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THE SINGULARITIES OF THE SOLUTIONS OF THE CAUCHY PROBLEM FOR SECOND ORDER EQUATIONS IN CASE THE INITIAL MANIFOLD INCLUDES CHARACTERISTIC POINTS

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1. Introduction. In the theory of linear ordinary differential equations in the complex domain, the only possible singular points of the solutions are those of the coefficients, namely, the points where Cauchy's theorem does not hold. In the theory of partial differential equations, the singularities of the solutions of the Cauchy problem propagate along the characteristics of the equations, and they emanate from the initial data or the inhomogeneous term or the initial manifold. The corresponding Cauchy-Kowalevskaya's theorem can not be applied to a Cauchy problem when either the initial data or the inhomogeneous term is singular or when there are characteristic points in the initial manifold. The corresponding situation in the theory of ordinary differential equations is the case of inhomogeneous equations with singular inhomogeneous term in the former case, or when the point in question is a singular point of the coefficient in the latter.

J. Leray [4] studied Cauchy problems with holomorphic initial data given on the manifold which includes characteristic points. He used singular transformations of the independent variables to reduce the characteristic initial value problem to a non-characteristic one. On the other hand, Hamada [2], Wagschal [5] and others studied non-characteristic problems when the initial data or the inhomogeneous term have singularities of the regular singular type. In this note, we give an answer to the natural questions: What will happen when the initial data or the inhomogeneous term have singularities of the regular singular type given on a manifold containing the characteristic points of the differential equation, when the order of the equation is two.

2. Statement of the results. We denote by $x = (x_1, x_2, \cdots, x_n)$ a point in the n-dimensional complex space C^n.

We consider a linear partial differential operator of order 2 whose coefficients are holomorphic in a neighborhood of $x = 0$:

$$a\left(x; \frac{\partial}{\partial x}\right) = \sum_{|\alpha| \leq 2} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where α stands for the multi-index $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ with a length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote the characteristic polynomial of $a(x; \partial/\partial x)$ by $h(x; \xi)$.

Throughout this paper, we assume the following three conditions (a), (b) and (c).

(a) Let T; $x_1 = x_2 = 0$ be the set of all characteristic points of the initial manifold S; $x_1 = 0$. Then any bicharacteristic curve issuing from T never becomes tangential to T.

(b) If $h(0, \cdots, 0; \xi_1, 1, 0, \cdots, 0) = 0$(resp. $h(0, \cdots, 0; 1, \xi_2, 0, \cdots, 0) = 0$), then $(\partial/\partial\xi_1)h(0, \cdots, 0; \xi_1, 1, 0, \cdots, 0) \neq 0$(resp. $(\partial/\partial\xi_2)h(0, \cdots, 0; 1, \xi_2, 0, \cdots, 0) \neq 0$).

From the assumption (b), we shall show in the next section that there exist two sheets of simple characteristic surfaces K_j:

$$K_j = \{x; \varphi_j(x) = 0\} \quad \text{grad}\, \varphi_j(x) \neq 0 \quad (j = 1, 2)$$

issuing from T in a neighborhood of $x = 0$.

(c) K_1 is the only characteristic surface which becomes tangential to S along T. Its degree of contact is $p - 1 (p \geqq 2)$.

In order to state our results, it is convenient to introduce the following notation $H(r, s, t)$ $(r, s, t) \in \boldsymbol{C} \times \boldsymbol{Z} \times \boldsymbol{N}$, $s \geqq 0$.

DEFINITION. A complex valued function $f(x)$ defined in a neighborhood of $x = 0$ in $\boldsymbol{C}^n$ belongs to $H(r, s, t)$ if and only if

$$f(x) = \begin{cases} [\varphi_1(x)]^{r/t} P_{1,s}([\varphi_1(x)]^{1/t}, x; \log \varphi_1(x)) + [\varphi_2(x)]^r P_{2,s}([\varphi_1(x)]^{1/t}, x; \\ \log \varphi_2(x)) \\ \quad \text{when} \quad r \in \boldsymbol{C} - \boldsymbol{Z} \quad \text{or} \quad r \geqq 0, \\ [\varphi_1(x)]^{r/t} P_{1,s-1}([\varphi_1(x)]^{1/t}, x; \log \varphi_1(x)) \\ + [\varphi_2(x)]^r P_{2,s-1}([\varphi_1(x)]^{1/t}, x; \log \varphi_2(x)) \\ + \sum_{j=1}^{2} P_{j,s}([\varphi_1(x)]^{1/t}, x; \log \varphi_j(x)) \\ \quad \text{when} \quad 0 > r \in \boldsymbol{Z}, s \geqq 1, \\ P_0([\varphi_1(x)]^{1/t}, x) \quad \text{when} \quad 0 > r \in \boldsymbol{Z}, s = 0, \end{cases}$$

where $P_{j,s}(\zeta, x; \omega) (j = 1, 2)$ are polynomials in ω of degree $\leqq s$ whose coefficients are holomorphic in a neighborhood of $(\zeta, x) = 0$ and $P_0(\zeta, x)$ is a function holomorphic in a neighborhood of $(\zeta, x) = 0$. Moreover, we simply denote $H(r, s, 1)$ by $H(r, s)$.

Now we state our theorem.

THEOREM. *Consider the Cauchy problem:*

$$(CP)_1 \qquad \begin{cases} a\left(x; \dfrac{\partial}{\partial x}\right)u(x) = v(x)\,, \\ \textit{all the derivatives of } u(x) - w(x) \textit{ up to order } 1 \\ \qquad \textit{vanish on } S - T\,. \end{cases}$$

(1) *The Cauchy problem* $(CP)_1$ *has a unique solution* $u(x) \in H(p(r-2)+1, s, p)$ *for* $w = 0$ *and*

$$v(x) = \begin{cases} [\varphi_1(x)]^{r-2}[\log \varphi_1(x)]^s F_1(x) & \textit{when } s \geqq 0 \textit{ and} \\ \qquad \textit{either } r-2 \in \boldsymbol{C} - \boldsymbol{Z} \textit{ or } r-2 \geqq 0\,, \\ [\varphi_1(x)]^{r-2}[\log \varphi_1(x)]^{s-1} F_1(x) & \textit{when } 0 > r-2 \in \boldsymbol{Z},\ s \geqq 1\,, \end{cases}$$

where $F_1(x)$ *is a holomorphic function defined in a neighborhood of* $x = 0$.

