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CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS BY COMPACT GROUPS

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Since the crossed products of operator algebras were first defined and investigated explicitly by T. Turumaru [14] in 1958, most of theories on them have been restricted to the crossed products by discrete groups (for the literature, see [11] and [5]). It was 1966 that the crossed products of C^*-algebras by continuous groups were first proposed by S. Doplicher, D. Kastler and D. Robinson [4] under the terminology "covariance algebra". (They discussed the case of abelian groups. For the non abelian case, see [13]). The definition of the crossed product of a von Neumann algebra by a locally compact group has been given recently by M. Takesaki [11]. Applying the Tomita-Takesaki theory [10], he proved a duality theorem for the crossed products and succeeded to show a remarkable structure theorem for von Neumann algebras of type III.

Now, though the presentation of the general theory of continuous crossed products is desirable, it is not such an easy task to analyze them mainly because we have no explicit description of elements in the crossed product contrary to the discrete case. Up to this time, little is known about continuous crossed products of von Neumann algebras except the fundamental results in [11]. In the present paper, we restrict our interest to the simple case where automorphism groups are compact, and we investigate the structure of the crossed products of von Neumann algebras.

Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$, equipped with a continuous action of a compact group G. The crossed product $\mathscr{R}(\mathscr{A}, G)$ is defined on the Hilbert space $L^2(\mathscr{H}, G)$, the space of all $\mathscr{H}$-valued square-integrable functions on G, as given in [11]. In addition, we define in §1 another crossed product $\mathscr{R}'(\mathscr{A}, G)$ on the same space $L^2(\mathscr{H}, G)$ which is spatially isomorphic to $\mathscr{R}(\mathscr{A}, G)$. In §2, making use of the Fourier expansion of vector-valued functions in $L^2(\mathscr{H}, G)$, we decompose the Hilbert space $L^2(\mathscr{H}, G)$ into a family of mutually orthogonal subspaces. Then, considering the second crossed product $\mathscr{R}'(\mathscr{A}, G)$ connected with this decomposition of the space, we get a structure theorem which asserts that we essentially need not extend the original Hilbert space $\mathscr{H}$ to $L^2(\mathscr{H}, G)$ so far as we concern with the crossed products by compact groups (Theorem 2.2). Subsequently in § 3, we consider the first crossed product $\mathscr{R}(\mathscr{A}, G)$ connected with the same decomposition of the space, and we have another structure theorem (Theorem 3.1) which, we have heard, is essentially a result proven by A. Connes.

The conditions sufficient for the conclusions of our theorems are conceived to be related deeply to the free action of the automorphisms. We discuss them to some extent. It seems difficult but of interest to make further investigation into the property of the free action of continuous automorphism groups.

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1. Crossed products. We treat only compact groups which satisfy the second axiom of countability and separable Hilbert spaces. Hence, omitting the adjectives for simplicity, a compact group means always such one that satisfies the second axiom of countability, and a Hilbert space means a separable one throughout this paper. Under these situations, we first quote the definition of the crossed product given in [11; § 3] in a simplified form. Given a compact group G and a Hilbert space $\mathscr{H}$, we denote by $\mathscr{K}(\mathscr{H}, G)$ the vector space of all $\mathscr{H}$-valued continuous functions on G. Consider the inner product in $\mathscr{K}(\mathscr{H}, G)$ defined by

$$(\xi \mid \eta) = \int_G (\xi(g) \mid \eta(g)) dg ,$$

where $\xi = \xi(g)$ and $\eta = \eta(g)$ are arbitrary functions in $\mathscr{K}(\mathscr{H}, G)$ and dg denotes the normalized Haar measure on G. The completion of $\mathscr{K}(\mathscr{H}, G)$ with respect to this inner product is denoted by $L^2(\mathscr{H}, G)$ which is also a separable Hilbert space. Therefore, each element ξ in $L^2(\mathscr{H}, G)$ is an $\mathscr{H}$-valued measurable function, i.e. $(\xi(g) \mid \eta)$ is measurable for each $\eta \in \mathscr{H}$ and

$$\|\xi\|^2 = \int_G \|\xi(g)\|^2 dg < +\infty .$$

Let $\mathscr{A}$ be a von Neumann algebra acting on $\mathscr{H}$, and G a compact group. A continuous action of G on $\mathscr{A}$ means a homomorphism of $\mathscr{A}$ into the group of all automorphisms of $\mathscr{A}$ such that for each $A \in \mathscr{A}$, the map $g \in G \rightarrow g(A) \in \mathscr{A}$ is σ-strongly* continuous. For simplicity, no notational distinction will be made between group element g and the corresponding automorphism $A \to g(A)$. On the Hilbert space $L^2(\mathscr{H}, G)$, we define representations π of $\mathscr{A}$ and λ of G as follows:

$$(\pi(A)\xi)(g) = g^{-1}(A)\xi(g)\,, \quad A \in \mathscr{A};$$
$$(\lambda(h)\xi)(g) = \xi(h^{-1}g) \qquad h \in G,\ \xi \in L^2(\mathscr{H}, G)\,.$$

