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A SATURATION RESULT FOR COMBINATIONS OF BERNSTEIN POLYNOMIALS

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1. Introduction. The saturation class for approximation by the Bernstein polynomials

$$B_n(f, x) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{1.1}$$

was investigated in [1], [4] and [6]. In these papers it is shown that the optimal rate of convergence of Bernstein polynomials $B_n(f, x)$ to $f(x)$ is $O(1/n)$ as $n \to \infty$. It was shown by Butzer [3] (see also [5]) that for an approximation process $B_n(f, k, x)$, constructed by certain combinations of Bernstein polynomials, the rate of convergence to $f(x)$ may be much faster than $O(1/n)$.

The approximation processes treated, $B_n(f, k, t)$, are defined inductively by:

$$\begin{aligned} B_n(f, k, t) &\equiv (2^k-1)^{-1}[2^k B_{2n}(f, k-1, t) - B_n(f, k-1, t)] , \\ B_n(f, 0, t) &\equiv B_n(f, t) . \end{aligned} \tag{1.2}$$

Similarly, using Szasz operators, one can define

$$S_\tau(f, k, t) \equiv (2^k-1)^{-1}[2^k S_{\tau/2}(f, k-1, t) - S_\tau(f, k-1, t)] \tag{1.3}$$

where $S_\tau(f, 0, t) \equiv S_\tau(f, t)$ is the Szasz operator

$$S_\tau(f, 0, t) \equiv S_\tau(f, t) \equiv e^{-t/\tau} \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{t}{\tau}\right)^k f(\tau k) . \tag{1.4}$$

The Szasz operator $S_\tau(f, t)$ was introduced by Szasz [9]. A local saturation theorem for $S_\tau(f, t)$ was proved by Suzuki [8].

A saturation theorem for a given approximation process determines its optimal rate of convergence and the class of functions for which that rate is achieved. The optimal rate here would mean that only for a fixed finite dimensional space of functions can we improve on that rate.

In this paper local saturation theorems for $B_n(f, k, t)$ and $S_\tau(f, k, t)$ will be derived. The following special case of our main theorem is representative for the type of results achieved:

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THEOREM 1. *Let* $f(x) \in C[0, 1]$ *and* $0<a<b<1$, *then* $\|n^2(2B_{2n}(f, x) - B_n(f, x) - f(x))\|_{C[\alpha,\beta]} \leqq M_1(\alpha, \beta)$ *for all* $[\alpha, \beta] \subset (a, b)$ *if and only if* $f^{(3)}(x) \in A.C.[\alpha, \beta]$ *and* $f^{(4)} \in L_\infty[\alpha, \beta]$ *for all* $[\alpha, \beta] \subset (a, b)$; $\|n^2(2B_{2n}(f, x) - B_n(f, x) - f(x))\|_{C[\alpha,\beta]} = o(1)$ *for all* $n \to \infty$ $[\alpha, \beta] \subset (a, b)$ *if and only if* $f \in C^4(a, b)$ *and* $(1 - 2x)f^{(3)}(x) - 3(x(1 - x))f^{(4)}(x) = 0$ *in* (a, b).

2. The main result. The saturation result for combinations of Bernstein polynomials and Szasz operators will be given in Theorems 2 and 3 respectively.

THEOREM 2. *For* $f \in C[0, 1]$, $0 < a < a_1 < b_1 < b < 1$ *and* $\{n_i\}$ *contains* $n_0 2^m$

(2.1) $$I(f; n_i, k, a, b) \equiv n_i^{k+1} \|B_{n_i}(f, k, \cdot) - f(\cdot)\|_{C[a,b]} \leqq M$$

implies

(2.2) $$f^{(2k+1)}(x) \in A.C.(a, b) \text{ and } f^{(2k+2)}(x) \in L_\infty[a, b] ,$$

and (2.2) *implies* $I(f; n, k, a_1 b_1) \leqq M_1$;

(2.3) $$I(f; n; k, a, b) = o(1) \quad n \to \infty \quad \text{implies} \sum^{2k+2} q_i(t) f^{(i)}(t) = 0$$

for $t \in (a, b)$, *and* $\sum^{2k} q_i(t) f^{(i)}(t)$ *in* (a, b) *implies* $I(f; n, k, a_1, b_1) = o(1)$ $n \to \infty$, *where* $q_i(t)$ *are fixed polynomials that depend on* k.

