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ANTI-INVARIANT SUBMANIFOLDS OF SASAKIAN SPACE FORMS I

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Introduction. In previous papers [14, 15] the present authors have studied totally real submanifolds of Kaehler manifolds, especially those of complex space forms.

Let $\bar{M}$ be a real $2m$-dimensional Kaehler manifold with almost complex structure J. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is said to be totally real or anti-invariant in $\bar{M}$ if $T_x(M) \perp JT_x(M)$ for each $x \in M$, where $T_x(M)$ denotes the tangent space to M at x. Here we have identified $T_x(M)$ with its image under the differential of the immersion. Since, if X is a vector tangent to M at x then JX is normal to M, we see that, the rank of J being $2m$, $n \leqq 2m - n$, that is, $n \leqq m$.

In [14] we have studied n-dimensional totally real submanifold of a real $2n$-dimensional complex space form $\bar{M}$ satisfying certain conditions on the second fundamental forms, and in [15] we have studied n-dimensional totally real submanifolds of a real $2m$-dimensional complex space form.

The purpose of the present paper is to study similar problems for submanifolds of almost contact metric manifolds, especially for those of Sasakian space forms (cf. [1], [6], [8], [11] and [12]).

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact metric manifold whose (1, 1)-type structure tensor field is ϕ. An $(n+1)$-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is said to be anti-invariant if $T_x(M) \perp \phi T_x(M)$ for each $x \in M$. Then we have $n \leqq m$. In the present paper, we study the case $n = m$.

1. Sasakian manifolds. In this section we would like to recall definitions and some fundamental properties of a Sasakian manifold.

Let $\bar{M}$ be a $(2m+1)$-dimensional differentiable manifold of class C^∞ and ϕ, ξ, η be a tensor field of type (1, 1), a vector field, a 1-form on $\bar{M}$ respectively such that

$$\phi^2 = -I + \eta \otimes \xi , \quad \phi\xi = 0 , \quad \eta(\phi X) = 0 , \quad \eta(\xi) = 1$$

for any vector field X on $\bar{M}$, where I denotes the identity tensor. Then

$\bar{M}$ is said to have an almost contact structure (ϕ, ξ, η) and is called an almost contact manifold. The almost contact structure is said to be normal if

$$N + d\eta \otimes \xi = 0 ,$$

where N denotes the Nijenhuis tensor formed with ϕ and $d\eta$ the differential of the 1-form η. When a Riemannian metric tensor field $\bar{g}$ is given on $\bar{M}$ and $\bar{g}$ satisfies the equations

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y) , \quad \eta(X) = \bar{g}(X, \xi)$$

for any vector fields X and Y, $(\phi, \xi, \eta, \bar{g})$-structure is called an almost contact metric structure and $\bar{M}$ an almost contact metric manifold. If

$$d\eta(X, Y) = \bar{g}(\phi X, Y)$$

for any vector fields X and Y, then an almost contact metric structure is called a contact metric structure. If moreover the structure is normal, then a contact metric structure is called a Sasakian structure and a manifold with Sasakian structure is called a Sasakian manifold. It is well known that in a Sasakian manifold with structure $(\phi, \xi, \eta, \bar{g})$ we have

$$\bar{\nabla}_X \xi = \phi X , \quad (\bar{\nabla}_X \phi) Y = - \bar{g}(X, Y)\xi + \eta(Y)X$$

for any vector fields X and Y, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to $\bar{g}$.

A plane section in the tangent space $T_x(\bar{M})$ at x of a Sasakian manifold $\bar{M}$ is called a ϕ-section if it is spanned by a vector X orthogonal to ξ and ϕX. The sectional curvature $K(X, \phi X)$ with respect to a ϕ-section determined by a vector X is called a ϕ-sectional curvature. It is easily verified that if a Sasakian manifold has a ϕ-sectional curvature k which does not depend on the ϕ-section at each point, then k is a constant in the manifold. A Sasakian manifold is called a Sasakian space form and is denoted by $\bar{M}(k)$ if it has the constant ϕ-sectional curvature k.

A typical example of Sasakian manifolds is an odd-dimensional sphere S^{2n+1} (cf. [7]).

2. Anti-invariant submanifolds. Let $\bar{M}$ be an almost contact metric manifold of dimension $2m + 1$ with structure tensors $(\phi, \xi, \eta, \bar{g})$. An $(n + 1)$-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is called an anti-invariant submanifold if $T_x(M) \perp \phi T_x(M)$ for each $x \in M$ where $T_x(M)$ denotes the tangent space to M at $x \in M$. Here we have identified $T_x(M)$ with its image under the differential of the immersion because our computation is local. By the definition, if $X \in T_x(M)$, then ϕX is a normal vector to M. Since the rank of ϕ is $2m$, we have $n \leqq (2m+1)-(n+1)$, from which $n \leqq m$. In the sequel, we shall study the case $m=n$.