Then π is a normal faithful representation and

$$\lambda(h)\pi(A)\lambda(h)^* = \pi(h(A))\,, \quad A \in \mathscr{A},\ h \in G\,.$$

Obviously, $\lambda(e) = \pi(I)$ is the identity operator on $L^2(\mathscr{H}, G)$, where e denotes the unity of G and I the identity operator on $\mathscr{H}$. The crossed product of $\mathscr{A}$ by G is defined as the von Neumann algebra on $L^2(\mathscr{H}, G)$ generated by $\pi(\mathscr{A})$ and $\lambda(G)$, and denoted by $\mathscr{R}(\mathscr{A}, G)$.

We resume the fundamental results in [11; § 5] which are available without the assumption of the commutativity of G. In [11], the relative invariance of faithful normal semi-finite weight ϕ on $\mathscr{A}$ under a locally compact group G plays an important role for the structure theorems. We note here, however, that under a compact group G the relative invariance coincides with the invariance. In fact, if ϕ is relatively invariant under G, i.e., if there exists a continuous positive character χ of G such that $\phi\circ g = \chi(g)\phi$ for any $g \in G$, then $\phi\circ g^n = (\chi(g))^n\phi$ for $n = 1, 2, \cdots$. Let g_0 be an accumulating point of the set $\{g^n\}_{n=1,2,\cdots}$. By the continuity of the action of G with respect to g, we have $\phi\circ g_0(A) = \chi(g_0)\phi(A) = 0$ or ∞ for all $A \geqq 0$, $A \in \mathscr{A}$ if $\chi(g) \neq 1$. But it is impossible since ϕ is faithful normal and semi-finite. Hence $\chi(g) = 1$ for all $g \in G$. This shows our assertion. Now, the algebraic structure of the crossed product $\mathscr{R}(\mathscr{A}, G)$ is independent of the particular representation space $\mathscr{H}$ of A to construct it ([11; Prop. 3.4]). Therefore, under the assumption of the existence of a faithful semi-finite normal weight ϕ which is invariant under G, we may and do assume that there exists a maximal Tomita algebra $\mathfrak{A}$ such that $\mathscr{A}$ is spatially isomorphic to the left von Neumann algebra $\mathscr{L}(\mathfrak{A})$ of $\mathfrak{A}$, that there exists a continuous representation U_g of G such that $g(A) = U_g A U_g^*$, and that there exists a unitary involution J in $\mathscr{H}$ such that $J\mathscr{A}J = \mathscr{A}'$ and $J\mathscr{A}'J = \mathscr{A}$. We denote by $\mathscr{K}(\mathfrak{A}, G)$ the vector space of all continuous $\mathfrak{A}$-valued functions $A(g)$ on G and define the inner product of $A = A(g)$ and $B = B(g)$ in $\mathscr{K}(\mathfrak{A}, G)$ by

$$(A \mid B) = \int_G (A(g) \mid B(g))dg\,.$$

The completion of $\mathscr{K}(\mathfrak{A}, G)$ is nothing but $L^2(\mathscr{H}, G)$. We consider the algebraic structure in $\mathscr{K}(\mathfrak{A}, G)$ defined by the following:

$$(AB)(g) = \int_G h^{-1}(A(gh^{-1}))B(h)dh\,,$$
$$A^{\#}(g) = g^{-1}A(g^{-1})^{\#}\,.$$

Then the involution algebra $\mathscr{K}(\mathfrak{A}, G)$ is a Tomita algebra and its left von Neumann algebra $\mathscr{L}(\mathscr{K}(\mathfrak{A}, G))$ coincides with the crossed product $\mathscr{R}(\mathscr{A}, G)$ ([11; Th. 5.11 and 5.12]).

The commutant $\mathscr{R}(\mathscr{A}, G)'$ of $\mathscr{R}(\mathscr{A}, G)$ is generated by the operators $\pi'(A)$, $A \in \mathscr{A}'$ and $\lambda'(h)$, $h \in G$ which are defined as follows:

$$\left.\begin{aligned}&(\pi'(A)\xi)(g) = A\xi(g);\\&(\lambda'(h)\xi)(g) = U_h\xi(gh), \quad \xi \in L^2(\mathscr{H}, G), \quad g \in G\end{aligned}\right\} \quad (1)$$