THEOREM 3. *For* $f \in C[0, \infty)$, $|f(t)| \leqq Ke^{Lt}$ *for some* K *and* L $0 < a < a_1 < b_1 < b < \infty$ *and* $\tau_m = \tau_0 2^{-m}$ *we have:*

(2.4) $$J(f; \tau_i, k, a, b) \equiv \tau_i^{-k-1} \|S_{\tau_i}(f, k, \cdot) - f(\cdot)\|_{C[a,b]} \leqq M$$

implies (2.2) *and* (2.2) *implies* $J(f; \tau, k, a_1, b_1) \leqq M_1$;

(2.5) $$J(f; \tau_i, k, a, b) = o(1) \quad \tau_i \to 0+ \quad \text{implies} \sum^{2k+2} Q_i(t) f^{(i)}(t) = 0$$

for $t \in (a, b)$ *where* $Q_i(t)$ *are fixed polynomials that depend on* k *and* i *and* $\sum^{2k+2} Q_i(t) f^{(i)}(t) = 0$ *in* (a, b) *implies* $J(f; \tau, k, a_1, b_1) = o(1)$.

We shall prove Theorem 2 and discuss afterwards only the points of the proof of Theorem 3 in which it differs from that of Theorem 2. The gap between the necessary and the sufficient conditions, that is the fact that the conditions are not on the same interval, though not a big gap is vital since even in the case of Bernstein polynomials approximation, the result will be wrong if $a_1 = a$ and $b_1 = b$.

One can write Theorems 2 and 3 in such a way that the conditions are necessary and sufficient as is common for saturation results. For example, Theorem 2 could be written as:

THEOREM 2*. *For* $f \in C[0, 1]$, $n_i = n_0 2^i$ *and* $I(f, n_i, k, \alpha, \beta) \equiv n_i^{k+1} \| B_{n_i}(f, k, \cdot) - f(\cdot) \|_{C[\alpha, \beta]}$, *we have for* $0 < a < b < 1$

1. $I(f, n_i, k, \alpha, \beta) \leqq M(\alpha, \beta)$ *for all* $[\alpha, \beta] \subset (a, b)$ *if and only if* $f^{(2k+1)}(x) \in A.C.(\alpha, \beta)$ *and* $f^{(2k+2)}(x) \in L_\infty(\alpha, \beta)$ *for all* $[\alpha, \beta] \subset (a, b)$.

2. $I(f, n_i, k, \alpha, \beta) = o(1)$ *for all* $[\alpha, \beta] \subset (a, b)$ *if and only if* $f \in C^{2k+2}(a, b)$ *and* $\sum^{2k+2} q_i(t) f^{(i)}(t) = 0$ *in* (a, b).

3. Main steps of the proof. We shall outline the proof of Theorem 2 in this section and actually prove it pending proof of Lemmas 3.1 and 3.4 which we shall prove in Sections 4, 5 and 6. We shall discuss in Section 7 the points in which the proof of Theorem 3 differes from that of Theorem 2.

I. We first observe, using the recursion relation (1.2), that $B_n(f, k, t) \rightarrow f(t)$ for all k and therefore is an approximation process. The following lemma will establish the equivalence of the conditions $n^{k+1} \| B_n(f, k, \cdot) - f(\cdot) \|_{C[a,b]} = O(1)$ (or $o(1)$) and $n^{k+1} \| B_{2n}(f, k, \cdot) - B_n(f, k, \cdot) \|_{C[a,b]} = O(1)$ (or $o(1)$).

LEMMA 3.1. *If* $f \in C[0, 1]$ *and* $n_i = 2^i n_0$, *then* $n_i^{k+1} \| B_{n_i}(f, k, \cdot) - f(\cdot) \|_{C[a,b]} \leqq M$ *for all* i *implies* $n_i^{k+1} \| B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot) \|_{C[a,b]} \leqq 2M$ *for all* i *and the latter implies* $n_i^{k+1} \| B_{n_i}(f, k, \cdot) - f(\cdot) \|_{C[a,b]} \leqq 4M$, *for any* $[a, b]$ *satisfying* $0 \leqq a < b \leqq 1$.

Therefore it is enough to prove our theorem for $\| B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot) \|_{C[a,b]}$ in place of $I(f; n, k, a, b)$.