(2) *The Cauchy problem* $(CP)_1$ *has a unique solution* $u(x) \in H(r-1, s, p)$ *for* $w = 0$ *and*

$$v(x) = \begin{cases} [\varphi_2(x)]^{r-2}[\log \varphi_2(x)]^s F_2(x) & \textit{when } s \geqq 0 \textit{ and} \\ \qquad \textit{either } r-2 \in \boldsymbol{C} - \boldsymbol{Z} \textit{ or } r-2 \geqq 0\,, \\ [\varphi_2(x)]^{r-2}[\log \varphi_2(x)]^{s-1} F_2(x) & \textit{when } 0 > r-2 \in \boldsymbol{Z},\ s \geqq 1\,, \end{cases}$$

where $F_2(x)$ *is a holomorphic function defined in a neighborhood of* $x = 0$.

(3) *The Cauchy problem* $(CP)_1$ *has a unique solution* $u(x) \in H(p(r-1), s, p) + H(pr-1, s, p) + H(r, s)$ *for* $v = 0$ *and*

$$w(x) = \begin{cases} [\varphi_1(x)]^{r}[\log \varphi_1(x)]^s G_1(x) & \textit{when } s \geqq 0 \textit{ and} \\ \qquad \textit{either } r \in \boldsymbol{C} - \boldsymbol{Z} \textit{ or } r \geqq 0\,, \\ [\varphi_1(x)]^{r}[\log \varphi_1(x)]^{s-1} G_1(x) & \textit{when } 0 > r \in \boldsymbol{Z},\ s \geqq 1\,, \end{cases}$$

where $G_1(x)$ *is a holomorphic function defined in a neighborhood of* $x = 0$.

(4) *The Cauchy problem* $(CP)_1$ *has a unique solution* $u(x) \in H(r-1, s, p) + H(r, s)$ *for* $v = 0$ *and*

$$w(x) = \begin{cases} [\varphi_2(x)]^{r}[\log \varphi_2(x)]^s G_2(x) & \textit{when } s \geqq 0 \textit{ and either} \\ \qquad r \in \boldsymbol{C} - \boldsymbol{Z} \textit{ or } r \geqq 0\,, \\ [\varphi_2(x)][\log \varphi_2(x)]^{s-1} G_2(x) & \textit{when } 0 > r \in \boldsymbol{Z},\ s \geqq 1\,, \end{cases}$$

where $G_2(x)$ *is a holomorphic function defined in a neighborhood of* $x = 0$.

Consequently, combining (1) $\sim$ (4), *the Cauchy problem* $(CP)_1$ *has a* *unique solution* $u(x) \in H(r-1, s, p) + H(p(r-2)+1, s, p) + H(p(r-1), s, p) + H(pr-1, s, p) + H(r, s)$ *for the given* $v(x) \in H(r-2, s)$ *and* $w(x) \in H(r, s)$.

3. Construction of the characteristic surfaces $K_j (j=1, 2)$. Our method of constructing the characteristic surfaces $K_j (j=1, 2)$ is quite similar to that of Hamada [2]. But it slightly differs in details. We first construct the characteristic surface K_1 issuing from T. For this purpose, consider the Cauchy problem:

$$\begin{cases} h(x; \operatorname{grad} \varphi_1(x)) = 0 \\ \varphi_1(x_1, 0, x_3, \cdots, x_n) = x_1 . \end{cases} \tag{3.1}$$

The Cauchy problem (3.1) can be solved by the well-known Cauchy's characteristic method. Before we explain this method, we note that, from the assumption (b), the algebraic equation $h(x; \xi) = 0$ with respect to ξ_2 has a simple root $\beta(x; \xi_1, \xi_3, \cdots, \xi_n)$ for sufficiently small $|x|$ and $|\xi_1 - 1| + |\xi_3| + \cdots + |\xi_n|$.

Pursuing Cauchy's characteristic method, we consider the bicharacteristic equation associated to (3.1):

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial}{\partial \xi_i} h(x; \xi) \\ \dfrac{d\xi_i}{dt} = -\dfrac{\partial}{\partial x_i} h(x; \xi) \\ \dfrac{d\varphi_1}{dt} = 2h(x; \xi) \end{cases} \qquad (i = 1, 2, \cdots, n) \tag{3.2}$$

with initial conditions:

$$\begin{cases} x_1(0) = y_1, x_2(0) = 0, x_3(0) = y_3, \cdots, x_n(0) = y_n , \\ \xi_1(0) = 1, \xi_2(0) = \beta(y_1, 0, y_3, \cdots, y_n; 1, 0, \cdots, 0) , \\ \xi_3(0) = 0, \cdots, \xi_n(0) = 0 , \\ \varphi_1(0) = y_1 . \end{cases}$$

Since $h(x; \xi)$ is the first integral for (3.2), we can write the solution of (3.2) in the form

$$\begin{cases} x_i = x_i(t; y_1, 0, y_3, \quad , y_n) \quad (i = 1, 2 \cdots, n) , \\ \varphi_1 = y_1 . \end{cases}$$

From (3.2) and the assumption (b), we obtain

$$\left.\frac{D(x_1, x_2, x_3, \cdots, x_n)}{D(t, y_1, y_3, \cdots, y_n)}\right|_{t=0}$$
$$= -\frac{\partial h}{\partial \xi_2}(y_1, 0, y_3, \cdots y_n; 1, \beta(y_1, 0, y_3, \cdots, y_n; 1, 0, \cdots, 0), 0, \cdots, 0)$$
$$\neq 0$$

for sufficiently small $|y_1| + |y_3| + \cdots + |y_n|$.

The implicit function theorem shows that there exists in a neighborhood of $x = 0$ a set of holomorphic functions:

$$\begin{cases} t = t(x_1, \cdots, x_n) \\ y_1 = y_1(x_1, \cdots, x_n) \\ y_3 = y_3(x_1, \cdots, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_n = y_n(x_1, \cdots, x_n) \end{cases}$$

which satisfy the previous relations $x_i = x_i(t; y_1, 0, y_3, \cdots, y_n)(i=1, 2, \cdots, n)$. Hence $\varphi_1(x) = y_1(x_1, \cdots, x_n)$ and it is obvious that $\operatorname{grad} \varphi_1(x) \neq 0$ holds in a neighborhood of $x = 0$. From the assumption (a), K_1 is generated by bicharacteristic curves issuing from T. So $K_1 = \{x; \varphi_1(x) = 0\}$. This also shows that the assumption (c) itself has a thorough meaning.