([11; Cor. 5.13]). The unitary involution $\tilde{J}$ in $L^2(\mathscr{H}, G)$ given by

$$(\tilde{J}(\xi))(g) = U_g^* J\xi(g^{-1})$$

satisfies

$$\tilde{J}\mathscr{R}(\mathscr{A}, G)\tilde{J} = \mathscr{R}(\mathscr{A}, G)' \quad \text{and} \quad \tilde{J}\mathscr{R}(\mathscr{A}, G)'\tilde{J} = \mathscr{R}(\mathscr{A}, G)\,.$$

Suggested by this expression of the commutant, we define another crossed product $\mathscr{R}'(\mathscr{A}, G)$ as the von Neumann algebra generated by the operators $\pi'(A)$ and $\lambda'(h)$ defined as (1), but $A \in \mathscr{A}$ this time. Hence,

$$\mathscr{R}(\mathscr{A}, G)' = \mathscr{R}'(\mathscr{A}', G)\,.$$

Thus we have defined two crossed products $\mathscr{R}(\mathscr{A}, G)$ and $\mathscr{R}'(\mathscr{A}, G)$. But direct calculations show the following lemma.

LEMMA 1.1. *The crossed products $\mathscr{R}(\mathscr{A}, G)$ and $\mathscr{R}'(\mathscr{A}, G)$ are spatially isomorphic by the unitary involution W defined by*

$$(W\xi)(g) = U_g^*\xi(g^{-1})\,.$$

The automorphisms $g \in G$ of $\mathscr{A}$ are extended to the automorphisms of $\mathscr{L}(\mathscr{H})$, the algebra of all bounded operators on $\mathscr{H}$, induced by the unitary representation U_g of G which in turn define the automorphisms of $\mathscr{A}'$ denoted also by g. Then we have

$$W\mathscr{R}(\mathscr{A}, G)W = \mathscr{R}'(\mathscr{A}, G) = \mathscr{R}(\mathscr{A}', G)'$$

and

$$W\mathscr{R}(\mathscr{A}', G)W = \mathscr{R}'(\mathscr{A}', G) = \mathscr{R}(\mathscr{A}, G)'\,.$$

Finally, if $\mathscr{A}$ is a von Neumann algebra with faithful semi-finite normal trace τ, then the trace τ_G defined by $\tau_G(A) = \int_G \tau(g(A))dg$, $A \in \mathscr{A}$

is invariant under G, and the discussion in §5 of [11] is reduced to such a trivial case that the modular operator with respect to τ_G is the identity. In this case, $\mathscr{K}(\mathfrak{A}, G)$ is a (unimodular) Hilbert algebra and its left von Neumann algebra coincides with the crossed product $\mathscr{R}(\mathscr{A}, G)$. Hence we have;

LEMMA 1.2. *If G is compact and $\mathscr{A}$ has a faithful semi-finite normal trace, then the crossed product $\mathscr{R}(\mathscr{A}, G)$ is also semi-finite.*

The canonical trace $\tilde{\tau}$ on $\mathscr{R}(\mathscr{A}, G)$ associated with the Hilbert algebra $\mathscr{K}(\mathfrak{A}, G)$ is said to be dual to the original trace τ on $\mathscr{A}$. Denoting by π_l the left regular representation with respect to τ, the dual trace $\tilde{\tau}$ is given by

$$\tilde{\tau}(A) = \begin{cases} \|\xi\|^2 & \text{if } A = \pi_l(\xi)^*\pi_l(\xi) \text{ for a bounded} \\ & \text{element } \xi \in L^2(\mathscr{H}, G)\,, \\ +\infty & \text{otherwise} \end{cases}$$

([11; Def. 5.14] and [3; I. 6.2]).

2. **Decomposition of the Hilbert space $L^2(\mathscr{H}, G)$.** The main tool we use is the fundamental Fourier analysis. We quote from [7; §27] what we need. See also [12]. Let Γ be the dual object of a compact group G, i.e. the set of all equivalence classes of continuous irreducible unitary representations of G. The identity representation of G is denoted by ι. Let $U_g^{(\alpha)}$ be a representation in the class $\alpha \in \Gamma$, d_α the finite dimension of the representation space H_α of $U^{(\alpha)} \in \alpha$, and $\{\zeta_1, \zeta_2, \cdots, \zeta_{d_\alpha}\}$ an arbitrary but fixed orthogonal basis in H_α. For $m, n \in \{1, 2, \cdots, d_\alpha\}$, let $u_{mn}^{(\alpha)}$ be the function on G defined by

$$u_{mn}^{(\alpha)}(g) = d_\alpha^{1/2}(U_g^{(\alpha)}\zeta_n \mid \zeta_m)\,.$$

Since $U^{(\alpha)}$ is a unitary representation, we have

$$u_{mn}^{(\alpha)}(gh) = d_\alpha^{-1/2} \sum_{r=1}^{d_\alpha} u_{mr}^{(\alpha)}(g) u_{rn}^{(\alpha)}(h)\,,$$

and

$$u_{mn}^{(\alpha)}(g^{-1}) = \overline{u_{nm}^{(\alpha)}(g)}\,, \quad g, h \in G\,.$$

The function on G defined by

$$\chi_\alpha(h) = d_\alpha^{-1/2} \sum_{m=1}^{d_\alpha} u_{mm}^{(\alpha)}(h)$$

is called the character of the equivalence class α.