Let g be the induced metric tensor field of M. We denote by $\bar{\nabla}$ (resp. ∇) the operator of covariant differentiation with respect to $\bar{g}$ (resp. g). Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + D_X N$$

for any tangent vector fields X, Y and a normal vector field N on M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle. Both A and B are called the second fundamental forms of M and satisfy

$$\bar{g}(B(X, Y), N) = g(A_N X, Y) .$$

A vector field N normal to M is said to be parallel if $D_X N = 0$ for any tangent vector field X on M. The mean curvature vector m of M is defined to be $m = (\mathrm{Tr}\, B)/(n+1)$ where $\mathrm{Tr}\, B = \sum_i B(e_i, e_i)$ for an orthonormal frame $\{e_i\}$. If $m=0$, then M is said to be minimal and if the second fundamental form of M is of the form $B(X, Y) = g(X, Y)m$, then M is said to be totally umbilical. If the second fundamental form of M vanishes identically, i.e., $B=0$, then M is said to be totally geodesic.

Let $T_x(M)^\perp$ be the normal space to M at $x \in M$. Since $m=n$, we see that $\phi T_x(M) = T_x(M)^\perp$ at each point $x \in M$. Since, for any tangent vector field X on M, we have $\bar{g}(\xi, \phi X) = -\bar{g}(\phi\xi, X) = 0$, we see that ξ is tangent to M. Thus we have

LEMMA 2.1. *Let $\bar{M}$ be an almost contact metric manifold of dimension $2n+1$ and let M be an anti-invariant submanifold of $\bar{M}$ of dimension $n+1$. Then the vector field ξ is tangent to M.*

In the sequel, we assume that the ambient manifold $\bar{M}$ is a Sasakian manifold.

We choose a local field of orthonormal frames $e_0 = \xi, e_1, \cdots, e_n$; $e_{1^*} = \phi e_1, \cdots, e_{n^*} = \phi e_n$ in $\bar{M}$ in such a way that, restricted to M, $e_0, e_1, \cdots, e_n$ are tangent to M. With respect to this frame field of $\bar{M}$, let $\omega^0 = \eta, \omega^1, \cdots, \omega^n; \omega^{1^*}, \cdots, \omega^{n^*}$ be the field of dual frames. Unless otherwise stated we use the conventions that the ranges of indices are respectively:

$$A, B, C, D = 0, 1, \cdots, n, 1^*, \cdots, n^* ,$$
$$t, s, i, j, k, l = 1, \cdots, n ,$$
$$a, b, c, d = 0, 1, \cdots, n ,$$

and that when an index appears twice in any term as a subscript and a superscript, it is understood that this index is summed over its range. Then the structure equations of $\bar{M}$ are given by

$$d\omega^A = \omega^A_B \wedge \omega^B , \quad \omega^A_B + \omega^B_A = 0 , \tag{2.1}$$

$$d\omega^A_B = - \omega^A_C \wedge \omega^C_B + \Phi^A_B , \quad \Phi^A_B = \frac{1}{2}K^A_{BCD}\omega^C \wedge \omega^D . \tag{2.2}$$

Restriction of these forms to M gives

$$\omega^{t^*} = 0 , \tag{2.3}$$

$$d\omega^a = - \omega^a_b \wedge \omega^b , \quad \omega^a_b + \omega^b_a = 0 , \tag{2.4}$$

$$\omega^i_j = \omega^{i^*}_{j^*} , \quad \omega^{i^*}_j = \omega^{j^*}_i , \quad \omega^i = \omega^{i^*}_0 , \tag{2.5}$$

$$d\omega^a_b = -\omega^a_c \wedge \omega^c_b + \Omega^a_b , \quad \Omega^a_b = \frac{1}{2}R^a_{bcd}\omega^c \wedge \omega^d . \tag{2.6}$$

Since $0 = d\omega^{t^*} = -\omega^{t^*}_a \wedge \omega^a$, by Cartan's lemma, we have

$$\omega^{t^*}_a = h^t_{ab}\omega^b , \quad h^t_{ab} = h^t_{ba} , \tag{2.7}$$

where we use h^t_{ab} instead of $h^{t^*}_{ab}$ to simplify the notation. From (2.5) and (2.7) we have

$$h^i_{jk} = h^j_{ik} = h^k_{ij} , \quad h^t_{00} = 0 , \quad h^t_{0b} = \delta^t_b . \tag{2.8}$$

Moreover we see that $g(A_te_a, e_b) = h^t_{ab}$ where $A_t = A_{\phi e_t}$. The Gauss equation is given by

$$R^a_{bcd} = K^a_{bcd} + \sum_t (h^t_{ac}h^t_{bd} - h^t_{ad}h^t_{bc}) . \tag{2.9}$$

We also have

$$d\omega^{i^*}_{j^*} = - \omega^{i^*}_{k^*} \wedge \omega^{k^*}_{j^*} + \Omega^{i^*}_{j^*} , \quad \Omega^{i^*}_{j^*} = \frac{1}{2}R^{i^*}_{j^*cd}\omega^c \wedge \omega^d , \tag{2.10}$$

and consequently the Ricci equation is given by

$$R^{i^*}_{j^*cd} = K^{i^*}_{j^*cd} + \sum_a (h^i_{ac}h^j_{ad} - h^i_{ad}h^j_{ac}) . \tag{2.11}$$