Next we construct K_2 so as to fit for the later purpose of constructing the formal solution. From the assumption (c) and the fact $K_1 \cap \{x; x_2 = 0\} = T$, there exists in a neighborhood of $x = 0$ a non-vanishing holomorphic function $\sigma(x)$ such that S is expressed in S; $\varphi_1(x) - x_2^p\sigma(x) = 0$. We choose a branch of $\sigma(x)^{1/p}$ and set $\tau(x) = \sigma(x)^{1/p}$. Then S is given by S; $\varphi_1(x) - (x_2\tau(x))^p = 0$. We note that from the assumption (b) the algebraic equation $h(x; \xi) = 0$ with respect to ξ_1 has a simple root $\alpha(x; \xi_2, \cdots, \xi_n)$ for sufficiently small $|x|$ and $|\xi_2 - 1| + |\xi_3| + \cdots + |\xi_n|$.

Now we consider the Cauchy problem:

$$\text{(3.3)} \qquad \begin{cases} h(x; \operatorname{grad} \varphi_2(x)) = 0 , \\ \varphi_2(0, x_2, \cdots, x_n) = x_2\tau(0, x_2, \cdots, x_n) . \end{cases}$$

Replacing φ_1 by φ_2, the bicharacteristic equation associated to (3.3) is the same as (3.2) under the initial conditions:

$$\begin{cases} x_1(0) = 0, x_2(0) = y_2, \cdots, x_n(0) = y_n , \\ \xi_1(0) = \alpha(0, y'; \tau(0, y') + y_2\dfrac{\partial \tau}{\partial y_2}(0, y'), y_2\dfrac{\partial \tau}{\partial y_3}(0, y') , \\ \quad \cdots, y_2\dfrac{\partial \tau}{\partial y_n}(0, y')) , \end{cases}$$

$$\begin{cases} \xi_2(0) = \tau(0, y') + y_2 \dfrac{\partial \tau}{\partial y_2}(0, y') , \\ \xi_3(0) = y_2 \dfrac{\partial \tau}{\partial y_3}(0, y') , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_n(0) = y_2 \dfrac{\partial \tau}{\partial y_n}(0, y') , \\ \varphi_2(0) = y_2 \tau(0, y') , \end{cases}$$

where $y' = (y_2, \cdots, y_n)$. The solution of this Cauchy problem can be written in the form

$$x_i = x_i(t; 0, y') \quad (i = 1, 2, \cdots, n) ,$$
$$\varphi_2 = \tau(0, y') y_2 .$$

Thus we obtain

$$\begin{aligned} \left.\frac{D(x_1, x_2, \cdots, x_n)}{D(t, y_2, \cdots, y_n)}\right|_{t=0} = \frac{\partial h}{\partial \xi_1}\Big(0, y'; \alpha\Big[0, y'; \tau(0, y') + y_2 \frac{\partial \tau}{\partial y_2}(0, y') , \\ y_2 \frac{\partial \tau}{\partial y_3}(0, y'), \cdots, y_2 \frac{\partial \tau}{\partial y_n}(0, y')\Big] , \\ \tau(0, y') + y_2 \frac{\partial \tau}{\partial y_2}(0, y'), y_2 \frac{\partial \tau}{\partial y_3}(0, y'), \cdots , \\ y_2 \frac{\partial \tau}{\partial y_n}(0, y')\Big) . \end{aligned}$$

Since $h(x; \xi)$ is homogeneous of degree 2, the right hand side becomes $\tau(0)(\partial h/\partial \xi_1)(0; \alpha(0; 1, 0, \cdots, 0), 1, 0, \cdots, 0)$ for $y' = 0$. Thus, from the assumption (b),

$$\left.\frac{D(x_1, x_2, \cdots, x_n)}{D(t, y_2, \cdots, y_n)}\right|_{t=0} \neq 0$$

for sufficiently small y'.

Repeating the same argument just we have done for K_1, we can construct the characteristic surface $K_2 = \{x; \varphi_2(x) = 0\}$, where grad $\varphi_2(x) \neq 0$.

We note that K_1 and K_2 are transversal in a neighborhood of $x = 0$.

In the following sections, we shall only prove the assertion (3) of the theorem. The others are proved in a similar manner.

4. The reduction to the normal form. We choose a local coordinate system:

(4.1) $$\hat{x}_1 = \varphi_1(x),\ \hat{x}_2 = x_2\tau(x),\ \hat{x}_3 = x_3,\ \cdots,\ \hat{x}_n = x_n\ .$$

Then K_1, S and T become

(4.2) $$K_1 = \{\hat{x};\ \hat{x}_1 = 0\},\ S = \{\hat{x};\ \hat{x}_1 - \hat{x}_2^p = 0\},\ T = \{\hat{x};\ \hat{x}_1 = \hat{x}_2 = 0\}\ .$$

For simplifying the notations, we rewrite these independent variables $\hat{x} = (\hat{x}_1, \cdots, \hat{x}_n)$ by $x = (x_1, \cdots, x_n)$. We assume that the Cauchy problem $(CP)_1$ is given in this local coordinate system x corresponding to $\hat{x}$. In this situation K_1 and K_2 are given by $K_1 = \{x;\ x_1 = 0\}$ and $K_2\{x;\ \varphi_2(x) = 0\}$, where grad $\varphi_2(x) \neq 0$ in a neighborhood of $x = 0$. Here we also emphasize the fact that K_1 and K_2 are transversal. Since the assumption (a) is invariant under the change of the independent variables, we may assume without loss of generality that $a_\alpha(x) \equiv 1$ for $\alpha = (1, 1, 0, \cdots, 0)$. Moreover, from the process of constructing the characteristic surface K_1, $a_\alpha(x) = 0$ holds in a neighborhood of $x = 0$ for $\alpha = (2, 0, \cdots 0)$.

In order to uniformize the solution of $(CP)_1$, we use the following singular transformation:

(4.3) $$x_1 = y_1^p,\ x_2 = y_2,\ \cdots,\ x_n = y_n\ .$$

Let J be the Jacobian $D(x_1, x')/D(y_1, y') = py_1^{p-1}$ of (4.3), then $\partial/\partial x_1 = (1/J)\partial/\partial y_1$.