The well-known Peter-Weyl theorem states that the set $\{u_{mn}^{(\alpha)} \mid \alpha \in \Gamma;\ m, n = 1, 2, \cdots, d_\alpha\}$ is an orthonormal basis for $L^2(G)$. Thus, we have

$$\int_G u_{mn}^{(\alpha)}(g)\overline{u_{pq}^{(\beta)}(g)}dg = \delta_{\alpha\beta}\delta_{mp}\delta_{nq} ,$$

and, for $x \in L^2(G)$, we have

$$x = \sum_{\alpha \in \Gamma} \sum_{m,n=1}^{d_\alpha} (x \mid u_{mn}^{(\alpha)})u_{mn}^{(\alpha)} ,$$

where

$$(x \mid u_{mn}^{(\alpha)}) = \int_G x(h)\overline{u_{mn}^{(\alpha)}(h)}\, dh .$$

Furthermore, a generalization of the Plancherel theorem holds as follows. If $\{a_{mn}^{(\alpha)} \mid \alpha \in \Gamma;\ m, n = 1, 2, \cdots, d_\alpha\}$ is any set of complex numbers such that

$$\sum_{\alpha \in \Gamma} \sum_{m,n=1}^{d_\alpha} |a_{mn}^{(\alpha)}|^2 < \infty ,$$

there is a unique function y in $L^2(G)$ such that $(y \mid u_{mn}^{(\alpha)}) = a_{mn}^{(\alpha)}$ for all α, m and n, and for which accordingly

$$y = \sum_{\alpha \in \Gamma} \sum_{m,n} a_{mn}^{(\alpha)}u_{mn}^{(\alpha)} .$$

Now, we begin with a decomposition of the Hilbert space $L^2(\mathscr{H}, G)$. Analogously to the Peter-Weyl theorem, we may consider the formal expansion of any function $\xi = \xi(g) \in L^2(\mathscr{H}, G)$:

$$\xi = \sum_{\alpha \in \Gamma} \sum_{m,n=1}^{d_\alpha} u_{mn}^{(\alpha)}\eta_{mn}^{(\alpha)} ,$$

where

$$\eta_{mn}^{(\alpha)} = \int_G \overline{u_{mn}^{(\alpha)}(h)}\, \xi(h)dh \in \mathscr{H} .$$

For any fixed $\xi \in \mathscr{H}$, we denote by $u_{mn}^{(\alpha)}\tilde{\xi}$ the function defined by $(u_{mn}^{(\alpha)}\tilde{\xi})(g) = u_{mn}^{(\alpha)}(g)\xi$. In particular, $\tilde{\xi}$ denotes the constant function $\tilde{\xi}(g) = \xi$. The set of (α, m, n)-components of all $\xi \in L^2(\mathscr{H}, G)$ is denoted by $\mathscr{H}_{mn}^{(\alpha)}$;

$$\mathscr{H}_{mn}^{(\alpha)} = \{u_{mn}^{(\alpha)}\tilde{\xi} \mid \xi \in \mathscr{H}\} , \quad \alpha \in \Gamma \quad \text{and} \quad m, n = 1, 2, \cdots, d_\alpha .$$

Then, the set of all $\mathscr{H}_{mn}^{(\alpha)}$ is a family of mutually orthogonal Hilbert subspaces of $L^2(\mathscr{H}, G)$ each of which is isometrically isomorphic to $\mathscr{H}$. In fact,

$$\begin{aligned}(u_{mn}^{(\alpha)}\tilde{\xi} \mid u_{pq}^{(\beta)}\tilde{\eta}) &= \int_G (u_{mn}^{(\alpha)}(g)\xi \mid u_{pq}^{(\beta)}(g)\eta)dg \\ &= \int_G u_{mn}^{(\alpha)}(g)\overline{u_{pq}^{(\beta)}(g)}dg(\xi \mid \eta) \\ &= \delta_{\alpha\beta}\delta_{mp}\delta_{nq}(\xi \mid \eta) .\end{aligned}$$