II. Using Lemma 3.1, we have $n_i^{k+1} \| B_{n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot) \|_{C[a,b]} \leqq M$. For any $g \in C_0^\infty$ such that $\operatorname{supp} g \subset (a, b)$, we have $g \in L_1[a, b]$ and since $L_1[a, b]^* = L_\infty[a, b]$, there exists using Alaoglu's theorem a function h, $h \in L_\infty[a, b]$, and a subsequence $\{n_{i_\rho}\}$ of $\{n_i\}$ such that for any g as above, we have

$$\langle n_{i_\rho}^{k+1}(B_{2n_{i_\rho}}(f, k, \cdot) - B_{n_{i_\rho}}(f, k, \cdot)), g(\cdot) \rangle \rightarrow \langle h(\cdot), g(\cdot) \rangle . \tag{3.1}$$

III. For $f \in C^{2k+2}$ we can calculate $\langle n^{k+1}(B_{2n}(f, k, \cdot) - B_n(f, k, \cdot)), g(\cdot) \rangle$ directly using the following asymptotic relation, (see also Butzer [3]) for $n^{k+1}(B_{2n}(f, k; t) - B_n(f, k, t))$.

LEMMA 3.2. *For* $f \in C[0, 1] \cap C^{2k+2}[a, b]$ *and* $t \in (a, b)$,

$$\begin{aligned} n^{k+1}\{B_{2n}(f, k, t) - B_n(f, k, t)\} &= \sum_{j=2}^{2k+2} C_{k,j} Q(k, j; t) f^{(j)}(t) + o(1) \\ &\equiv P_{2k+2}(D) f + o(1) \end{aligned} \tag{3.2}$$

where $Q(k, j; t) \equiv q_j(t)$ *are polynomials in* t *and in particular*

$$(3.3)\qquad \begin{aligned} Q(k, 2k+2, t) &= C_1(t(1-t))^{k+1} \\ Q(k, 2k+1, t) &= C_2(t(1-t))^k \cdot (1-2t) . \end{aligned}$$

Formula (3.2) follows Butzer's paper [3] but our method here yields the proof for other approximation processes (see the proof of Theorem 3).

IV. Using Lemma 3.2 we have for $f \in C^{2k+2}[a, b]$

$$(3.4)\qquad \begin{aligned} &\lim_{n_i\to\infty} \langle n_i^{k+1}(B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)), g(\cdot)\rangle \\ &= \langle P_{2k+2}(D)f(\cdot), g(\cdot)\rangle = \langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle \end{aligned}$$

and the last expression is a continuous linear functional on $f(t)$. In order to compare (3.1) and (3.4) one has to prove (3.4) for all $f \in C[0, 1]$ satisfying

$$(3.5)\qquad \| n_i^{k+1}(B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot))\|_{C[a,b]} \leqq M .$$

V. One first observes the following implications:

LEMMA 3.3. *For* $f \in C[0, 1]$ $\delta > 0$ *and* $n_i = n_0 2^i$, $n_i^{k+\delta} \| B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)\|_{C[a,b]} \leqq M$ *for all* n_i *implies* $n_i^k \| B_{2n_i}(f, k-1, \cdot) - B_{n_i}(f, k-1, \cdot)\|_{C[a,b]} \leqq M_1$.

Using induction and since our theorem is well-known for $k = 0$, it is clear that (3.5) implies $f^{(2k)}(x) \in L_\infty[a, b]$ and therefore $f^{(2k)}(x) \in L_1[a, b]$. Finally we will prove the crucial lemma.

LEMMA 3.4. *For* $f \in C[0, 1]$ *and* $f^{(2k)}(x) \in L_1[a, b]$,

$$(3.6)\qquad \begin{aligned} &| n^{k+1}\langle (B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)), g(\cdot)\rangle | \\ &\leqq K(\| f\|_{C[0,1]} + \| f^{(2k)}\|_{L_1[a,b]}) \end{aligned}$$

where K *depends on* g (*and its derivatives*).