We have:

(4.4) $$\frac{\partial}{\partial y_2}\Big(\frac{1}{J}\frac{1}{\partial y_1}\Big)\tilde{u}(y) + \Big\{\sum_{\substack{|\alpha'|\leqq 1 \\ \alpha'\neq(1,0,\cdots,0)}} \tilde{a}_{(1,\alpha')}(y)\frac{\partial^{|\alpha'|}}{\partial y'^{\alpha'}}\Big(\frac{1}{J}\frac{\partial}{\partial y_1}\Big)\Big\}\tilde{u}(y) + \Big\{\sum_{|\alpha'|\leqq 2} \tilde{a}_{(0,\alpha')}\frac{\partial^{|\alpha'|}}{\partial y'^{\alpha'}}\Big\}\tilde{u}(y) = 0$$

where $\alpha' = (\alpha_2, \cdots, \alpha_n)$ and $\tilde{a}_\alpha(y)$, $\tilde{u}(y)$ are simply the functions $a_\alpha(x)$, $u(x)$ expressed in the variable y, respectively. In the following, we use the symbol "$\sim$" to indicate the y-space interpretation. We write (4.4) in the form:

$$\frac{\partial}{\partial y_2}\Big(\frac{1}{J}\frac{\partial}{\partial y_1}\Big)\tilde{u}(y) + B_1\Big(y;\frac{\partial}{\partial y'}\Big)\Big(\frac{1}{J}\frac{\partial}{\partial y_1}\Big)\tilde{u}(y) + B_0\Big(y;\frac{\partial}{\partial y'}\Big)\tilde{u}(y) = 0$$

where

$$B_0\Big(y;\frac{\partial}{\partial y'}\Big) = \sum_{|\alpha'|\leqq 2} \tilde{a}_{(0,\alpha')}(y)\frac{\partial^{|\alpha'|}}{\partial y'^{\alpha'}}$$

and

$$B_1\Big(y;\frac{\partial}{\partial y'}\Big)=\sum_{\substack{|\alpha'|\leq 1\\ \alpha'\neq(1,0,\cdots,0)}} a_{(1,\alpha')}(y)\frac{\partial^{|\alpha'|}}{\partial y'^{\alpha'}}\,.$$

Next we introduce new dependent variables $\tilde{u}_\mu(y)=((1/J)\partial/\partial y_1)^\mu\tilde{u}(y)$ ($\mu=0,1$) and reduce the problem $(CP)_1$ to the following Cauchy problem $(CP)_2$ for a system of equations in the y-space:

$$(4.5)\quad (CP)_2\qquad \begin{cases}\begin{pmatrix}\dfrac{\partial}{\partial y_1} & -J\\ B_0 & B_1+\dfrac{\partial}{\partial y_2}\end{pmatrix}\begin{pmatrix}\tilde{u}_0\\ \tilde{u}_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\\ \text{all the derivatives of } \tilde{u}_\mu(y)-\tilde{w}_\mu(y)\\ \text{up to order } 1-\mu \text{ vanish on } \tilde{S}-\tilde{T}\,,\end{cases}$$

where $\tilde{w}_\mu(y)$ is the y-space interpretation of $(\partial^\mu/\partial x_1^\mu)w(x)$. The system (4.5) is a Volevič system associated with the operator matrix

$$\begin{pmatrix}a_\mu^\nu; & \mu\downarrow 0,1\\ & \nu\rightarrow 0,1\end{pmatrix}=\begin{pmatrix}\dfrac{\partial}{\partial y_1} & -J\\ B_0 & B_1+\dfrac{\partial}{\partial y_2}\end{pmatrix}$$

in regard to the set of integers:

$$n_\mu=\mu+1,\ m_\nu=\nu\ (0\leqq\mu,\nu\leqq 1)\,.$$

The characteristic polynomial $g(y;\eta)$ of the system (4.5) is

$$g(y;\eta)=J\Big]\frac{\eta_1\eta_2}{J}+b_1(y;\eta'')+b_0(y;\eta')\Big]$$

where $b_0(y;\eta')$ and $b_1(y;\eta'')$ are the principal symbols of the operators $B_0(y;\partial/\partial y')$ and $B_1(y;\partial/\partial y)$, respectively. Here we used the convention $\eta'=(\eta_2,\cdots,\eta_n)$ and $\eta''=(\eta_3,\cdots,\eta_n)$. Taking account of the facts that $b_1(y;\eta'')$ is a homogeneous polynomial in η'' of degree 1 and J vanishes on $\tilde{T}$, we can easily see that the initial manifold $\tilde{S}=\{y;\,y_1-y_2=0\}$ is non-characteristic. Since $\eta_1\eta_2/J+b_1(y;\eta'')\eta_1/J+b_0(y;\eta')$ corresponds to the y-space interpretation of the characteristic polynomial in the x-space, the form of $g(y;\eta)$ shows that the characteristic surfaces through $\tilde{T}$ are $\tilde{K}_1=\{y;\,y_1=0\}$ and $K_2=\{y;\,\tilde{\varphi}_2(y)=0\}$. These surfaces are regular. In fact, from the transversality of K_1 and K_2, we have on $\tilde{T}$

$$(4.6)\qquad \mathrm{grad}_y\,\hat{\varphi}_2(y)={}^t\Big(\frac{\partial x}{\partial y}\Big)\widetilde{\mathrm{grad}_x\,\varphi_2(x)}$$

$$= \begin{pmatrix} J & & 0 \\ 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} \overset{*}{\widetilde{\dfrac{\partial \varphi_2}{\partial x_2}}}(y) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \widetilde{\dfrac{\partial \varphi_2}{\partial x_2}}(y) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq 0 \,.$$

There is an another important fact we have to remark here. Namely, from the process of constructing K_1 and K_2, we have

$$\tilde{\varphi}_2(y) = y_1 \quad \text{on} \quad \tilde{S} \,. \tag{4.7}$$

We shall later use this.

We need one more transformation of the dependent variables to reduce the system (4.5) to a normal system. Let $G_\nu^\mu(y; \eta)$ be the (μ, ν)-cofactor of the matrix $(a_\mu^\nu(y; \eta))$. Introduce a new set of the dependent variables $(U_0(y), U_1(y))$ by

$$\tilde{u}_\nu(y) = \sum_{\substack{\mu=0 \\ (\nu=0,1)}}^{1} G_\nu^\mu\Big(y; \frac{\partial}{\partial y}\Big) U_\mu(y) + \tilde{w}_\nu(y) \,. \tag{4.8}$$

Then the system (4.5) becomes:

$$g\Big(y; \frac{\partial}{\partial y}\Big) U_\mu(y) + \sum_{\nu=0}^{1} C_\mu^\nu\Big(y; \frac{\partial}{\partial y}\Big) U_\nu(y) = V_\mu(y) \quad (\mu = 0, 1) \,, \tag{4.9}$$

where $C_\mu^\nu(y; \partial/\partial y)(\mu, \nu = 0, 1)$ are linear differential operators of the respective order $\leqq \mu - \nu + 1$ whose coefficients are holomorphic in a neighborhood of $y = 0$, and the inhomogeneous terms $V_\mu(y)(\mu = 0, 1)$ are defined by

$$\begin{cases} V_0(y) = 0 \,, \\ V_1(y) = -\displaystyle\sum_{\nu=0}^{1} a_1^\nu\Big(y; \frac{\partial}{\partial y}\Big) \tilde{w}_\nu(y) \,. \end{cases} \tag{4.10}$$

Now consider the Cauchy problem $(CP)_3$ for the system (4.9) with initial condition: All the derivatives of $U_\mu(y)$ up to order 1 vanish on

$\tilde{S}-\tilde{T}$. Then we have the following elementary lemma (cf. Wagschal [5]).