The projection onto each $\mathscr{H}_{mn}^{(\alpha)}$ is not in $\mathscr{R}(\mathscr{A}, G)$, since $\mathscr{H}_{mn}^{(\alpha)}$ is not invariant by $\mathscr{R}(\mathscr{A}, G)'$. However, the subspaces $\mathscr{H}_\alpha$ defined for each $\alpha \in \Gamma$ by

$$\begin{aligned}\mathscr{H}_\alpha &= \sum_{m,n=1}^{d_\alpha} \oplus \mathscr{H}_{mn}^{(\alpha)} \\ &= \{\sum_{m,n} u_{mn}^{(\alpha)} \tilde{\xi}_{mn}^{(\alpha)} \mid \xi_{mn}^{(\alpha)} \in \mathscr{H} \quad \text{for} \quad m, n = 1, 2, \cdots, d_\alpha\}\end{aligned}$$

are invariant by $\mathscr{R}(\mathscr{A}, G)'$. In fact, the generators of $\mathscr{R}(\mathscr{A}, G)'$ act on each $\mathscr{H}_\alpha$ as follows:

$$\begin{aligned}(\pi'(A')\sum_{m,n} u_{mn}^{(\alpha)}\tilde{\xi}_{mn}^{(\alpha)})(g) &= \sum_{m,n} u_{mn}^{(\alpha)}(g)A'\xi_{mn}^{(\alpha)} \\ &= \sum_{m,n} u_{mn}^{(\alpha)}(A'\xi_{mn}^{(\alpha)})^\sim(g) , \quad A' \in \mathscr{A}' ; \\ (\lambda'(h)\sum_{m,n} u_{mn}^{(\alpha)}\tilde{\xi}_{mn}^{(\alpha)})(g) &= U_h \sum_{m,n} u_{mn}^{(\alpha)}(gh)\xi_{mn}^{(\alpha)} \\ &= U_h \sum_{m,n}\sum_{r} d_\alpha^{-1/2} u_{mr}^{(\alpha)}(g) u_{rn}^{(\alpha)}(h)\xi_{mn}^{(\alpha)} \\ &= \sum_{m,r} u_{mr}^{(\alpha)}(d_\alpha^{-1/2}\sum_{n} u_{rn}^{(\alpha)}(h) U_h \xi_{mn}^{(\alpha)})^\sim(g) , \quad h \in G .\end{aligned}$$

Hence the projection onto each $\mathscr{H}_\alpha$ belongs to $\mathscr{R}(\mathscr{A}, G)$. Explicitly, the projection onto $\mathscr{H}_\alpha$ is given by

$$P_\alpha = d_\alpha \int_G \chi_\alpha(h)\lambda(h)dh , \quad \alpha \in \Gamma ,$$

where the integration is taken in the strong operator topology. Furthermore, $\sum_{\alpha\in\Gamma} \oplus \mathscr{H}_\alpha = L^2(\mathscr{H}, G)$. To show this, suppose that $\eta = \eta(g)$ is an element of $L^2(\mathscr{H}, G)$ which satisfies $(\eta \mid u_{mn}^{(\alpha)}\tilde{\xi}) = 0$ for all $\xi \in \mathscr{H}$ and (α, m, n). Then

$$\int_G (\eta(g) \mid \xi)\overline{u_{mn}^{(\alpha)}(g)}dg = 0 .$$

Hence all of the Fourier coefficients of the numerical function $(\eta(g) \mid \xi)$ are zero, and hence $(\eta(g) \mid \xi) = 0$ for almost all $g \in G$. Since $\xi \in \mathscr{H}$ is arbitrary, $\eta(g) = 0$ for almost all $g \in G$, which means $\eta = 0$. This proves our assertion.

Summarizing these facts, we get the following lemma.

LEMMA 2.1. *In the crossed product* $\mathscr{R}(\mathscr{A}, G)$, $\{P_\alpha \mid \alpha \in \Gamma\}$ *is a family of mutually orthogonal non zero projections of* $L^2(\mathscr{H}, G)$ *onto* $\mathscr{H}_\alpha$ *with sum* $\pi(I)$.

Now, we consider the action of the crossed product $\mathscr{R}'(\mathscr{A}, G)$ on $\mathscr{H}_\iota$. From direct computations follows that

$$\pi'(A)\tilde{\xi} = (A\xi)^{\sim}\,, \quad A \in \mathscr{A};$$
$$\lambda'(h)\tilde{\xi} = (U_h\xi)^{\sim}\,, \quad h \in G\,.$$

That is, $\pi'(A)$ and $\lambda'(h)$ act on $\mathscr{H}_\iota$ just like the operators A and U_h on $\mathscr{H}$ respectively. Hence, $\mathscr{R}'(\mathscr{A}, G)_{\mathscr{H}_\iota}$ is isomorphic to the von Neumann algebra $\mathscr{R}(\mathscr{A}, \mathscr{U}_G)$ on the original space $\mathscr{H}$ generated by $\mathscr{A}$ and $\mathscr{U}_G$, the set of all unitaries U_g, $g \in G$.

