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ALMOST CONTACT STRUCTURES ON BRIESKORN MANIFOLDS

SHIGEO SASAKI AND TOSHIO TAKAHASHI

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1. Introduction. Let $a_0, a_1, \cdots, a_n$ be positive integers and let $X^{2n} = X^{2n}(a_0, a_1, \cdots, a_n)$ be the algebraic variety given by $X^{2n}(a_0, a_1, \cdots, a_n) = \{z = (z_0, z_1, \cdots, z_n) \in C^{n+1} | z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0\}$. The only possible singularity of X^{2n} is the origin 0 of C^{n+1}, and $B^{2n} = B^{2n}(a_0, a_1, \cdots, a_n) = X^{2n}(a_0, a_1, \cdots, a_n) - \{0\}$ is a complex hypersurface of C^{n+1}. Let $\Sigma^{2n-1} = \Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ be the intersection of B^{2n} and the unit sphere $S^{2n+1} = \{z \in C^{n+1} | z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = 1\}$, which we call a Brieskorn manifold.

A Brieskorn manifold Σ^{2n-1} is a real hypersurface of the complex manifold B^{2n}, and it is also a submanifold of the unit sphere S^{2n+1} with codimension 2. K. Abe [1] introduced an almost contact structure on Σ^{2n-1} by using a property that $\Sigma^{2n-1} \times R$ is diffeomorphic with B^{2n}, and discussed about the non-regularity of the almost contact structure. On the other hand, C. J. Hsu and one of the authors [6] introduced a contact structure on Σ^{2n-1} by using a property that Σ^{2n-1} is a submanifold of B^{2n} and S^{2n+1}, and gave a necessary and sufficient condition that the almost contact structure given by K. Abe and the almost contact structure given in [6] coincide. K. Abe and J. Erbacher [2] also introduced contact structures on a wide class of submanifolds which contain Brieskorn manifolds as a special case.

In this note, first we give a simplified definition of the almost contact structure introduced by K. Abe, and the criterions for its non-regularity and regularity. Secondly we show that our almost contact structure is normal.

2. Definition of an almost contact structure for Brieskorn manifold Σ^{2n-1}. Let $\{f_s\}(s \in R)$ be the 1-parameter group of holomorphic transformations of the complex manifold B^{2n} given by

$$f_s(z_0, z_1, \cdots, z_n) = (e^{b_0 s}z_0, e^{b_1 s}z_1, \cdots, e^{b_n s}z_n) , \tag{2.1}$$

where $b_0 = m/a_0, b_1 = m/a_1, \cdots, b_n = m/a_n$ and m is the L.C.M. of $a_0, a_1, \cdots, a_n$. Let α be the vector field on B^{2n} which is induced by the 1-parameter group $\{f_s\}(s \in R)$. It is easy to see that α is transversal and $J\alpha$ is tangent to the Brieskorn manifold Σ^{2n-1}, where J is the induced complex structure of B^{2n} from the standard complex structure of C^{n+1}. Hence we get a direct sum decomposition of tangent spaces along Σ^{2n-1}:

$$T_x B^{2n} = T_x \Sigma^{2n-1} \oplus \{\mathfrak{a}_x\}\ , \quad x \in \Sigma^{2n-1}\ . \tag{2.2}$$

Let ξ be a vector field on Σ^{2n-1} defined by

$$\xi_x = J\mathfrak{a}_x\ , \quad x \in \Sigma^{2n-1}\ , \tag{2.3}$$

and let ϕ be a (1, 1) type tensor field and η be a 1-form on Σ^{2n-1} given by

$$JX = \phi X - \eta(X)\mathfrak{a} \tag{2.4}$$

for any tangent vector X of Σ^{2n-1}, where the right hand side of (2.3) is the direct sum decomposition of JX according to (2.2). Applying J to the both sides of (2.4), we get

$$\begin{cases} \phi^2 X = -X + \eta(X)\xi \\ \eta(\phi X) = 0\ . \end{cases} \tag{2.5}$$