LEMMA 4.1. *If $U_\mu(y)(\mu=0,1)$ are the solutions of $(CP)_3$, then the solutions $\tilde{u}_\nu(y)(\nu=0,1)$ of $(CP)_2$ are given by the relations* (4.8).

PROOF. What we only have to prove is whether $\tilde{u}_\nu(y)(\nu=0,1)$ defined by the relations (4.8) satisfy the initial conditions of the Cauchy problem $(CP)_2$. Since the respective order of the differential operators $G^\mu_\nu(y;\partial/\partial y)(0\leqq \mu,\nu\leqq 1)$ is not greater than $1-\mu+\nu$, it is sufficient to prove that all the derivatives of $U_\mu(y)$ up to order $2-\mu$ vanish on $\tilde{S}-\tilde{T}$. This is obvious for $U_1(y)$. We assume that all the derivatives of $U_0(y)$ up to order λ vanish on $\tilde{S}-\tilde{T}$. Since $\tilde{S}$ is non-characteristic with respect to the operator $g(y;\partial/\partial y)$, $\lambda-2\geqq\min\{\lambda-1,1\}$ follows from the equation (4.9). Rewriting the inequality $\lambda-2\geqq\min\{\lambda-1,1\}$ in the form $\lambda-3\geqq\min\{\lambda-2,0\}$, we obtain $\lambda\geqq 2$. q.e.d.

Our next aim is to show that the Cauchy problem $(CP)_3$ admits a solution $(U_0(y),U_1(y))\in(H(pr,s)+H(p(r-1)+1,s), H(pr-1,s)+H(p(r-1),s))$.

5. **Construction of a formal solution of the Cauchy problem $(CP)_3$.** First we clarify the type of the inhomogeneous term $V_1(y)$. By (4.1) the initial data $w(x)$ is transformed into

$$w(x)=\begin{cases} x_1^r(\log x_1)^s G_1'(x) & \text{when } s\geqq 0 \text{ and either } \\ & r\in \boldsymbol{C}-\boldsymbol{Z} \text{ or } r\geqq 0\,, \\ x_1^r(\log x_1)^{s-1}G_1'(x) & \text{when } 0>r\in\boldsymbol{Z},\ s\geqq 1\,, \end{cases}$$

where $G_1'(x)$ is holomorphic at $x=0$. From this we can easily see that the y-space interpretation $\tilde{w}_\mu(y)$ of $(\partial^\mu/\partial x_1^\mu)w(x)$ does not include the function $\tilde{\varphi}_2(y)$ and it belongs to the space $H(p(r-\mu),s)$. Consequently the inhomogeneous term $V_1(y)=-\sum_{\nu=0}^1 a_1^\nu(y;\partial/\partial y)\tilde{w}_\nu(y)$ does not include $\tilde{\varphi}_2(y)$ and it belongs to the space $H(pr-2,s)+H(p(r-1),s)$.

According to the principle of superposition, we can simplify our problem stated at the end of the Section 4. Namely it is sufficient to prove that $(CP)_3$ admits a unique solution $(U_0(y),U_1(y))\in(H(q,s),H(q-1,s))$ for the given inhomogeneous terms $V_0(y)\equiv 0$ and

$$V_1(y)=\begin{cases} y_1^{q-2}(\log y_1)^s V_1^0(y) & \text{when } s\geqq 0 \text{ and either } \\ & q-2\in\boldsymbol{C}-\boldsymbol{Z} \text{ or } q-2\geqq 0\,, \\ y_1^{q-2}(\log y_1)^{s-1}V_1^0(y) & \text{when } 0>q-2\in\boldsymbol{Z},\ s\geqq 1\,, \end{cases}$$

where $V_1^0(y)$ is holomorphic at $y=0$. In this section we construct a formal solution for this simplified form of $(CP)_3$.

In accordance with the above $V_1(y)$, we define the wave forms $(f_k)_{k\in Z}$ as follows. Namely we define $(f_k)_{k\in Z}$ by the following relations:

$$f_{-2}(\zeta)=\begin{cases}\zeta^{q-2}(\log\zeta)^s & \text{when } s\geqq 0 \text{ and either} \\ & q-2\in \boldsymbol{C}-\boldsymbol{Z} \text{ or } q-2\geqq 0\,, \\ \zeta^{q-2}(\log\zeta)^{s-1} & \text{when } 0>q-2\in \boldsymbol{Z},\, s\geqq 1\,,\end{cases}$$

$$\frac{df_k}{d\zeta}(\zeta)=f_{k-1}(\zeta)\quad (k\in \boldsymbol{Z})\,.$$

Though there are many wave forms $(f_k)_{k\in Z}$ which satisfy the above relations, the explicit forms of $(f_k)_{k\in Z}$ are not so important in the following arguments.

Next we show that we can obtain a formal solution $U_\mu(y)$ of the form

$$U_\mu(y)=\sum_{k=0}^{\infty} f_{k-\mu}(y_1)U_{\mu,k}^{(1)}(y)+\sum_{k=0}^{\infty} f_{k-\mu}(\tilde{\varphi}_2(y))U_{\mu,k}^{(2)}(y)\,. \tag{5.1}$$

Since $y_1=\tilde{\varphi}_1(y)$, this becomes

$$U_\mu(y)=\sum_{j=1}^{2}\sum_{k=0}^{\infty} f_{k-\mu}(\tilde{\varphi}_j(y))U_{\mu,k}^{(j)}(y)\,. \tag{5.2}$$

In the following we shall frequently use this notation to simplify the discriptions.