Therefore, if f satisfies (3.5), there exists a sequence $f_l \in C^{2k+2}$ such that $f_l \to f$ in the norm $\| f\|_{C[0,1]} + \| f^{(2k)}\|_{L_1[a,b]}$ and using (3.6) we have

$$\begin{aligned} &\lim_{n_i\to\infty} \langle n_i^{k+1}(B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)), g(\cdot)\rangle \\ &= \lim_{l\to\infty} \lim_{n_i\to\infty} \langle n_i^{k+1}(B_{2n_i}(f_l, k, \cdot) - B_{n_i}(f_l, k, \cdot)), g(\cdot)\rangle = I . \end{aligned}$$

But since $f_l \in C^{2k+2}$, we have, using (3.4) for f_l,

$$I = \lim_{l\to\infty} \langle f_l(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle = \langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle$$

and therefore (3.4) is valid for any f satisfying (3.5).

VI. Combining (3.1) and (3.4), we obtain

$$\langle h(\cdot), g(\cdot)\rangle = \langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle .$$

This implies $P_{2k+2}(D)f(t) = \sum^{2k+2} q_i(t)f^{(i)}(t) = h(t)$ in the distribution sense. Since the above equation is solvable for $h \in L_\infty$, we obtain $f^{(2k+2)}(t) \in L_\infty(a, b)$. (Actually we need to treat just a second order differential equation, since we know $f^{(2k)}(t) \in L_\infty(a, b)$ by the induction step.)

VII. The "small o" part is similar with only one difference: instead of $\langle h(\cdot), g(\cdot)\rangle = \langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle$, we have $\langle f(\cdot), P^*_{2k+2}(D)g(\cdot)\rangle = 0$.

VIII. The second implication, that is $f^{(2k+2)}(x) \in L_\infty[a, b]$ and $f^{(2k+1)}(x) \in A.C.[a, b]$, implies $I(f, n_i, k, a_1, b_1) \leqq M$ and $P_{2k+2}(D)f(x) = 0$ in $[a, b]$ implies $I(f, n, k, a_1, b_1) = o(1)$ $n \to \infty$ is computational and will be omitted.

Lemmas 3.1–3.4 will be proved in the following sections.

4. **Proof of Lemmas 3.1 and 3.3.** Wherever a norm is written in this section, it is the $C[a, b]$ norm.

PROOF OF LEMMA 3.1. The first implication follows

$$\begin{aligned} n_i^{k+1}\|B_{2n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)\| &\leqq (2^{k+1})^{-1}(2n_i)^{k+1}\|B_{2n_i}(f, k, \cdot) - f(\cdot)\| \\ &+ n_i^{-k-1}\|B_{n_i}(f, k, \cdot) - f(\cdot)\| \leqq 2M . \end{aligned}$$

The second implication follows

$$\begin{aligned} n_i^{k+1}\|f(\cdot) - B_{n_i}(f, k, \cdot)\| &= \lim_{m\to\infty} n_i^{k+1}\|B_{2^m n_i}(f, k, \cdot) - B_{n_i}(f, k, \cdot)\| \\ &\leqq \lim_{m\to\infty}\sum_{l=1}^{m} 2^{-(l-1)(k+1)}\{(2^{l-1}n_i)^{k+1}\|B_{2^l n_i}(f, k, \cdot) - B_{2^{l-1}n_i}(f, k, \cdot)\|\} \\ &\leqq 4M . \end{aligned}$$

PROOF OF LEMMA 3.3. Using the recursion relation (1.2) and $n_i = n_0 2^i$, we have

$$\begin{aligned} I(N) &\equiv (2^k - 1)\sum_{i=0}^{N} 2^{ki}(B_{2n_i}(f, k, t) - B_{n_i}(f, k, t)) \\ &= \sum_{i=0}^{N} 2^{ki}(2^k B_{4n_i}(f, k^{-1}, t) - B_{2n_i}(f, k^{-1}, t)) - \sum_{i=0}^{N} 2^{ki}(B_{2n_i}(f, k-1, t) \\ &\quad - B_{n_i}(f, k-1, t)) \\ &= 2^{k(N+1)}[B_{2n_{N+1}}(f, k-1, t) - B_{n_{N+1}}(f, k-1, t)] \\ &\quad - 2^k[B_{2n_0}(f, k-1, t) - B_{n_0}(f, k-1, t)] . \end{aligned}$$

Using the assumption of our lemma,

$$\|I(N)\| \leqq (2^k - 1)\sum_{i=0}^{N} M(2^i n_0)^{-k-\delta} \leqq 2(2^k - 1)Mn_0^{-k-\delta} ,$$

and therefore

$$\left\| 2^{k(N+1)}\left[B_{2n_{N+1}}(f, k-1, \cdot) - B_{n_{N+1}}(f, k-1, \cdot)\right]\right\| \leqq 2(2^k - 1)n_0^{-k-\delta} + K = K_1$$

which concludes the proof.