If we put $X = \xi$ in (2.4), we get

$$\begin{cases} \eta(\xi) = 1 \\ \phi\xi = 0 \end{cases}\ . \tag{2.6}$$

Hence (ϕ, ξ, η) is an almost contact structure of the Brieskorn manifold Σ^{2n-1}, which is essentially the same as the one given by K. Abe [1]. As the group of diffeomorphisms of Σ^{2n-1} generated by the vector field ξ is the restriction of the 1-parameter group of holomorphic transformations $\{g_t\}(t \in R)$ of B^{2n} given by

$$g_t(z_0, z_1, \cdots, z_n) = (e^{b_0 it} z_0, e^{b_1 it} z_1, \cdots, e^{b_n it} z_n)\ , \tag{2.7}$$

leaves of the foliation determined by orbits of the group (=integral curves of ξ) are closed curves.

An almost contact structure (ϕ, ξ, η) is said to be regular if the foliation determined by maximal integral curves of ξ is regular. Otherwise, it is said to be non-regular. In the next section, we shall study the problem of regularity of the almost contact structure on Σ^{2n-1} defined in this section.

3. A criterion for regularity. First, we explain some notational conventions: For positive integers $p_1, p_2, \cdots, p_k (k \geqq 2)$, the L. C. M. of $p_1, p_2, \cdots, p_k$ and the G. C. M. of $p_1, p_2, \cdots, p_k$ are respectively denoted by $(p_1, p_2, \cdots, p_k)$ and $\langle p_1, p_2, \cdots, p_k \rangle$.

LEMMA 1. *Let $a_0, a_1, \cdots, a_n$ be positive integers and put $m = (a_0, a_1, \cdots, a_n)$, $b_0 = m/a_0$, $b_1 = m/a_1, \cdots, b_n = m/a_n$. Then $(a_\lambda, a_\mu) = (a_\lambda, a_\mu, a_\nu)$*

holds for any triplet a_λ, a_μ, a_ν *among* $a_0, a_1, \cdots, a_n$ *if and only if* $\langle b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\rangle = \langle b_0, b_1, \cdots, b_n\rangle \equiv 1$ *holds for any subset* $\{b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\}$ *of* $\{b_0, b_1, \cdots, b_n\}$, $k \geqq 1$.

PROOF. Suppose $(a_\lambda, a_\mu) = (a_\lambda, a_\mu, a_\nu)$ holds for any triplet a_λ, a_μ, a_ν. Take any subset $\{b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\}$ of $\{b_0, b_1, \cdots, b_n\}$, $k \geqq 1$, and put $l = (a_{\lambda_0}, a_{\lambda_1})$. Then we get $l \leqq m$. On the other hand, since we have $(a_{\lambda_0}, a_{\lambda_1}) = (a_{\lambda_0}, a_{\lambda_1}, a_\mu)$ for any a_μ, we see that l is a common multiple of $a_0, a_1, \cdots, a_n$, and hence we get $l = m$. Thus we get $1 = \langle b_{\lambda_0}, b_{\lambda_1}\rangle \geqq \langle b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\rangle \geqq \langle b_0, b_1, \cdots, b_n\rangle \equiv 1$.

Conversely, suppose $\langle b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\rangle = 1$ holds for any subset $\{b_{\lambda_0}, b_{\lambda_1}, \cdots, b_{\lambda_k}\}$ of $\{b_0, b_1, \cdots, b_n\}$. Then we have, in particular, $\langle b_\lambda, b_\mu\rangle = 1$ for any pair b_λ, b_μ. Take arbitrary triplet a_λ, a_μ, a_ν. Then $\langle b_\lambda, b_\mu\rangle = 1$ and $a_\lambda b_\lambda = a_\mu b_\mu = m$ imply $(a_\lambda, a_\mu) = m$. Hence we get $m = (a_\lambda, a_\mu) \leqq (a_\lambda, a_\mu, a_\nu) \leqq (a_0, a_1, \cdots, a_n) = m$, which completes the proof.

Now, we state the criterion for the non-regularity of our almost contact structure.

THEOREM 1. *Let* $\Sigma^{2n-1} = \Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ *be a Brieskorn manifold and let* (ϕ, ξ, η) *be the almost contact structure on* Σ^{2n-1} *defined by* (2.3) *and* (2.4). *A necessary and sufficient condition for the almost contact structure* (ϕ, ξ, η) *to be non-regular is that there exist three positive integers* a_λ, a_μ, a_ν *among* $a_0, a_1, \cdots, a_n$ *such that the L. C. M. of* a_λ, a_μ *is different from the L. C. M. of* a_λ, a_μ, a_ν.