Now, if we substitute (5.2) into the equations (4.9) and let the coefficients of $f_k(\tilde{\varphi}_j(y))(k\in \boldsymbol{Z};\, j=1,2)$ equal zero, we have the following transport equations (5.3) for the distortion factors $U_{\mu,k}^{(j)}(y)(\mu=0,1;\, j=1,2;\, k\in\boldsymbol{Z})$. Namely, with the convention $U_{\mu,k}^{(j)}(y)\equiv 0$ for $k<0$, $U_{\mu,k}^{(j)}(y)(\mu=0,1;\, j=1,2;\, k\in \boldsymbol{Z})$ satisfy

$$\begin{cases} M_1^{(j)}\Big(y;\dfrac{\partial}{\partial y}\Big)U_{\mu,0}^{(j)}(y)+\sum_{\nu=0}^{1} M_{\mu,0}^{(\nu,j)}\Big(y;\dfrac{\partial}{\partial y}\Big)U_{\nu,0}^{(j)}(y)=\delta_1^\mu\delta_1^j V_1^0(y)\,, \\ M_1^{(j)}\Big(y;\dfrac{\partial}{\partial y}\Big)U_{\mu,k}^{(j)}(y)+M_2^{(j)}\Big(y;\dfrac{\partial}{\partial y}\Big)U_{\mu,k-1}^{(j)}(y) \\ \quad +\sum_{\nu=0}^{1}\sum_{l=0}^{\mu-\nu+1} M_{\mu,l}^{(\nu,j)}\Big(y;\dfrac{\partial}{\partial y}\Big)U_{\nu,k-l}^{(j)}(y)=0\quad (k\geqq 1)\,,\end{cases} \tag{5.3}$$

where $M_l^{(j)}(y;\partial/\partial y)$ and $M_{\mu,l}^{(\nu,j)}(y;\partial/\partial y)$ are linear differential operators of order $\leqq l$ whose coefficients are holomorphic in a neighborhood of $y=0$ and, especially, $M_1^{(j)}(y;\partial/\partial y)=\sum_{i=1}^n(\partial/\partial\eta_i)g(y;\operatorname{grad}\tilde{\varphi}_j(y))(\partial/\partial y_i)+a^{(j)}(y)$.

We also reduce the initial conditions of $(CP)_3$ to the conditions for the distortion factors $U_{\mu,k}^{(j)}(y)(\mu=0,1;\, j=1,2;\, k\in\boldsymbol{Z})$. Taking account of the fact (4.7), we have from the form (5.2) the following conditions (5.4) for the distortion factors $U^{(j)}_{\mu,k}(y)(\mu = 0, 1;\ j = 1, 2;\ k \in \boldsymbol{Z})$, by letting the coefficients of $f_k(\tilde{\varphi}_j(y))(k \in \boldsymbol{Z};\ j = 1, 2)$ equal zero. Namely $U^{(j)}_{\mu,k}(y)(\mu = 0, 1;\ j = 1, 2;\ k \in \boldsymbol{Z})$ satisfy

$$(5.4)\qquad \begin{cases} \sum_{j=1}^{2} [\tilde{D}_0\tilde{\varphi}_j(y)]^h U^{(j)}_{\mu,0}(y) = 0 \quad \text{on} \quad \tilde{S} \quad (h = 0, 1)\,, \\ \begin{cases} \sum_{j=1}^{2} U^{(j)}_{\mu,k}(y) = 0 \\ \sum_{j=1}^{2} [\tilde{D}_0\tilde{\varphi}_j(y)] U^{(j)}_{\mu,k} = - \sum_{j=1}^{2} P^{(j)}_1(y;\ D_0) U^{(j)}_{\mu,k-1}(y) \\ \qquad\qquad\qquad \text{on} \quad \tilde{S} \quad (k \geqq 1)\,, \end{cases} \end{cases}$$

where $\tilde{D}_0$ is the normal derivative of $\tilde{S}$ and $P^{(j)}_l(y;\ \tilde{D}_0)$ is a linear differential operator of order $\leqq l$ whose coefficients are holomorphic at $y = 0$.

It is a routine calculation to derive the equations (5.3) and (5.4). For further details, see Hamada [2] and Wagschal [5].

Now let us complete the proof of our assertion. It is enough to show that we can determine the distortion factors $U^{(j)}_{\mu,k}(y)$ successively from (5.3) and (5.4). To see this, we denote the characteristic polynomials of (5.3) by $m^{(j)}_1(y;\ \eta)(j = 1, 2)$. From the facts (4.6) and $g(y;\ \eta) = \eta_1\eta_2 + b_1(y;\ \eta'')\eta_1 + Jb_0(y;\ \eta')$ we have on $\tilde{T}$:

$$m^{(j)}_1(y;\ \eta)|_{\eta=(1,-1,0,\cdots,0)} = \begin{cases} -1 & \text{when} \quad j = 1\,, \\ \widetilde{\dfrac{\partial\varphi_2}{\partial x_2}}(y) \neq 0 & \text{when} \quad j = 2\,. \end{cases}$$

Thus the initial manifold $S;\ y_1 - y_2 = 0$ is non-characteristic for (5.3). On the other hand we have on $\tilde{T}$:

$$\tilde{D}_0\tilde{\varphi}_j(y) = \frac{\partial}{\partial y_1}\tilde{\varphi}_j(y) = \begin{cases} 1 & \text{when} \quad j = 1\,, \\ 0 & \text{when} \quad j = 2\,. \end{cases}$$

Thus the coefficient matrix $\begin{bmatrix} 1 & 1 \\ \tilde{D}_0\tilde{\varphi}_1(y) & \tilde{D}_0\tilde{\varphi}_2(y) \end{bmatrix}$ of (5.4) is non-singular in a neighborhood of $\tilde{T}$. Hence, from Cauchy-Kowalevskaya's theorem, we can determine the distortion factors $U^{(j)}_{\mu,k}(y)$ from (5.3) and (5.4).

6. Convergence of the formal solution. An argument analogous to Wagschal ([5] p. 385 ~ 391) shows that the formal solutions $U_\mu(y)(\mu = 0, 1)$ defined by (5.1) belong to the space $H(q - \mu,\ s)$, respectively. We shall reproduce its essence for the convenience of the readers. For further details, the readers should refer to Wagschal's paper [5].

First, we choose a suitable local coordinate system and rewrite the equations (5.3) and (5.4) in the normal form. In order to explain this we also denote by y an another local coordinate system in which the initial manifold $\tilde{S}$ is given by $y_1 = 0$. For simplifying the notations we shall use the same symbols $U^{(j)}_{\mu,k}(y)(\mu=0, 1; j=1, 2; k \in \boldsymbol{Z})$ and $\tilde{\varphi}_j(y)(j=1, 2)$ to denote the corresponding distortion factors and phase functions, respectively. In the following the symbol y always stands for the new local coordinate system introduced above.