5. An asymptotic relation. We shall prove here results that we shall utilize also in later sections.

Define

$$W(n, t, u) \equiv \sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} \delta\left(u - \frac{m}{n}\right), \tag{5.1}$$

$$\langle W(n, t, u), f \rangle = B_n(f, t) .$$

LEMMA 5.1. *$\frac{\partial}{\partial t} W(n, t, u) = \left(\frac{n}{p(t)}\right) W(n, t, u)(u-t)$ where $p(t) = t(1-t)$.*

PROOF.

$$\begin{aligned}
\frac{\partial}{\partial t} W(n, t, u) &= \sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} \delta\left(u - \frac{m}{n}\right)\frac{m}{t} \\
&\quad - \sum_{m=0}^{n} \binom{n}{m} t^m (1-t)^{n-m} \delta\left(n - \frac{m}{n}\right)\frac{n-m}{1-t} \\
&= \frac{n}{t} W(n, t, u)u - \frac{n}{1-t} W(n, t, u) + \frac{n}{1-t} W(n, t, u)u \\
&= \left(\frac{n}{t(1-t)}\right) W(n, t, u)(u - t) .
\end{aligned}$$

LEMMA 5.2. *Let $A_m(n, t)$ be given by*

$$A_m(n, t) = n^m \int W(n, t, u)(u - t)^m du , \tag{5.2}$$

then:

a) *$A_{m+1}(n, t) = p(t)nA_{m-1}(n, t) + p(t)(\partial/\partial t)A_m(n, t)$;*

b) *$A_m(n, t)$ is a polynomial in t and n;*

c) *The degree of $A_m(n, t)$ in n is $[m/2]$;*

d) *The coefficient of n^m in the polynomial $A_{2m}(n, t)$ is $C_1 p(t)^m$ and in the polynomial A_{2m-1} is $C_2 p(t)^m p'(t)$.*

PROOF. We first observe that the recursion formula (a) follows Lemma 5.1. The remainder of the conclusion is derived from (a) by computation.

Using the recursion relation defining $B_n(f, k, t)$ (1.2), we obtain

$$B_n(f, k, t) = \sum_{j=0}^{k} \alpha(j, k) B_{2^j n}(f, t) \tag{5.3}$$

and

$$B_{2n}(f, k, t) - B_n(f, k, t) = \sum_{j=0}^{k+1} C(j, k) B_{2^j n}(f, t) . \tag{5.4}$$

Obviously, $\alpha(j, k)$ and $C(j, k)$ are constants that depend only on (1.2), and satisfy $\sum_{j=0}^{k} \alpha(j, k) = 1$.

PROOF OF LEMMA 3.2. We have, using Taylor's formula,

$$\begin{aligned} &n^{k+1}[B_{2n}(f, k, t) - B_n(f, k, t)] \\ &\quad = n^{k+1} \sum_{j=0}^{k+1} C(j, k) \int W(n2^j, t, u) \Big[\sum_{m=0}^{2k+2} \frac{f^{(m)}(t)}{m} (u - t)^m \\ &\qquad + \varepsilon(u, t)(u - t)^{2k+2} \Big] du . \end{aligned}$$

The rest of the proof follows Lemma 5.2 and some computations.

6. Proof of the crucial step. In this section we shall prove Lemma 3.4 but for the proof we shall need a preliminary lemma.

LEMMA 6.1. *Let $C(j, k)$ be defined by* (5.4), *then*

$$\sum_{j=0}^{k+1} C(j, k) 2^{-mj} = 0 \qquad \text{for } m = 0, 1, \cdots, k . \tag{6.1}$$