We define the covariant derivative h^t_{abc} of h^t_{ab} by putting

$$h^t_{abc}\omega^c = dh^t_{ab} - h^t_{ad}\omega^d_b - h^t_{db}\omega^d_a + h^s_{ab}\omega^{t^*}_{s^*} . \tag{2.12}$$

The Laplacian Δh^t_{ab} of h^t_{ab} is defined to be

$$\Delta h^t_{ab} = \sum_c h^t_{abcc} , \tag{2.13}$$

where we have put

$$h^t_{abcd}\omega^d = dh^t_{abc} - h^t_{dbc}\omega^d_a - h^t_{adc}\omega^d_b - h^t_{abd}\omega^d_c + h^s_{abc}\omega^{t^*}_{s^*} . \tag{2.14}$$

The Riemannian connection of M is defined by (ω_b^a). The form $(\omega_{j^*}^{i^*})$ defines a connection induced in the normal bundle of M from that of $\bar{M}$. The second fundamental form of M is represented by $h_{ab}^t\omega^a\omega^b e_{t^*}$ and is sometimes denoted by its components h_{ab}^t. If $h_{abc}^t = 0$ for all t, a, b and c, the second fundamental form of M is said to be parallel. If $\sum_a h_{aa}^t = 0$ for all t, then M is a minimal submanifold of $\bar{M}$.

If a Sasakian manifold $\bar{M}$ is of constant ϕ-sectional curvature k, then we have

$$(2.15)\quad K_{BCD}^A = \frac{1}{4}(k+3)(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + \frac{1}{4}(k-1)(\eta_B\eta_C\delta_{AD} - \eta_B\eta_D\delta_{AC} + \eta_A\eta_D\delta_{BC} - \eta_A\eta_C\delta_{BD} + \phi_{AC}\phi_{BD} - \phi_{AD}\phi_{BC} + 2\phi_{AB}\phi_{CD}),$$

where δ_{AC} denotes the Kronecker delta. This is a Sasakian space form and is denoted by $\bar{M}(k)$. If a Riemannian manifold M is of constant curvature c, then we call such a manifold a real space form and denote it by $M(c)$.

3. Fundamental properties. Let $\bar{M}$ be a Sasakian manifold of dimension $2n+1$ with structure tensors $(\phi, \xi, \eta, \bar{g})$ and M be an anti-invariant submanifold of $\bar{M}$ of dimension $n+1$. For any tangent vector field X to M we have

$$\phi X = \bar{\nabla}_X\xi = \nabla_X\xi + B(X, \xi).$$

Consequently, comparing the tangential part and the normal part, we have $\nabla_X\xi = 0$ and $\phi X = B(X, \xi)$. Putting $X = \xi$ in the second equation, we obtain $B(\xi, \xi) = 0$. Thus we have

LEMMA 3.1. *Let $\bar{M}$ be a Sasakian manifold of dimension $2n+1$ and M be an anti-invariant submanifold of $\bar{M}$ of dimension $n+1$. Then the vector field ξ restricted to M is parallel.*

PROPOSITION 3.1. *Let $\bar{M}$ be a Sasakian manifold of dimension $2n+1$ and M be an anti-invariant submanifold of $\bar{M}$ of dimension $n+1$. Then M is not totally umbilical when $n \geqq 1$.*

PROOF. Let us assume that M is totally umbilical. Then $B(X, Y) = g(X, Y)m$ for any tangent vectors X, Y to M, where m denotes the mean curvature vector. Since $B(\xi, \xi) = 0$, we have $g(\xi, \xi)m = 0$, which shows that M is minimal. Therefore M is totally geodesic. Then we have $\phi X = B(X, \xi) = 0$ for any tangent vector X to M. But this is a contradiction, and Proposition 3.1 is proved.

Next we shall study the second fundamental form of an anti-invariant submanifold. For each t $(=1, \cdots, n)$ the second fundamental form A_t is represented by the symmetric $(n+1, n+1)$-matrix $A_t = (h^t_{ab})$. Equations (2.8) show that

$$
\text{(3.1)} \qquad A_t = \begin{array}{c} \\ \\ \\ \\ t \\ \\ \\ \end{array}\left[\begin{array}{c|c} 0 & 0\cdots 0\overset{t}{1}0\cdots 0 \\ \hline 0 & \\ \vdots & \\ 0 & \\ 1 & h^t_{ij} \\ 0 & \\ \vdots & \\ 0 & \end{array}\right] .
$$

Hereafter we put $H_t = (h^t_{ij})$, which is a symmetric (n, n)-matrix. Let S denote the square of the length of the second fundamental form of M, i.e.,

$$S = \sum_t \operatorname{Tr} A_t^2 = \sum_{t,a,b} (h^t_{ab})^2 .$$

Putting $T = \sum_t \operatorname{Tr} H_t^2 = \sum_{t,i,j} (h^t_{ij})^2$, we obtain

$$\text{(3.2)} \qquad S = T + 2n .$$

On the other hand, we see from (2.8) that

$$\operatorname{Tr} A_t = \sum_a h^t_{aa} = \sum_i h^t_{ii} = \operatorname{Tr} H_t .$$

Thus M is minimal if and only if $\operatorname{Tr} H_t = 0$ for all t.