$$\mathscr{R}'(\mathscr{A}, G)_{\mathscr{H}_\iota} = \mathscr{R}(\mathscr{A}, \mathscr{U}_G)\,.$$

Since P_ι belongs to the commutant of $\mathscr{R}'(\mathscr{A}, G)$ as well as to $\mathscr{R}(\mathscr{A}, G)$, we have by [3; I. 2.1. Prop. 2]

$$\mathscr{R}'(\mathscr{A}, G) = \mathscr{R}'(\mathscr{A}, G)_{\mathscr{H}_\iota}\,,$$

provided that the central support $C'(P_\iota)$ of P_ι in $\mathscr{R}'(\mathscr{A}, G)$ coincides with the identity $\pi(I)$. Therefore, we have the following theorem.

THEOREM 2.2. *Let $\mathscr{A}$ be a von Neumann algebra on $\mathscr{H}$ and G a compact group of automorphisms of $\mathscr{A}$. Suppose that the central support $C'(P^\iota)$ in $\mathscr{R}'(\mathscr{A}, G)$ coincides with $\pi(I)$. Then, the crossed product $\mathscr{R}(\mathscr{A}, G)$ on $L^2(\mathscr{H}, G)$ is isomorphic to the von Neumann algebra $\mathscr{R}(\mathscr{A}, G)$ on the original space $\mathscr{H}$ generated by $\mathscr{A}$ and $\mathscr{U}_G$.*

In other words, we essentially need not extend the space $\mathscr{H}$ to $L^2(\mathscr{H}, G)$ so far as we concern with the crossed product by a compact group.

COROLLARY 2.3. *If the crossed product $\mathscr{R}(\mathscr{A}, G)$ by a compact group G is a factor, it is isomorphic to the factor $\mathscr{R}(\mathscr{A}, \mathscr{U}_G)$ on the original space $\mathscr{H}$.*

Next we investigate the condition $C'(P_\iota) = \pi(I)$ of Theorem 2.2. If G is a countable discrete freely acting group of automorphisms of $\mathscr{A}$, the center of $\mathscr{R}(\mathscr{A}, G)$ is $\mathscr{Z}^G$, the fixed point subalgebra of the center $\mathscr{Z}$ of $\mathscr{A}$ ([6; Cor. to Lemma 4]). In case of continuous group G, however, we do not know the analogous result, chiefly because we have no explicit description of the elements of $\mathscr{R}(\mathscr{A}, G)$. We give here a converse of [6; Lemma 4].

LEMMA 2.4. *Let G be a group of automorphisms of a von Neumann algebra $\mathscr{A}$. If*

$$\mathscr{R}(\mathscr{A}, G) \cap \pi(\mathscr{A})' \subset \pi(\mathscr{A})\,,$$

then each $g \in G$ except the unity e is freely acting on $\mathscr{A}$.

PROOF. If an automorphism g in G is not freely acting on $\mathscr{A}$, there exists a non zero central projection C such that g is inner on $C\mathscr{A}$ ([8; Theorem 1.11]). That is, there exists a unitary operator U in $\mathscr{A}$ which implements the automorphism g on $C\mathscr{A}$. Then, we have

$$(\lambda(g)\pi(U^*C))\pi(A) = \lambda(g)\pi(U^*AC) ,$$

and

$$\begin{aligned}\pi(A)(\lambda(g)\pi(U^*C)) &= \lambda(g)\pi(g^{-1}(A)U^*C) \\ &= \lambda(g)\pi(U^*AC) .\end{aligned}$$

Hence $\lambda(g)\pi(U^*C)$ is an element of $\mathscr{R}(\mathscr{A}, G)$ which commutes with every element of $\pi(\mathscr{A})$ while it is not in $\pi(\mathscr{A})$ if $g \neq e$. This completes the proof.

THEOREM 2.5. *If G is compact and*

$$\mathscr{R}(\mathscr{A}, G) \cap \pi(\mathscr{A})' \subset \pi(\mathscr{A}) ,$$

then we have

$$C'(P_\iota) = \pi(I) .$$

PROOF. The center of $\mathscr{R}'(\mathscr{A}, G)$ is included in

$$\begin{aligned}\mathscr{R}'(\mathscr{A}, G) \cap \pi'(\mathscr{A})' &= W(\mathscr{R}(\mathscr{A}, G) \cap \pi(\mathscr{A})')W \\ &\subset W\pi(\mathscr{A})W = \pi'(\mathscr{A}) .\end{aligned}$$

Hence, $C'(P_\iota)$ has the form $\pi'(C)$, $C \in \mathscr{Z}$. It acts on $\mathscr{H}_\iota$ as follows:

$$(\pi'(C)\tilde{\xi})(g) = C\xi = (C\xi)^\sim(g) .$$

On the other hand, as an extension of the projection P_ι, $\pi'(C)$ leaves any vector of $\mathscr{H}_\iota$ invariant; $\pi'(C)\tilde{\xi} = \tilde{\xi}$. Hence we have $C\xi = \xi$ for all $\xi \in \mathscr{H}$. Then, as easily seen, $\pi'(C)$ leaves any vector of $\mathscr{H}_{mn}^{(\alpha)}$ invariant for all $\alpha \in \Gamma$ and $m, n = 1, 2, \cdots, d_\alpha$. Hence we have $C'(P_\iota) = \pi'(C) = \pi(I)$.