PROOF. Suppose $(a_\lambda, a_\mu) \neq (a_\lambda, a_\mu, a_\nu)$ holds for some a_λ, a_μ, a_ν. Without any loss of generality, we may assume that $\lambda = 0, \mu = 1, \nu = 2$. Applying Lemma 1 to the case when $n = 2$, we see that $(a_0, a_1) \neq (a_0, a_1, a_2)$ implies $\langle b_0, b_1\rangle \neq \langle b_0, b_1, b_2\rangle$. Now, the orbit of ξ passing through the point $(z_0, z_1, 0, 0, \cdots, 0)$ of Σ^{2n-1} is of the form $(e^{b_0 it}z_0, e^{b_1 it}z_1, 0, 0, \cdots, 0)$, and its period is $2\pi/\langle b_0, b_1\rangle$. Similarly, the orbit of ξ passing through the point $(z_0', z_1', z_2', 0, \cdots, 0)$ which is very close to the first point $(z_0, z_1, 0, 0, \cdots, 0)$, is of the form $(e^{b_0 it}z_0', e^{b_1 it}z_1', e^{b_2 it}z_2', 0, \cdots, 0)$ and lies very close to the first orbit. Its period is $2\pi/\langle b_0, b_1, b_2\rangle$. Hence our almost contact structure is non-regular.

Conversely, suppose $(a_\lambda, a_\mu) = (a_\lambda, a_\mu, a_\nu)$ holds for any triplet a_λ, a_μ, a_ν among $a_0, a_1, \cdots, a_n$. Then, since the orbit of ξ passing through the point $(z_0, z_1, \cdots, z_n)$ of Σ^{2n-1} is of the form $(e^{b_0 it}z_0, e^{b_1 it}z_1, \cdots, e^{b_n it}z_n)$, Lemma 1 implies that any orbit of ξ has the same period 2π, and hence our almost contact structure is regular. q.e.d.

The following is the criterion for the regularity, which is equivalent to Theorem 1.

COROLLARY. *The almost contact structure in Theorem* 1 *is regular if and only if for each triplet* a_λ, a_μ, a_ν *among* $a_0, a_1, \cdots, a_n$, *the L. C. M. s of* a_λ, a_μ *and* a_λ, a_μ, a_ν *are the same.*

EXAMPLE. Let $p_1, p_2, \cdots, p_n$ be mutually prime positive integers. Put $a_0 = p_1p_2 \cdots p_n$, $a_1 = p_2p_3 \cdots p_n$, $a_2 = p_1p_3p_4 \cdots p_n$, $\cdots$, $a_n = p_1p_2 \cdots p_{n-1}$. Then for each triplet a_λ, a_μ, a_ν, the L. C. M. s of a_λ, a_μ and a_λ, a_μ, a_ν are the same number $p_1p_2 \cdots p_n$. Thus, in this case, our almost contact structure of the Brieskorn manifold $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ is regular.

4. **The normality.** For vector fields X and Y on a differentiable manifold with an almost contact structure (ϕ, ξ, η), we put

$$N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] - \{X\cdot\eta(Y) - Y\cdot\eta(X)\}\xi \,. \tag{4.1}$$

When the tensor field N vanishes everywhere, we call the almost contact structure (ϕ, ξ, η) to be normal (S. Sasaki and Y. Hatakeyama [5]). To show that the almost contact structure (ϕ, ξ, η) on the Brieskorn manifold Σ^{2n-1} constructed by (2.3) and (2.4) is normal, we first extend the structure tensors ϕ, ξ, η to the tensors $\tilde{\phi}, \tilde{\xi}, \tilde{\eta}$ on the complex manifold B^{2n}.