PROPOSITION 3.2. *Let $\bar{M}$ be a Sasakian manifold of dimension $2n+1$ and M be an anti-invariant submanifold of $\bar{M}$ of dimension $n+1$. Then M is flat if and only if the normal connection of M is flat, i.e., $R^{i^*}_{j^*cd} = 0$.*

PROOF. Since $\bar{M}$ is a Sasakian manifold and M is anti-invariant, we have

$$\text{(3.3)} \qquad K^{i^*}_{j^*cd} = K^i_{jcd} - (\delta_{ic}\delta_{jd} - \delta_{id}\delta_{jc}) .$$

On the other hand, from Lemma 3.1, we have

$$\text{(3.4)} \qquad R^0_{bcd} = R^a_{0cd} = 0 .$$

From (2.8), (2.11) and (3.3) we obtain

$$\text{(3.5)} \qquad \begin{aligned} R^{i^*}_{j^*cd} &= K^{i^*}_{j^*cd} + \sum_a (h^i_{ac}h^j_{ad} - h^i_{ad}h^j_{ac}) \\ &= K^i_{jcd} + \sum_t (h^t_{ic}h^t_{jd} - h^t_{id}h^t_{jc}) . \end{aligned}$$

Equations (2.9) and (3.5) imply that $R^i_{jcd} = R^{i*}_{j*cd}$. This combined with (3.4) proves our assertion.

Next we assume that the ambient manifold $\bar{M}$ is of constant ϕ-sectional curvature k. Since M is anti-invariant, (2.15) implies that

$$K^a_{bcd} = \frac{1}{4}(k+3)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \frac{1}{4}(k-1)(\eta_b\eta_c\delta_{ad} - \eta_b\eta_d\delta_{ac} + \eta_a\eta_d\delta_{bc} - \eta_a\eta_c\delta_{bd}) . \tag{3.6}$$

If $A_tA_s = A_sA_t$ for all t and s, then the second fundamental form of M is said to be commutative, which is equivalent to $\sum_b h^t_{ab}h^s_{bc} = \sum_b h^s_{ab}h^t_{bc}$. If we assume that the second fundamental form of M is commutative, then by a direct computation and (2.8), we have

$$\sum_t (h^t_{ac}h^t_{bd} - h^t_{ad}h^t_{bc}) = -(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) . \tag{3.7}$$

From the Gauss equation (2.9) and (3.7) we obtain

$$R^a_{bcd} = K^a_{bcd} - (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) . \tag{3.8}$$

When $\bar{M}$ is of constant ϕ-sectional curvature k, substituting (3.6) into (3.8), we find

$$R^a_{bcd} = \frac{1}{4}(k-1)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + \eta_b\eta_c\delta_{ad} - \eta_b\eta_d\delta_{ac} + \eta_a\eta_d\delta_{bc} - \eta_a\eta_c\delta_{bd}) . \tag{3.9}$$

From this we have

PROPOSITION 3.3. *Let M be an $(n+1)$-dimensional $(n \geqq 2)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with commutative second fundamental form. Then M is flat if and only if $\bar{M}$ is of constant curvature* 1, *i.e.*, $k = 1$.

By Lemma 3.1, ξ is parallel with respect to the induced connection on M. Therefore, by (3.9) and a theorem in [9; p. 274], we have

THEOREM 3.1. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. If the second fundamental form of M is commutative, then M is locally a Riemannian direct product $M^n \times R^1$, where M^n is a hypersurface of M^{n+1} of constant curvature* $(1/4)(k-1)$ *and is totally geodesic in M^{n+1}.*

4. **Anti-invariant submanifolds of a sphere.** In this section we shall study the Laplacian for the square of the length of the second fundamental form of anti-invariant submanifolds. In the first place, we prove the following

LEMMA 4.1. *Let M be an $(n+1)$-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then we have*

$$\begin{aligned}(4.1)\quad \sum_{t,a,b} h^t_{ab}\Delta h^t_{ab} = \sum_{t,a,b,c} h^t_{ab}h^t_{ccab} + \frac{1}{4}(k+3)(n+1)\sum_t \operatorname{Tr} H_t^2 \\ - \frac{1}{2}(k+3)\sum_t (\operatorname{Tr} H_t)^2 + \sum_{t,s}\{\operatorname{Tr}(H_tH_s - H_sH_t)^2 \\ - [\operatorname{Tr}(H_tH_s)]^2 + \operatorname{Tr} H_s \operatorname{Tr}(H_tH_sH_t)\}\,.\end{aligned}$$