Since the initial manifold is non-characteristic for the transport equations (5.3), we have from (5.3) the following equations for the distortion factors $U^{(j)}_{\mu,k}(y)(\mu = 0, 1; j = 1, 2; k \in \boldsymbol{Z})$. Namely, with the convention $U^{(j)}_{\mu,k}(y) \equiv 0$ for $k < 0$, the distortion factors $U^{(j)}_{\mu,k}(y)$ satisfy:

$$
(6.1)\quad \begin{cases} \dfrac{\partial}{\partial y_1} U^{(j)}_{\mu,0}(y) = N^{(j)}_1\Big(y; \dfrac{\partial}{\partial y'}\Big) U^{(j)}_{\mu,0}(y) \\ \qquad\qquad + \sum\limits_{\nu=0}^{1} N^{(\nu,j)}_{\mu,0}\Big(y; \dfrac{\partial}{\partial y}\Big) U^{(j)}_{\nu,0}(y) + \delta^{\mu}_1 \delta^{j}_1 W^0_1(y)\,, \\ \dfrac{\partial}{\partial y_1} U^{(j)}_{\mu,k}(y) = N^{(j)}_1\Big(y; \dfrac{\partial}{\partial y'}\Big) U^{(j)}_{\mu,k}(y) + N^{(j)}_2\Big(y; \dfrac{\partial}{\partial y}\Big) U^{(j)}_{\mu,k-1}(y) \\ \qquad\qquad + \sum\limits_{\nu=0}^{1} \sum\limits_{l=0}^{\mu-\nu+1} N^{(\nu,j)}_{\mu,l}\Big(y; \dfrac{\partial}{\partial y}\Big) U^{(j)}_{\nu,k-l}(y) \quad (k \geqq 1)\,, \end{cases}
$$

where $N^{(j)}_1(y; \partial/\partial y')$, $N^{(j)}_2(y; \partial/\partial y)$ and $N^{(\nu,j)}_{\mu,l}(y; \partial/\partial y)$ are linear differential operators of order $\leqq l$ whose coefficients are holomorphic in a neighborhood of $y = 0$ and $W^0_1(y)$ is a holomorphic function defined in a neighborhood of $y=0$.

If we solve the equations (5.4) with respect to the distortion factors by Cramer's method, we have the following conditions for $U^{(j)}_{\mu,k}(y)$.

$$
(6.2)\quad \begin{cases} U^{(j)}_{\mu,0}(0, y') = 0\,, \\ U^{(j)}_{\mu,k}(0, y') = \sum\limits_{j=1}^{2} Q^{(j)}_1\Big(y'; \dfrac{\partial}{\partial y_1}\Big) U^{(j)}_{\mu,k-1}(y) \quad (k \geqq 1)\,, \end{cases}
$$

where $Q^{(j)}_1(y'; \partial/\partial y_1)$ is a linear differential operator of order $\leqq 1$ whose coefficients are holomorphic at $y' = 0$.

For the later use we denote here without proof an essential lemma due to Wagschal [5] paraphrased in case of our recurrent formulas (6.1) and (6.2).

LEMMA 6.1. *Let*

$$
\psi_k(\zeta) = \Big(\frac{d}{d\zeta}\Big)^k \frac{1}{r - \zeta} = \frac{k!}{(r - \zeta)^{k+1}} \quad (k = 0, 1, 2, \cdots)\,.
$$

Then there exist constants $R > 0$, $\rho \geqq 1$ *and* $c_1 \geqq 1$ *such that the relations*

$$U^{(j)}_{\mu,k}(y) \ll c_1^{k+1}\psi_k\Big(\rho y_1 + \sum_{i=2}^{n} y_i\Big) \qquad (\mu = 0, 1;\ j = 1, 2;\ k \in \boldsymbol{Z})$$

are valid for any $0 < r < R$.

Now, from the definition of the forms $f_k(\zeta)(k \in \boldsymbol{Z})$, we can easily see that each $f_{k-\mu}(\zeta)(k \geqq K)$ takes the form

$$f_{k-\mu}(\zeta) = \zeta^{q+k-\mu} \sum_{l=0}^{s} \frac{1}{l!} c_{k-\mu,l} (\log \zeta)^l\,, \quad c_{k-\mu,l} \in \boldsymbol{C} \tag{6.3}$$

when K is an integer such that $\operatorname{Re} q + K - \mu \geqq 0$ is valid. In the following we fix K for a while. If we substitute (6.3) into $\sum_{k=K}^{\infty} f_{k-\mu}(\tilde{\varphi}_j(y)) U^{(j)}_{\mu,k}(y)$ and rearrange the terms formally, we have

$$\sum_{k=K}^{\infty} f_{k-\mu}(\tilde{\varphi}_j(y)) U^{(j)}_{\mu,k}(y) = [\tilde{\varphi}_j(y)]^{q-\mu} \sum_{l=0}^{s} \frac{1}{l!} [\log \tilde{\varphi}_j(y)]^l \times \Big\{\sum_{k=K}^{\infty} c_{k-\mu,l} [\tilde{\varphi}_j(y)]^k U^{(j)}_{\mu,k}(y)\Big\}\,.$$

Next we prove the convergence of the series

$$\sum_{k=K}^{\infty} c_{k-\mu,l} [\tilde{\varphi}_j(y)]^k U^{(j)}_{\mu,k}(y)\,. \tag{6.4}$$

To see this we also denote without proof an elementary lemma due to Wagschal [5] which gives an estimate of the coefficients $c_{k-\mu,l}(k \geqq K;\ 0 \leqq l \leqq s)$.

LEMMA 6.2. *Define* T_k *by* $T_k = \max_{0 \leqq l \leqq s} |c_{k-\mu,l}|$ *and denote the greatest integer* $\operatorname{Re} q - \mu$ *by* q_0. *There exist a constant* $c_2 > 0$ *such that*

$$M_k \leqq \frac{c}{(k + q_0)!} \qquad (k \geqq K)\,.$$

Combining the Lemmas 6.1 and 6.2, we can easily see the convergence of the series (6.4). In fact, from those lemmas, we obtain:

$$|c_{k-\mu,l}[\tilde{\varphi}_j(y)]^k U^{(j)}_{\mu,k}(y)| \leqq \frac{k!}{(k + q_0)!} \Big\{\frac{c_1 c_2}{r - (\rho|y_1| + \sum_{i=2}^{n} |y_i|)}\Big\}^{k+1} |\tilde{\varphi}_j(y)|^k$$

$$(j = 1, 2;\ k \geqq K;\ 0 \leqq l \leqq s)\,.$$

Hence it immediately follows that (6.4) converges compact uniformly in a neighborhood of $y = 0$ where $c_1 c_2 |\tilde{\varphi}_j(y)| < r - (\rho|y_1| + \sum_{i=2}^{n} |y_i|)$ holds.

Finally we shall prove that the formal solution $U_\mu(y)(\mu = 0, 1)$ belongs to the space $H(q - \mu, s)$, respectively. To see this, we consider three cases in comparison with the definition of the space $H(q - \mu, s)$. Namely,

1. the case when $q - \mu \in \boldsymbol{C} - \boldsymbol{Z}$ or $q - \mu \geqq 0$,

2. the case when $0 > q - \mu \in \boldsymbol{Z}$ and $s \geqq 1$,

and

3. the case when $0 > q - \mu \in \boldsymbol{Z}$ and $s = 0$.

