha_1, \cdots, \alpha_m$ this implies that $\langle a_i, y\rangle = 0$ for $1 \leqq i \leqq m$. Thus $D^* \subset (U + \boldsymbol{Z}^n)^*$ and since $U + \boldsymbol{Z}^n$ is dense in $\boldsymbol{R}^n$ we have $(U + \boldsymbol{Z}^n)^* = 0$. Hence $D^* = 0$ and the proof is completed.

Recall the following criterion of Malcev for closedness of an analytic subgroup A of G: We have $\bar{A} = A$ if and only if for every 1-dimensional analytic subgroup P of A its closure in G is contained in A. Our Theorem 2 gives an explanation for the validity of that criterion.

Now we can state our main result.

THEOREM 3. *Let G, A and $B = \bar{A}$ be as in Theorem* 1. *Then* $B \cong (A \ltimes S)/N$ *as an analytic group, where*

(i) *S is a torus acting on A via a morphism $\varphi\colon S \to \mathrm{Aut}(A)$;*

(ii) *There is a closed vector subgroup V of A and an injective morphism $g\colon V \to S$ such that $g(V)$ is dense in S and N is the graph of g;*

(iii) *N is contained in the center of $A \ltimes S$ or, equivalently, $\varphi(s)(v) = v$ and $\varphi(g(v))(a) = v^{-1}av$ for all $s \in S$, $v \in V$, $a \in A$.*

Conversely, if A is an analytic group and φ, g, V satisfy (i), (ii) *and* (iii) *then the canonical image of A is dense in $(A \ltimes S)/N$.*

PROOF. Let V be as in Theorem 1 (iii) and let $S = \bar{V}$ (as in the proof of that theorem). Since S is a torus and $V = A \cap S$ it is clear that V is closed in A. We let S act on A by conjugation, i.e., $\varphi(s)(a) = sas^{-1}$ for $a \in A$, $s \in S$. Then we have the canonical morphism $A \ltimes S \to AS = B$ whose kernel N consists of all elements (v, v^{-1}) for $v \in V$. Hence if we define $g\colon V \to S$ by $g(v) = v^{-1}$ then N is the graph of g and $g(V)$ is dense in S. Since in $A \ltimes S$ we have

$$(a, 1)(v, v^{-1}) = (av, v^{-1}) = (vv^{-1}av, v^{-1}) = (v, v^{-1})(a, 1)\,,$$
$$(1, s)(v, v^{-1}) = (svs^{-1}, sv^{-1}) = (v, v^{-1}s) = (v, v^{-1})(1, s)$$

for all $a \in A$, $v \in V$, $s \in S$ we conclude that N is contained in the center of $A \ltimes S$. Thus, all three conditions have been verified.

For the converse it suffices to note that the closure of the canonical image of V under the composite map $V \to A \to (A \ltimes S)/N$ is the canonical image of S in $(A \ltimes S)/N$.

2. **The Lie algebra of** (A, B). Let G be a group and A, B two subgroups of G. By (A, B) we denote the subgroup of G generated by all commutators $(a, b) = aba^{-1}b^{-1}$ with $a \in A$ and $b \in B$. Since

$$\begin{aligned} a_1(a, b)a_1^{-1} &= a_1aba^{-1}b^{-1}a_1^{-1} \\ &= (a_1a)b(a_1a)^{-1}b^{-1}ba_1b^{-1}a_1^{-1} \\ &= (a_1a, b)(a_1, b)^{-1} \end{aligned}$$

is valid for all $a, a_1 \in A$ and $b \in B$ it follows that A normalizes (A, B) and similarly B normalizes (A, B).

From now on G will be a real Lie group, A an analytic subgroup of G and B an arbitrary subgroup of G. Let Ad be the adjoint representation of G in $L(G)$ and put

$$P = \sum_{b \in B} (1 - \mathrm{Ad}\,(b))(L(A))\,.$$

THEOREM 4. *Let G, A, B, P be as just defined above. Then (A, B) is an analytic subgroup of G and $L((A, B))$ coincides with the smallest subalgebra M of $L(G)$ such that $M \supset P$ and $[L(A), M] \subset M$.*

PROOF. Since (A, B) is clearly arcwise connected it follows from [3] that it is an analytic subgroup of G.

If $\alpha \in L(A)$, $t \in \boldsymbol{R}$, $b \in B$ then

$$\exp(t\alpha)b\exp(-t\alpha)b^{-1} = \exp(t\alpha)\exp(-t\operatorname{Ad}(b)(\alpha)) \in (A, B)\ .$$

By taking the derivative at $t = 0$ we get

$$\alpha - \operatorname{Ad}(b)(\alpha) \in L((A, B))\ ,$$

which proves that $P \subset L((A, B))$. Since A normalizes (A, B) we must have $[L(A), L((A, B))] \subset L((A, B))$. By definition of M we must have $M \subset L((A, B))$.