PROOF. By the assumption, the second fundamental form of M satisfies the Codazzi equation, i.e., $h^t_{abc} = h^t_{acb}$. Therefore, by a straightforward computation, we have

$$\begin{aligned}\sum_{t,a,b} h^t_{ab}\Delta h^t_{ab} = \sum_{t,a,b,c}(h^t_{ab}h^t_{ccab} - K^{t*}_{s*ac}h^s_{bc}h^t_{ab} + K^d_{cac}h^t_{db}h^t_{ab} \\ + K^d_{abc}h^t_{dc}h^t_{ab}) - \sum_{t,s,a,b,c,d}[(h^t_{ac}h^s_{bc} - h^t_{bc}h^s_{ac})(h^t_{ad}h^s_{bd} - h^t_{bd}h^s_{ad}) \\ + h^t_{ab}h^t_{cd}h^s_{ab}h^s_{cd} - h^t_{ab}h^t_{ca}h^s_{cb}h^s_{dd}]\,.\end{aligned}$$

Substituting (2.15) into this equation, we have

$$\begin{aligned}(4.2)\quad \sum_{t,a,b} h^t_{ab}\Delta h^t_{ab} = \sum_{t,a,b,c} h^t_{ab}h^t_{ccab} + \frac{1}{4}(k+3)(n+1)\sum_t \operatorname{Tr} A_t^2 \\ - \frac{1}{2}(k+1)\sum_t (\operatorname{Tr} A_t)^2 - \frac{1}{2}(k-1)n(n+1) \\ + \sum_{t,s}\{\operatorname{Tr}(A_tA_s - A_sA_t)^2 - [\operatorname{Tr}(A_tA_s)]^2 + \operatorname{Tr} A_s \operatorname{Tr}(A_tA_sA_t)\}\,.\end{aligned}$$

On the other hand, by (2.8) and (3.1), we have

$$\begin{aligned}(4.3)\quad \sum_{t\neq s}\operatorname{Tr}(A_tA_s - A_sA_t)^2 = -2\sum_{t\neq s}\sum_{i,j,k,l}(h^t_{ij}h^t_{jk}h^s_{kl}h^s_{li} \\ - h^t_{ij}h^s_{jk}h^t_{kl}h^s_{li} - 2\sum_{t\neq s}\{1 + \sum_i (2h^i_{tt}h^i_{ss} - 2h^t_{si}h^t_{si})\} \\ = \sum_{t\neq s}\operatorname{Tr}(H_tH_s - H_sH_t)^2 + 4\sum_t [\operatorname{Tr} H_t^2 - (\operatorname{Tr} H_t)^2] - 2n(n-1)\,,\end{aligned}$$

$$(4.4)\quad \sum_{t,s}[\operatorname{Tr}(A_tA_s)]^2 = \sum_t (\operatorname{Tr} A_t^2)^2 = \sum_t (\operatorname{Tr} H_t^2)^2 + 4\sum_t \operatorname{Tr} H_t^2 + 4n\,,$$

$$(4.5)\quad \sum_{t,s}\operatorname{Tr} A_s \operatorname{Tr}(A_tA_sA_t) = \sum_{t,s}\operatorname{Tr} H_s \operatorname{Tr}(H_tH_sH_t) + 3\sum_t (\operatorname{Tr} H_t)^2\,.$$

Substituting (3.2), (4.3), (4.4) and (4.5) into (4.2), we have (4.1), and Lemma 4.1 is proved.

LEMMA 4.2. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then we have*

$$(4.6)\qquad \sum_{t,a,b} h^t_{ab}\Delta h^t_{ab} = \frac{1}{4}(k+3)(n+1)\sum_t \operatorname{Tr} H_t^2 + \sum_{t,s} \operatorname{Tr}(H_tH_s - H_sH_t)^2 - [\operatorname{Tr}(H_tH_s)]^2 .$$

In the sequel, we need the following lemma proved in [3].

LEMMA 4.3 ([3]). *Let A and B be symmetric (n, n)-matrices. Then*

$$-\operatorname{Tr}(AB-BA)^2 \leqq 2\operatorname{Tr} A^2 \operatorname{Tr} B^2 ,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, where

$$\bar{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline 0 & & 0 \end{array}\right], \qquad \bar{B} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline 0 & & 0 \end{array}\right].$$

Moreover, if A_1, A_2, A_3 are symmetric (n, n)-matrices such that

$$-\operatorname{Tr}(A_aA_b - A_bA_a)^2 = 2\operatorname{Tr} A_a^2 \operatorname{Tr} A_b^2 , \quad 1 \leqq a, b \leqq 3 , \quad a \neq b ,$$

then at least one of the matrices A_a must be zero.

In the following, we put $T_{ts} = \sum_{i,j} h^t_{ij}h^s_{ij}$ and $T_t = T_{tt}$. Then we have $T = \sum_t T_t$.