We only illustrate $U_\mu(y) \in H(q-\mu, s)(\mu = 0, 1)$ in the second case. The others can be proved much easily.

Set $K = -q + \mu$. Then we have from the definition of the wave forms that we can write $f_{k-\mu}(\zeta)(k \geqq 0)$ in the following forms (6.5). Namely, taking account of the facts $0 > q + k - \mu \in \boldsymbol{Z}$ for $0 \leqq k \leqq K-1$ and $0 \leqq q + k - \mu \in \boldsymbol{Z}$ for $k \geqq K$, we have with $c_{k-\mu,l} \in \boldsymbol{Z}$

$$f_{k-\mu}(\zeta) = \begin{cases} \zeta^{q+k-\mu} \sum_{l=0}^{s-1} \frac{1}{l!} c_{k-\mu,l} (\log \zeta)^l\,, & (0 \leqq k \leqq K-1) \\ \zeta^{q+k-\mu} \sum_{l=0}^{s} \frac{1}{l!} c_{k-\mu,l} (\log \zeta)^l & (k \geqq K)\,. \end{cases} \tag{6.5}$$

If we substitute (6.5) into (5.1) interpreted by the new local coordinate system y, we have

$$\begin{aligned} U_\mu(y) = \sum_{j=1}^{2} \Big\{ &[\tilde{\varphi}_j(y)]^{q-\mu} \sum_{l=0}^{s-1} \frac{1}{l!} [\log \tilde{\varphi}_j(y)]^l \Big(\sum_{k=0}^{K-1} c_{k-\mu,l} [\tilde{\varphi}_j(y)]^k \times U_{\mu,k}^{(j)}(y) \Big) \\ &+ \sum_{l=0}^{s} \frac{1}{l!} [\log \tilde{\varphi}_j(y)]^l \Big(\sum_{k=K}^{\infty} c_{k-\mu,l} [\tilde{\varphi}_j(y)]^{q+k-\mu} U_{\mu,k}^{(j)}(y) \Big) \Big\}\,. \end{aligned} \tag{6.6}$$

Thus, taking account of the convergence of the series (6.4), the assertion $U_\mu(y) \in H(q-\mu, s)$ immediately follows from (6.6). This is also true with respect to the old local coordinate system y since the space $H(q-\mu, s)$ is invariant under coordinate transformations.

7. Return to the proof of the theorem. We are now in the position to complete the proof of our theorem. Let's start by reminding what we have proved in the previous sections. We have proved that $(CP)_3$ admits a solution $(U_0(y), U_1(y)) \in (H(pr, s) + H(p(r-1)+1, s), H(pr-1, s) + H(p(r-1), s))$ for the given inhomogeneous terms $(V_0(y), V_1(y))$ defined by (4.10). Consequently we can easily see from the Lemma 4.1 that $(CP)_2$ admits a unique solution $(\tilde{u}_0(y), \tilde{u}_1(y))$ such that

$$\tilde{u}_\nu(y) - \tilde{w}_\nu(y) \in H(pr-1-\nu, s) + H(p(r-1)-\nu, s) \qquad (\nu = 0, 1)\,.$$

Here the uniqueness of our solution follows from Cauchy-Kowalevskaya's theorem for Volevič systems generalized by Gårding, Kotake and Leray [1].

Since $(CP)_1$ and $(CP)_2$ are equivalent, Cauchy-Kowalevskaya's theorem insures that the x-space interpretation $u(x)$ of $\tilde{u}_0(y)$ is a unique solution of $(CP)_1$ for the given data mentioned in the assertion (3) of the theorem.

Taking account of the coordinate transformation (4.1) and the facts $\tilde{\varphi}_1(y) = \varphi_1(x)^{1/p}$ and $\tilde{\varphi}_2(y) = \varphi_2(x)$, we obtain a unique solution $u(x) \in H(p(r-1), s, p) + H(pr-1, s, p) + H(r, s)$ of $(CP)_1$. This completes the proof of our theorem.

8. **Some examples.** We give two examples when the number of the independent variables is two and K, S, T are given respectively by $K = \{(x, y); y = 0\}$, $S = \{(x, y); y - x^2 = 0\}$, $T = \{(x, y); x = y = 0\}$.

EXAMPLE 1. Consider the Cauchy problem:

$$(*) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{y^l}, \\ u = \dfrac{\partial u}{\partial y} = 0 \quad \text{on} \quad S, \end{cases}$$

where $l \in N$.

Since the fact that the inhomogeneous term belongs to $H(-l, 0)$ is invariant under any coordinate transformation, the part (1) of our theorem shows that the Cauchy problem $(*)$ has a unique solution in the class $H(-2l+1, 0, 2)$.

On the other hand a simple calculation shows that the solution $u(x, y)$ of the Cauchy problem $(*)$ is given by

$$u(x, y) = -\frac{1}{l-1}\frac{x}{y^{l-1}} + \frac{1}{l-\frac{3}{2}}\frac{1}{y^{l-3/2}} - \frac{1}{(l-1)(2l-3)}\frac{1}{x^{2l-3}}.$$

This also belongs to $H(-2l+1, 0, 2)$.

EXAMPLE 2. Consider the Cauchy problem:

$$(**) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 0 \\ u|_S = 0, \quad \left.\dfrac{\partial u}{\partial y}\right|_S = \dfrac{1}{y^l}, \end{cases}$$

where $l \in N$.

We introduce the following local coordinate system $z = y - x^2$, $w = x$. In this system $(**)$ reduces to the following:

$$\begin{cases} \text{a second order equation}, \\ \text{all the derivatives of } u(z, w) - \dfrac{z}{(z+w^2)^l} \text{ up to order } 1 \\ \text{vanishes on } z = 0 \text{ except where } z = w = 0. \end{cases}$$

Here we have denoted the unknown function in the (z, w)-space by $u(z, w)$.

Since the surface $z+w^2=0$ is the characteristic which touches the initial manifold $z=0$, the part (3) of our theorem shows that $(**)$ has a unique solution in the class $H(2(-l-1), 0, 2)+H(-2l-1, 0, 2)+H(-l, 0)$.

On the other hand a simple calculation shows that the solution $u(x, y)$ of $(**)$ is given by

$$u(x, y)=-\frac{1}{l-1}\frac{1}{y^{l-1}}+\frac{1}{l-1}\frac{1}{x^{2(l-1)}}.$$

This also belongs to $H(2(-l-1), 0, 2)+H(-2l-1, 0, 2)+H(-l, 0)$.

REMARK. Before we end this note, we should like to give a remark about our results. For an equation whose order is higher than two, one can no longer determine the distortion factors from the transport equations. For this reason, we can not apply our method to a higher order equation.

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