Now let C be the analytic subgroup of G such that $L(C) = M$. For fixed $b \in B$ and sufficiently small $\alpha \in L(A)$ we have

$$\begin{aligned}\exp(\alpha)b\exp(-\alpha)b^{-1} &= \exp(\alpha)\exp(-\operatorname{Ad}(b)(\alpha)) \\ &= \exp(H(\alpha, -\operatorname{Ad}(b)(\alpha)))\end{aligned}$$

where $H(\alpha, \beta)$ is the Campbell-Hausdorff series. If $\beta = \alpha - \operatorname{Ad}(b)(\alpha)$ then $\beta \in P$ so that

$$H(\alpha, -\operatorname{Ad}(b)(\alpha)) = H(\alpha, -\alpha + \beta) \in M$$

because every homogeneous part of this series is in M. This proves that $(\exp(\alpha), b) \in C$ for sufficiently small $\alpha \in L(A)$. If $a_1, a_2 \in A$ are such that $(a_1, b) \in C$ and $(a_2, b) \in C$ then since A normalizes C we have also

$$\begin{aligned}(a_1a_2, b) &= a_1a_2ba_2^{-1}a_1^{-1}b^{-1} \\ &= a_1(a_2, b)ba_1^{-1}b^{-1} \\ &= a_1(a_2, b)a_1^{-1}(a_1, b) \in C\ .\end{aligned}$$

Since A is connected these facts clearly imply that $(a, b) \in C$ for all $a \in A$. Since $b \in B$ was chosen arbitrarily we conclude that $(A, B) \subset C$. Therefore we have $L((A, B)) \subset L(C) = M$.

This concludes the proof.

THEOREM 5. *Let G be a real Lie group and A, B analytic subgroups of G. Then $L((A, B))$ coincides with the smallest subspace N of $L(G)$ such that $N \supset [L(A), L(B)]$, $[L(A), N] \subset N$ and $[L(B), N] \subset N$.*

PROOF. By Theorem 4 we have $\alpha - \operatorname{Ad}(b)(\alpha) \in L((A, B))$ for all $\alpha \in L(A)$ and $b \in B$. By taking $b = \exp(t\beta)$ where $t \in \boldsymbol{R}$ and $\beta \in L(B)$ we obtain

$$\alpha - \mathrm{Ad}\,(\exp(t\beta))(\alpha) = \alpha - \mathrm{Exp}\,(t\,\mathrm{ad}\,(\beta))(\alpha) \in L((A, B))\,.$$

By taking derivative at $t = 0$ we get $[\alpha, \beta] \in L((A, B))$, i.e., $[L(A), L(B)] \subset L((A, B))$. Since A and B normalize (A, B) we have that $[L(A), L((A, B))] \subset L((A, B))$ and $[L(B), L((A, B))] \subset L((A, B))$. Hence, by definition of N we have $N \subset L((A, B))$.

On the other hand it is clear that N is a subalgebra of $L(G)$. Let C be the analytic subgroup of G such that $L(C) = N$. Then by Bourbaki [2], Chap. III, § 9, Proposition 4 we have $(A, B) \subset C$ so that $L((A, B)) \subset N$.

This completes the proof of $L((A, B)) = N$.

3. **On the exponential map.** It is well-known that for simply connected nilpotent real analytic group G the exponential map $\exp: L(G) \to G$ is an isomorphism of analytic manifolds [5], Theorem 2.1, p. 136. When G is simply connected and solvable then it is known that the following are equivalent:

(i) exp is an isomorphism of analytic manifolds;

(ii) exp is injective;

(iii) exp is surjective.

This list of equivalent statements can be extended by another five statements. We refer to [7] or [2], p. 278–279 for more details.

THEOREM 6. *Let G be a simply connected solvable real analytic group such that G is the topological identity component of a real algebraic group. Then* exp: $L(G) \to G$ *is an isomorphism of analytic manifolds.*

PROOF. Since real algebraic groups have only finitely many topological connected components the theorem follows from the next lemma and the equivalence of statements (i), (ii) and (iii) above. Note also that in an algebraic group the centralizer of an element is also algebraic.

LEMMA. *Let G be a solvable, simply connected, real analytic group such that* exp *is not surjective. For $a \in G$ let $Z(a)$ be the centralizer of a in G and $Z_1(a)$ the identity component of $Z(a)$. If $a \notin \exp(L(G))$ then the image of a in $Z(a)/Z_1(a)$ has infinite order.*

PROOF. Assume that $a^n \in Z_1(a)$ for some $n \geqq 1$. Then a^n belongs to the center of $Z_1(a)$ and by [5], Theorem 1.2, p. 189 there exists $\alpha \in L(Z_1(a))$ such that $a^n = \exp(\alpha)$. If $b = \exp(-\alpha/n)$ then $(ab)^n = 1$. Since G is solvable and simply connected it has no nontrivial compact subgroups and we get $ab = 1$. Thus $a \in \exp(L(G))$, contradicting our hypothesis.

The proof is completed.

Since simply connected nilpotent groups are algebraic, Theorem 6 generalizes the result mentioned in the beginning of this part of the paper.

The condition stated in Theorem 6 for exp: $L(G) \to G$ to be an isomorphism of analytic manifolds (assuming that G is solvable and simply connected) is not necessary. An example is the group G such that $L(G)$ has a basis x, y, z such that $[x, y] = 0$, $[z, x] = x + y$, $[z, y] = y$. The Lie algebra $L(G)$ is not algebraic but exp: $L(G) \to G$ is bijective.

ACKNOWLEDGEMENT. I am grateful to Prof. G. Hochschild for his advice and to Canada Council for financial support.

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DEPARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO, CANADA N2L 3G1