THEOREM 4.1. *Let M be an $(n+1)$-dimensional compact orientable anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2n+1}(1)$. Then we have*

$$(4.7)\qquad \int_M \sum_{t,a,b,c} (h^t_{abc})^2 {*1} \leqq \int_M \left[\left(2 - \frac{1}{n}\right)T - (n+1)\right]T{*1} .$$

PROOF. We can write (4.6) as

$$(4.8)\qquad \sum_{t,a,b} h^t_{ab}\Delta h^t_{ab} = (n+1)T + \sum_{t,s}\operatorname{Tr}(H_tH_s - H_sH_t)^2 - \sum_t T_t^2 .$$

Applying Lemma 4.3, we obtain

$$(4.9)\qquad \begin{aligned} -\sum_{t,s}\operatorname{Tr}(H_tH_s - H_sH_t)^2 + \sum_t T_t^2 - (n+1)T &\leqq 2\sum_{t\neq s} T_tT_s + \sum_t T_t^2 - (n+1)T \\ &= \left[\left(2 - \frac{1}{n}\right)T - (n+1)\right]T - \frac{1}{n}\sum_{t,s}(T_t - T_s)^2 . \end{aligned}$$

Since M is compact orientable, we have

$$\int_M \sum_{t,a,b,c} (h^t_{abc})^2 *1 = -\int_M \sum_{t,a,b} h^t_{ab} \Delta h^t_{ab} *1 .$$

Therefore (4.8) and (4.9) imply (4.7) and Theorem 4.1 is proved.

COROLLARY 4.1. *Let M be an $(n+1)$-dimensional compact orientable anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2n+1}(1)$. Then either $T=0$, or $T=n(n+1)/(2n-1)$ or at some point $x \in M$, $T(x) > n(n+1)/(2n-1)$.*

Next we shall study the case in which $T = n(n+1)/(2n-1)$, that is, the square of the length of the second fundamental form of M satisfies $S = n(5n-1)/(2n-1)$.

THEOREM 4.2. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of a Sasakian space form $\bar{M}^{2n+1}(1)$. If $S = n(5n-1)/(2n-1)$, then $n=2$ and M is flat. With respect to an adapted dual orthonormal frame field $\omega^0, \omega^1, \omega^2, \omega^{1*}, \omega^{2*}$, the connection form (ω^A_B) of $\bar{M}^5(1)$, restricted to M, is given by*

$$\begin{bmatrix} 0 & 0 & 0 & -\omega^1 & -\omega^2 \\ 0 & 0 & 0 & \omega^0 + \lambda\omega^2 & \lambda\omega^1 \\ 0 & 0 & 0 & \lambda\omega^1 & \omega^0 - \lambda\omega^2 \\ \omega^1 & \omega^0 + \lambda\omega^2 & \lambda\omega^1 & 0 & 0 \\ \omega^2 & \lambda\omega^1 & \omega^0 - \lambda\omega^2 & 0 & 0 \end{bmatrix}, \quad \lambda = \frac{1}{\sqrt{2}} .$$

PROOF. From the assumption we have $T = n(n+1)/(2n-1)$. Then the second fundamental form of M is parallel by (4.8) because $\sum_{t,a,b} h^t_{ab} \Delta h^t_{ab} = -\sum_{t,a,b,c} (h^t_{abc})^2$ in this case. From Lemma 4.3 and (4.9) we have

$$(4.10) \qquad \sum_{t>s} (T_t - T_s)^2 = 0 ,$$

$$(4.11) \qquad -\operatorname{Tr}(H_t H_s - H_s H_t)^2 = 2 \operatorname{Tr} H_t^2 \operatorname{Tr} H_s^2 ,$$

and hence $T_t = T_s$ for all t, s and we may assume that $H_t = 0$ for $t = 3, \cdots, n$. Therefore we must have $n = 2$ and we obtain

$$(4.12) \qquad H_1 = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_2 = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

by putting $h^1_{12} = \lambda = h^2_{11}$. From (3.1) and (4.12) we have

$$(4.13) \qquad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & \lambda & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}.$$

On the other hand, by (2.12), we have

$$dh^t_{ab} = h^t_{ad}\omega^d_b + h^t_{db}\omega^d_a - h^s_{ab}\omega^t_s \,. \tag{4.14}$$

Putting $t = 1$, $a = 1$ and $b = 0$, we see that $d\lambda = \omega^2_0 = 0$, which shows that λ is a constant. Since $T = 2$, we get $4\lambda^2 = 2$. Thus we may assume that $\lambda = 1/\sqrt{2}$. Moreover (4.13) and (4.14) imply the equations:

$$\begin{aligned} &\omega^0_1 = \omega^{1*} = 0 \,, \quad \omega^0_2 = \omega^{2*} = 0 \,, \quad \omega^0_{1*} = -\omega^1 \,, \quad \omega^0_{2*} = -\omega^2 \,, \\ &\omega^{2*}_1 = \lambda\omega^1 \,, \quad \omega^{1*}_1 = \omega^0 + \lambda\omega^2 \,, \quad \omega^{2*}_2 = \omega^0 - \lambda\omega^2 \,, \quad \omega^2_1 = 0 \,. \end{aligned} \tag{4.15}$$

From the Gauss equation (2.9) and (4.13) we see easily that M is flat. From these considerations we obtain our assertion.