3. Fixed algebra and crossed product. Let $\mathscr{A}$ and G be as in the preceding sections. We consider in this section the crossed product $\mathscr{R}(\mathscr{A}, G)$ connected with the decomposition of the space $L^2(\mathscr{H}, G) = \sum_{\alpha \in \Gamma} \oplus \mathscr{H}_\alpha$. Let $\mathscr{A}^G$ denote the fixed point subalgebra of $\mathscr{A}$ under G;

$$\mathscr{A}^G = \{A \in \mathscr{A} \mid g(A) = A \quad \text{for all} \quad g \in G\} ,$$

and $\mathscr{L}(L^2(G))$ the algebra of all bounded linear operators on $L^2(G)$.

THEOREM 3.1. *Let $\mathscr{A}$ be a von Neumann algebra on $\mathscr{H}$ and G a compact group. Suppose that each projection P_α onto $\mathscr{H}_\alpha$, $\alpha \in \Gamma$ is* *decomposed into d_α^2 number of mutually orthogonal projections each of which is equivalent to P_ι. Then the crossed product $\mathscr{R}(\mathscr{A}, G)$ is spatially isomorphic to the tensor product $\mathscr{A}^G \otimes \mathscr{L}(L^2(G))$.*

PROOF. By [3; I. 2.4. Prop. 5] and the assumption on P_α, $\mathscr{R}(\mathscr{A}, G)$ is spatially isomorphic to the tensor product $\mathscr{R}(\mathscr{A}, G)_{\mathscr{H}_\iota} \otimes \mathscr{L}(L^2(X))$, where $X = \{(\alpha, m, n) \mid \alpha \in \Gamma \text{ and } m, n = 1, 2, \cdots, d_\alpha\}$. By the Peter-Weyl theorem, $\mathscr{L}(L^2(X))$ is spatially isomorphic to $\mathscr{L}(L^2(G))$. Therefore, it suffices to show that $\mathscr{R}(\mathscr{A}, G)_{\mathscr{H}_\iota}$ is spatially isomorphic to $\mathscr{A}^G$ on $\mathscr{H}$. For any $h \in G$,

$$\begin{aligned} P_\iota\lambda(h)P_\iota &= \int_G \lambda(g)dg\lambda(h)\int_G \lambda(k)dk \\ &= \iint_{G\times G} \lambda(ghk)dgdk \\ &= \int_G \lambda(g)dg\int_G dk \\ &= P_\iota \in \pi(\mathscr{A}^G)_{\mathscr{H}_\iota}, \end{aligned}$$

and, for any $A \in \mathscr{A}$, we have

$$\begin{aligned} P_\iota\pi(A)P_\iota &= \int_G \lambda(g)dg\pi(A)\int_G \lambda(k)dk \\ &= \iint_{G\times G} \lambda(gk)\pi(k^{-1}(A))dgdk \\ &= \int_G \lambda(g)dg\int_G \pi(k^{-1}(A))dk \in \pi(\mathscr{A}^G)_{\mathscr{H}_\iota} . \end{aligned}$$

That is, all of the generators of $\mathscr{R}(\mathscr{A}, G)$ restricted to the subspace $\mathscr{H}_\iota$ are in $\pi(\mathscr{A}^G)_{\mathscr{H}_\iota}$. Hence, we have $\mathscr{R}(\mathscr{A}, G)_{\mathscr{H}_\iota} \subset \pi(\mathscr{A}^G)_{\mathscr{H}_\iota}$. Since the inverse inclusion is obvious, we have $\mathscr{R}(\mathscr{A}, G)_{\mathscr{H}_\iota} = \pi(\mathscr{A}^G)_{\mathscr{H}_\iota}$. Further, since $(\pi(A)\tilde{\xi})(g) = g^{-1}(A)\xi = A\xi$ for $A \in \mathscr{A}^G$ and $\xi \in \mathscr{H}$, $\pi(\mathscr{A}^G)_{\mathscr{H}_\iota}$ is spatially isomorphic to $\mathscr{A}^G$ by the mapping $\tilde{\xi} \in \mathscr{H}_\iota \to \xi \in \mathscr{H}$. Therefore, $\mathscr{R}(\mathscr{A}, G)_{\mathscr{H}_\iota}$ is spatially isomorphic to $\mathscr{A}^G$. This completes the proof.