Put $\Sigma_0 = \Sigma^{2n-1}$ and put $\Sigma_s = f_s\Sigma_0$, a real hypersurface of B^{2n} for each $s \in R$, where f_s, $s \in R$, is a holomorphic transformation of B^{2n} defined by (2.1). Then we get $B^{2n} = \bigcup_{s\in R}\Sigma_s$. Since the vector field $\mathfrak{a}$, defined in §2, is invariant by the 1-parameter group of holomorphic transformations $\{f_s\}$ generated by $\mathfrak{a}$, $\mathfrak{a}_{\tilde{x}}$ is transversal and $J\mathfrak{a}_{\tilde{x}}$ is tangent to the hypersurface Σ_s such that $\tilde{x} \in \Sigma_s$, because $\mathfrak{a}_x$ is transversal and $J\mathfrak{a}_x$ is tangent to $\Sigma_0 = \Sigma^{2n-1}$ if $x \in \Sigma_0$. Hence, for each $\tilde{x} \in \Sigma_s \subset B^{2n}$, we get the direct sum decomposition

$$T_{\tilde{x}}B^{2n} = T_{\tilde{x}}\Sigma_s \oplus \{\mathfrak{a}_{\tilde{x}}\} \,. \tag{4.2}$$

Now we define a vector field $\tilde{\xi}$, (1, 1) type tensor field $\tilde{\phi}$ and 1-form $\tilde{\eta}$ on B^{2n}, which are extensions of ξ, ϕ and η, as follows:

$$\begin{cases} \tilde{\xi} = J\mathfrak{a} \\ J\tilde{X} = \tilde{\phi}\tilde{X} - \tilde{\eta}(\tilde{X})\mathfrak{a} \end{cases} \tag{4.3}$$

for a tangent vector $\tilde{X}$ of B^{2n}, where the right hand side of the second equality is the direct sum decomposition of $J\tilde{X}$ according to (4.2). $\tilde{\phi}, \tilde{\xi}$ and $\tilde{\eta}$ satisfy analogous equations with those of (2.5), (2.6) and

$$\tilde{\phi}\mathfrak{a} = \tilde{\xi} \,, \qquad \tilde{\eta}(\mathfrak{a}) = 0 \,. \tag{4.4}$$

LEMMA 2. *$\tilde{\phi}$, $\tilde{\xi}$, $\tilde{\eta}$ satisfy*

$$\mathcal{L}(\mathfrak{a})\tilde{\phi} = 0 , \quad \mathcal{L}(\mathfrak{a})\tilde{\xi} = 0 , \quad \mathcal{L}(\mathfrak{a})\tilde{\eta} = 0 . \tag{4.5}$$

PROOF. Since the complex structure J and the vector field $\mathfrak{a}$ are invariant by each holomorphic transformation f_s, we have $f_{s*}\tilde{\xi} = f_{s*}J\mathfrak{a} = Jf_{s*}\mathfrak{a} = J\mathfrak{a} = \tilde{\xi}$, i.e. $\tilde{\xi}$ is invariant by f_s, and this implies $(4.5)_2$.

Next, applying (4.3) to both sides of $f_{s*}J\tilde{X} = Jf_{s*}\tilde{X}$, we see that $\tilde{\phi}$ and $\tilde{\eta}$ are invariant by f_s, and hence $(4.5)_1$ and $(4.5)_3$ hold good. q.e.d.

Now, we shall prove the following:

THEOREM 2. *The almost contact structure (ϕ, ξ, η), constructed by* (2.3) *and* (2.4), *of a Brieskorn manifold Σ^{2n-1} is normal.*

PROOF. Take an arbitrary point x of $\Sigma = \Sigma^{2n-1}$ and a small neighborhood V of x on Σ, and let X, Y be vector fields on V tangent to Σ. Then, to prove the normality of the almost contact structure (ϕ, ξ, η) on Σ, it is sufficient to show $N(X, Y) = 0$.

To show it, we first take a small neighborhood U of x on B^{2n} so that $V = U \cap \Sigma$ and extend X, Y to vector fields $\tilde{X}$, $\tilde{Y}$ on U so that their restriction to V coincide with X, Y. We remark that the restriction of $[\tilde{X}, \tilde{Y}]$ to V coincides with $i_*[X, Y]$, where $i: V \to U$ is the inclusion map.