EXAMPLE 1. We give an example of an anti-invariant submanifold in S^5. Let $J = (a_{ts})$ $(t, s = 1, \cdots, 6)$ be the almost complex structure of C^3 such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1, 2, 3)$ the other components being zero. Let $S^1(1/\sqrt{3}) = \{z \in C: |z|^2 = 1/3\}$, a plane circle of radius $1/\sqrt{3}$. We consider $S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3}) \times S^1(1/\sqrt{3})$ in S^5 in C^3, which is obviously flat. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in C^3 has components given by

$$X = \frac{1}{\sqrt{3}}(\cos u^1, \sin u^1, \cos u^2, \sin u^2, \cos u^3, \sin u^3) \,,$$

u^1, u^2 and u^3 being parameters on each $S^1(1/\sqrt{3})$. Putting $X_i = \partial_i X = \partial X/\partial u^i$, we have

$$X_1 = \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, 0, 0, 0, 0) \,,$$

$$X_2 = \frac{1}{\sqrt{3}}(0, 0, -\sin u^2, \cos u^2, 0, 0) \,,$$

$$X_3 = \frac{1}{\sqrt{3}}(0, 0, 0, 0, -\sin u^3, \cos u^3) \,.$$

The vector field ξ on S^5 is given by

$$\xi = JX = \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, -\sin u^2, \cos u^2, -\sin u^3, \cos u^3) \,.$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. On the other hand, the structure tensors (ϕ, ξ, η) of S^5 satisfy

$$\phi X_i = JX_i + \eta(X_i)X \,, \quad i = 1, 2, 3,$$

which shows that ϕX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Therefore $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5. Moreover $S^1 \times$

$S^1 \times S^1$ is a minimal submanifold of S^5 with $S = 6$ and the normal connection of this is flat (see [3, 5]).

THEOREM 4.3. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of S^{2n+1}. If M is compact orientable and if $S = (5n^2 - n)/(2n-1)$, then*

$$M = S^1\left(\frac{1}{\sqrt{3}}\right) \times S^1\left(\frac{1}{\sqrt{3}}\right) \times S^1\left(\frac{1}{\sqrt{3}}\right) \quad in \quad S^5 .$$

5. Flat normal connection. Let $S^1(1/\sqrt{2})$ be a plane circle of radius $1/\sqrt{2}$. By a similar method as that in Example 1, we see that $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ is an anti-invariant submanifold of S^3, which is flat and minimal. Moreover this has flat normal connection and $S = 2$.

In this section, we characterize $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ of S^3. First we have the following

LEMMA 5.1 ([3]). *Let M be an $(n+1)$-dimensional minimal submanifold of S^{2n+1}. Then*

$$\sum_{t,a,b} h^t_{ab} \Delta h^t_{ab} = (n+1) \sum_t \operatorname{Tr} A_t^2 + \sum_{t,s} \{\operatorname{Tr}(A_t A_s - A_s A_t)^2 - [\operatorname{Tr}(A_t A_s)]^2\} . \tag{5.1}$$

THEOREM 5.1. *Let M be an $(n+1)$-dimensional anti-invariant minimal submanifold of S^{2n+1} with flat normal connection. If $S = n+1$, then M is flat and $n = 1$.*

PROOF. From the assumption and (2.11) we see that the second fundamental form of M is commutative, i.e., $A_t A_s = A_s A_t$. Putting

$$S_{ts} = \sum_{a,b} h^t_{ab} h^s_{ab} , \quad S_{tt} = S_t , \quad S = \sum_t S_t ,$$

we obtain, from (5.1),

$$\begin{aligned} \sum_{t,a,b} h^t_{ab} \Delta h^t_{ab} &= (n+1)S - \sum_t S_t^2 \\ &= (n+1)S - (\sum_t S_t)^2 + \sum_{t \neq s} S_t S_s . \end{aligned} \tag{5.2}$$

Since $S =$ constant, we have

$$\sum_{t,a,b} h^t_{ab} \Delta h^t_{ab} = - \sum_{t,a,b,c} (h^t_{abc})^2 .$$

From this and (5.2) we have

$$0 \leqq \sum_{t,a,b,c} (h^t_{abc})^2 = S[S-(n+1)] - \sum_{t \neq s} S_t S_s \leqq S[S-(n+1)] . \tag{5.3}$$

If $S = n+1$, we have $h^t_{abc} = 0$ and $\sum_{t \neq s} S_t S_s = 0$. Thus we may assume that $S_1 = n+1$ and $S_t = 0$ for $t = 2, \cdots, n$. Thus we obtain the following

$$\sum_{a,b} (h^1_{ab})^2 = n + 1 \,, \qquad h^t_{ab} = 0 \quad \text{for all } t > 1 \text{ and for all } a, b. \tag{5.4}$$

Using (3.1) we see that $\sum_{a,b} (h^1_{ab})^2 = 2 + \sum_{i,j} (h^1_{ij})^2$. Since $h^1_{ij} = h^i_{1j} = h^j_{1i} = 0$ unless $i = j = 1$ by (5.4), we have

$$\sum_{a,b} (h^1_{ab})^2 = 2 + (h^1_{11})^2 \,. \tag{5.5}$$

If M is minimal, the second fundamental form of M satisfies $\sum_i h^1_{ii} = 0$, which implies that $h^1_{11} = 0$. Consequently we must have $n + 1 = 2$, that is, $n = 1$. Moreover, Proposition 3.2 shows that M is flat.