PROOF. It is easily seen that $C(0, 0) = -1$, $C(1, 0) = 1$, $C(0, 1) = 1$, $C(1, 1) = -3$ and $C(2, 1) = 2$ and therefore (6.1) is valid for $k = 0$ and $k = 1$. Proceeding by induction,

$$\begin{aligned} &(2^k - 1)\{B_{2n}(f, k, t) - B_n(f, k, t)\} \\ &\quad = (2^k - 1) \sum_{j=0}^{k+1} C(j, k) B_{2^j n}(f, t) \\ &\quad = 2^k B_{2n}(f, k - 1, t) - B_n(f, k - 1, t) \\ &\quad = 2^k \sum_{j=0}^{k} C(j, k - 1) B_{2^{j+1} n}(f, t) - \sum_{j=0}^{k} C(j, k - 1) B_{2^j n}(f, t) , \end{aligned}$$

and therefore $(2^k - 1)C(j, k) = 2^k C(j - 1, k - 1) - C(j, k - 1)$, $j \geqq 1$ which yields

$$(2^k - 1) \sum_{j=0}^{k+1} C(j, k) 2^{-jm} = 2^k \sum_{j=1}^{k+1} C(j - 1, k - 1) 2^{-jm} - \sum_{j=0}^{k} C(j, k - 1) 2^{-jm}$$

which is equal to zero by the induction hypothesis for $m = 1, \cdots k - 1$. For $m = k$, we have

$$2^k \sum_{j=1}^{k+1} C(j-1, k-1)2^{-jk} - \sum_{j=0}^{k} C(j, k-1)2^{-jk} = 0 .$$

PROOF OF LEMMA 3.4.

$$\begin{aligned} n^{k+1}\langle B_{2n}(f, k, \cdot) &- B_n(f, k, \cdot), g(\cdot)\rangle \\ &= n^{k+1}\int_0^1\int_0^1 \sum_{j=0}^{k+1} C(j, k)W(2^j n, t, u)f(u)g(t)dudt \\ &= n^{k+1}\int_a^b\int_0^1 \sum_{j=0}^{k+1} C(j, k) \sum_{l=0}^{2k+2} W(2^j n, t, u)f(u)\frac{1}{l!}g^{(l)}(u)(t-u)^l dtdu \\ &+ n^{k+1}\int_0^1\int_0^1 \sum_{j=0}^{k+1} C(j, k)W(2^j n, t, u)f(u)\varepsilon(t, u)(t-u)^{2k+2}dtdu \\ &\equiv I_1 + I_2 \equiv \sum_{l=0}^{2k+1} J_l + I_2 . \end{aligned}$$

We estimate I_2 first,

$$\begin{aligned} ||I_2|| &\leqq \max|\varepsilon(t, u)|\cdot \sum_{j=1}^{k+1}|C(j, k)|\max_j n^{k+1}\int_0^1\int_0^1 W(2^j n, t, j)(t-u)^{2k+2}dtdu \\ &\leqq K\max|\varepsilon(t, u)| \leqq K_2 . \end{aligned}$$

To estimate the J_l, we evaluate the following typical expression

$$I_\gamma = n^{k+1}\int_a^b\int_0^1 \sum_{j=0}^{k+1} C(j, k)W(2^j n, t, u)[f(u)u^r g^{(s)}(u)]t^i dtdu .$$

We write $\phi_i(u) \equiv f(u)u^r g^{(s)}(u)$, $n_j \equiv 2^j n$ and recall $i \leqq 2k+2$. Since

$$\int_0^1 W(n, t, u)t^i dt = \sum_{m=0}^{n} \frac{(m+1)\cdots(m+i)}{(n+1)\cdots(n+1+i)}\delta\left(u - \frac{m}{n}\right) ,$$

$$\begin{aligned} I_\gamma &= n^{k+1}\sum_{j=0}^{k+1} C(j, k) \sum_{(m/n_j)\in(a,b)} \phi_i\left(\frac{m}{n_j}\right)\frac{(m+1)\cdots(m+i)}{(n_j+1)\cdots(n_j+1+i)} \\ &= n^{k+1}\sum_{j=0}^{k+1} C(j, k)\frac{n_j^{i+1}}{(n_j+1)\cdots(n_j+1+i)} \\ &\times \sum_{(m/n_j)\in(a,b)} \frac{1}{n_j}\phi_i\left(\frac{m}{n_j}\right)\left(\frac{m}{n_j}+\frac{1}{n_j}\right)\cdots\left(\frac{m}{n_j}+\frac{i}{n_j}\right) . \end{aligned}$$

We can write

$$\frac{n_j^{i+1}}{(n_j+1)\cdots(n_j+1+i)} = 1 + \frac{d_1}{n_j} + \cdots + \frac{d_k}{(n_j)^k} + O\left(\frac{1}{n_j^{k+1}}\right)$$

and

$$\left(\frac{m}{n_j}+\frac{1}{n_j}\right)\cdots\left(\frac{m}{n_j}+\frac{i}{n_j}\right) = \left(\frac{m}{n_j}\right)^i + \left\{\sum_{l=0}^{i-1} e_l\left(\frac{m}{n_j}\right)^l\left(\frac{1}{n_j}\right)^{i-l}\right\}$$

where neither $d_1, \cdots, d_k$ nor $e_0, \cdots, e_{i-1}$ depends on j.