REMARK 3.2. The conclusions of Theorems 2.2 and 3.1 are in a dual relation. Explicitly speaking, if the conclusion of Theorem 3.1 holds, then we have

$$\begin{aligned} \mathscr{R}'(\mathscr{A}', G) = \mathscr{R}(\mathscr{A}, G)' &\cong (\mathscr{A}^G \otimes \mathscr{L}(L^2(G)))' \\ &= (\mathscr{A}^G)' \otimes \mathscr{L}(L^2(G))' \quad \text{([10; Th. 12.3])} \\ &= (\mathscr{A} \cap \mathscr{U}_G')' \otimes C_{L^2(G)} \\ &\cong \mathscr{R}(\mathscr{A}', \mathscr{U}_G) , \end{aligned}$$

which shows the validity of the conclusion of Theorem 2.2 for the crossed product $\mathscr{R}'(\mathscr{A}', G)$.

THEOREM 3.3. *Let $\mathscr{A}$ be semi-finite and G compact. Suppose that the crossed product $\mathscr{R}(\mathscr{A}, G)$ is a factor. Then,*

$$\mathscr{R}(\mathscr{A}, G) = \mathscr{A}^G \otimes \mathscr{L}(L^2(G)) .$$

PROOF. By Lemma 1.2, $\mathscr{R}(\mathscr{A}, G)$ is a semi-finite factor. For any element $A = A(g)$ in $\mathscr{K}(\mathfrak{A}, G)$, we have

$$\begin{aligned} P_\alpha A(g) &= d_\alpha \int_G \chi_\alpha(h)\lambda(h)dhA(g) \\ &= d_\alpha \int_G \chi_\alpha(h)A(h^{-1}g)dh \\ &= d_\alpha \int_G \chi_\alpha(gh^{-1})A(h)dh \\ &= (\pi_l(d_\alpha\chi_\alpha)A)(g) . \end{aligned}$$

Hence, $P_\alpha = P_\alpha^* P_\alpha$ coincides with $\pi_l(d_\alpha\chi_\alpha(h))^*\pi_l(d_\alpha\chi_\alpha(h))$. Accordingly, we have

$$\tilde{\tau}(P_\alpha) = \tilde{\tau}(P_\alpha^* P_\alpha) = \| d_\alpha\chi_\alpha(h)\|^2\tau(I) = d_\alpha^2\tau(I)$$

for any $\alpha \in \Gamma$. Hence $P_\alpha(\alpha \in \Gamma)$ can be decomposed into subprojections in such a way as they satisfy the assumption of Theorem 3.1.

REMARK 3.4. In a factor of type III, all projections are equivalent to each other. Hence Theorem 3.3 remains valid if $\mathscr{R}(\mathscr{A}, G)$ is a factor of type III. The case that $\mathscr{A}$ is of type III and $\mathscr{R}(\mathscr{A}, G)$ a semi-finite factor is remained unsolved. We do not know whether this case, i.e., a converse of Lemma 1.2 occurs or not.

H. Choda [1] has given a proof for the case of finite groups. We remark in addition that the compactness of G is not a necessary condition for the theorem as shown in [2].

EXAMPLE 3.5. Let G be a compact abelian group and $\mathscr{A} = L^\infty(G)$ the multiplication algebra on the Hilbert space $L^2(G)$. Define the action of G on $\mathscr{A}$ by

$$(h(x))(t) = x(h^{-1}t) \quad \text{for} \quad x \in L^\infty(G) .$$

For each $\alpha \in \Gamma$, $\chi_\alpha(t)$ is a function in $L^2(G)$ and

$$(h(\chi_\alpha))(t) = \overline{\chi_\alpha(h)}\chi_\alpha(t) .$$

Define the isometry V_α in $\mathscr{R}(\mathscr{A}, G)$ by

$$V_\alpha = \int_G \lambda(h)\pi(h^{-1}(\chi_\alpha))dh \ .$$

Then we have $V_\alpha^* V_\alpha = P_\iota$ and $V_\alpha V_\alpha^* = P_\alpha$, i.e. $P_\alpha \sim P_\iota$ for any $\alpha \in \Gamma$ and hence we may apply Theorem 3.1. Since G acts ergodically on $\mathscr{A}$, $\mathscr{A}^G = CI$ and hence the crossed product $\mathscr{R}(\mathscr{A}, G)$ is isomorphic to $\mathscr{L}(L^2(G))$. This is a factor of type I_∞ if the order of G is infinite. Hence the crossed product of a finite von Neumann algebra by a compact group G need not be finite, contrary to the case of discrete groups.

This example is obtained also as a direct consequence of the Mackey-Stone-von Neumann theorem ([9]).

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