Now, as the torsion tensor field N_J of the complex structure J of B^{2n} vanishes identically, we have

$$N_J(\tilde{X}, \tilde{Y}) = [\tilde{X}, \tilde{Y}] + J[J\tilde{X}, \tilde{Y}] + J[\tilde{X}, J\tilde{Y}] - [J\tilde{X}, J\tilde{Y}] = 0 .$$

Putting (4.3) into the last equation and making use of (4.4), we get

$$\begin{aligned}[\tilde{X}, \tilde{Y}] &+ \tilde{\phi}[\tilde{\phi}\tilde{X}, \tilde{Y}] + \tilde{\phi}[\tilde{X}, \phi\tilde{Y}] - [\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}] - \{\tilde{X}\cdot\tilde{\eta}(\tilde{Y}) - \tilde{Y}\cdot\tilde{\eta}(\tilde{X})\}\tilde{\xi} \\ &+ \tilde{\eta}(\tilde{X})(\mathcal{L}(\mathfrak{a})\tilde{\phi})\tilde{Y} - \tilde{\eta}(\tilde{Y})(\mathcal{L}(\mathfrak{a})\tilde{\phi})\tilde{X} + (*)\mathfrak{a} = 0 ,\end{aligned}$$

where $(*)$ is a function on U such that we do not need to know its exact value. If we consider vector fields on the left hand side of the last equation only on Σ, we see by a remark stated above and (4.5) that

$$i_*N(X, Y) + (*)\mathfrak{a} = 0$$

holds good. This gives us $N(X, Y) = 0$. q.e.d.

Looking carefully at the proof of Theorem 2, we see that, in general, the following theorem holds good:

THEOREM 3. *Let M^{2n-1} be a real hypersurface of a complex manifold W^{2n} with complex structure J. Suppose there exists a transversal vector field $\mathfrak{a}$ along M^{2n-1} such that $J\mathfrak{a}$ is tangent to M^{2n-1} and $\mathfrak{a}$ is induced by a local* 1*-parameter group of holomorphic transformations* *of* W^{2n}. *Then* M^{2n-1} *admits a normal almost contact structure.*

EXAMPLE. The weighted homogeneous manifold.

Let $w_0, w_1, \cdots, w_n$ be positive rational numbers. Let the polynomial $F(z_0, z_1, \cdots, z_n)$ be weighted homogeneous of type $(w_0, w_1, \cdots, w_n)$; that is, it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$. Let $X = X(w_0, w_1, \cdots, w_n)$ be the algebraic variety given by $X(w_0, w_1, \cdots, w_n) = \{(z_0, z_1, \cdots, z_n) \in C^{n+1} \mid F(z_0, z_1, \cdots, z_n) = 0\}$. Suppose X has only possible singularity at the origin 0 of C^{n+1}, then $V^{2n} = V^{2n}(w_0, w_1, \cdots, w_n) = X(w_0, w_1, \cdots, w_n) - \{0\}$ is a Kählerian submanifold of C^{n+1}. The weighted homogeneous manifold $\Sigma^{2n-1} = \Sigma^{2n-1}(w_0, w_1, \cdots, w_n)$ is by definition the intersection of V^{2n} with the sphere S^{2n+1} of sufficiently small radius ε having the origin 0 as its center. Let $\{f_s\}(s \in R)$ be the 1-parameter group of holomorphic transformations of V^{2n} given by $f_s(z_0, z_1, \cdots, z_n) = (e^{s/w_0} z_0, e^{s/w_1} z_1, \cdots, e^{s/w_n} z_n)$. Let α be the vector field induced by $\{f_s\}(s \in R)$. It is easy to see that α is transversal and $J\alpha$ is tangent to the weighted homogeneous manifold Σ^{2n-1}, where J is the induced complex structure of V^{2n}. Using the same method as in the case of the Brieskorn manifold, we can define an almost contact structure on Σ^{2n-1}, and by Theorem 3 we see that it is normal.

According to A. Morimoto [4], a product manifold of two manifolds with normal almost contact structures admits a complex structure. Hence Theorem 2 implies a theorem of E. Brieskorn and A. Van de Ven [3] to the effect that a product manifold of two Brieskorn manifolds admits a complex structure.

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KUMAMOTO UNIVERSITY