Using the Euler-Mclaurin formula [2, pp. 268-275] we obtain

$$\frac{1}{n_j} \sum_{(m/n_j) \in [a,b]} \phi_i\left(\frac{m}{n_j}\right)\left(\frac{m}{n_j}\right)^i = \int_a^b \phi_i(u) u^i du + R$$

$$|R| = \left| \frac{1}{n_j^{2k+1}} \sum_{k=k_0}^{n_0} \int_0^t P_{2k}(t) h^{(2k)}(a + n_j^{-1}(t + k)) dt \right|$$

where $h(u) = \phi_i(u)u^i$, which, since $P_{2k}(t)$ is a fixed polynomial and $||P_{2k}(t)||_{C[0,1]} = M$, implies

$$|R| \leqq \frac{M}{n_j^{2k}} \int_a^b |h^{(2k)}(t)| \, dt \leqq \frac{M}{n_j^{2k}} ||h^{(2k)}||_{L_1} .$$

Therefore,

$$\begin{aligned} |I_r| \leqq n^{k+1} \Big| \sum_{j=0}^{k+1} C(j, k)\Big(1 + \frac{d_1}{n_j} + \cdots + \frac{d_k}{n_j^k}\Big)\Big[\int_a^b \phi_i(u) u^i du \\ + \sum_{l=0}^{i-1} \Big(\frac{1}{n_j}\Big)^{i-l} e_l \int_a^b \phi_i(u) u^l du\Big]\Big| \\ + n^{k+1} \cdot \sum^{k+1} |C(j, k)| \, K \frac{2k}{n^{2k}} \max_{l \leq i} \left\| \frac{d^{2k}}{du^{2k}}(\phi_i(u) u^l) \right\|_{L_1[a,b]} + o(1) . \end{aligned}$$

The second term is $O(n^{-k+1})$, since $||f^{(m)}||_{L_1[a,b]} \leqq C(||f||_{L_1[a,b]} + ||R^{(2k)}||_{L_1[a,b]})$ for $m \leqq 2k$. Recalling Lemma 6.1, $\sum_{j=0}^{k+1} C(j, k)(1/n_j^m) = 0$ for $m = 0, 1, \cdots, k$ we observe that the first term is $O(1)$ at most.

7. The Szasz operator. The difference in the statement of Theorem 3 is that $|f(t)| \leqq Ke^{Lt}$ (instead of $f(t) \in C$) and this assures us of the convergence of the operator $S_\tau(f, k, t)$ locally. Lemmas corresponding to Lemmas 3.1, 3.3 and 3.4 are stated and proved similarly. In the corresponding lemma to Lemma 3.2, (3.3) is replaced by

$$Q(k, 2k + 2, t) = C_3 t^{k+1} , \quad Q(k, 2k + 1, t) = C_4 t^k . \tag{7.1}$$

To show (7.1), one recalls that for the Szasz operators,

$$W_1(\tau, t, u) = e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \delta(u - k), \ \langle W_1(\tau, t, \cdot), f(\cdot) \rangle \equiv S_\tau(f, t) , \tag{7.2}$$

and therefore

$$\frac{\partial}{\partial t} W_1(\tau, t, u) = (t\tau)^{-1} W_1(\tau, t, u)(u - t) . \tag{7.3}$$

Defining $A_m^*(\tau, t)$ by

$$A_m^*(\tau, t) = \tau^{-m} \langle W_1(\tau, t, u), (u - t)^m \rangle , \tag{7.4}$$

we derive

$$A_m^*(\tau, t) = p(t)^{-1}A_{m-1}^*(\tau, t) + p(t)\frac{\partial}{\partial t}A_m^*(\tau, t) \tag{7.5}$$

for $p(t) = t$ (instead of $t(1 - t)$) and the rest follows similar steps where τ^{-1} takes the place of n.

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