From the theorem of Chern-Do Carmo-Kobayashi [3] and Theorem 5.1, we have

THEOREM 5.2. *Let M be an $(n + 1)$-dimensional compact orientable anti-invariant minimal submanifold of S^{2n+1} with flat normal connection. If $S = n + 1$, then*

$$M = S^1\left(\frac{1}{\sqrt{2}}\right) \times S^1\left(\frac{1}{\sqrt{2}}\right) \quad in \quad S^3 \,.$$

6. Anti-invariant submanifolds with parallel mean curvature vector. First of all, we consider the following example.

EXAMPLE 2. Let $J = (a_{ts})$ $(t, s = 1, \cdots, 2n + 2)$ be the almost complex structure of C^{n+1} such that $a_{2i,2i-1} = 0$, $a_{2i-1,2i} = -1$ $(i = 1, \cdots, n + 1)$ the other components being zero. Let $S^1(r_i) = \{z_i \in C : |z_i|^2 = r_i^2\}$, $i = 1, \cdots, n + 1$. We consider $M = S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{n+1})$ in C^{n+1} such that $r_1^2 + \cdots + r_{n+1}^2 = 1$. Then M is a flat submanifold of S^{2n+1} with parallel mean curvature vector and with flat normal connection (see [13]). The position vector X of M in C^{n+1} has components given by

$$X = (r_1 \cos u^1, r_1 \sin u^1, \cdots, r_{n+1} \cos u^{n+1}, r_{n+1} \sin u^{n+1}) \,,$$
$$r_1^2 + \cdots + r_{n+1}^2 = 1 \,.$$

Then X is an outward unit normal vector of S^{2n+1} in C^{n+1}. Putting $X_i = \partial_i X = \partial X / \partial u^i$, we have

$$X_1 = r_1(-\sin u^1, \cos u^1, 0, \cdots, 0) \,,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$X_{n+1} = r_{n+1}(0, \cdots, 0, -\sin u^{n+1}, \cos u^{n+1}) \,.$$

The vector field ξ on S^{2n+1} is given by its components

$$\xi = JX = (-r_1 \sin u^1, r_1 \cos u^1, \cdots, -r_{n+1} \sin u^{n+1}, r_{n+1} \cos u^{n+1}) \,.$$

Therefore we see that $\xi = X_1 + \cdots + X_{n+1}$, which means that the vector field ξ is tangent to M. And the structure tensors (ϕ, ξ, η) of S^{2n+1} satisfy

$$\phi X_i = JX_i + \eta(X_i)X , \quad i = 1, \cdots, n+1 .$$

Thus ϕX_i is normal to M for all i. Therefore M is an anti-invariant submanifold of S^{2n+1}.

LEMMA 6.1 ([13]). *Let M be an $(n+1)$-dimensional submanifold of S^{2n+1} with parallel mean curvature vector and with flat normal connection and we let λ_a^α, $1 \leqq a \leqq n+1$, be the eigenvalues of A_α corresponding to eigenvectors E_a (recall that the flat normal connection of M implies the A_α's are simultaneously diagonalizable). Then we have*

$$\sum_{\alpha,a,b} h_{ab}^\alpha \Delta h_{ab}^\alpha = \sum_\alpha \sum_{a>b} (\lambda_a^\alpha - \lambda_b^\alpha)^2 K_{ab} , \tag{6.1}$$

where K_{ab} denotes the sectional curvature of M determined by $\{E_a, E_b\}$.

THEOREM 6.1. *Let M be an $(n+1)$-dimensional compact orientable anti-invariant submanifold of S^{2n+1} with parallel mean curvature vector and with flat normal connection. Then*

$$M = S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{n+1}) , \quad r_1^2 + \cdots + r_{n+1}^2 = 1 .$$

PROOF. Since the normal connection of M is flat, by Proposition 3.2, M is flat. Moreover, the square of the length of the second fundamental form of M is constant since the mean curvature vector of M is parallel. Thus we have $\sum_{\alpha,a,b} h_{ab}^\alpha \Delta h_{ab}^\alpha = -\sum_{\alpha,a,b,c} (h_{abc}^\alpha)^2$. Since $K_{ab} = 0$, (6.1) implies $h_{abc}^\alpha = 0$, that is, the second fundamental form of M is parallel. Consequently, Theorem 4.1 of Yano-Ishihara [13] implies our assertion.

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