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BEHAVIOUR OF LEAVES OF CODIMENSION-ONE FOLIATIONS

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1. Introduction and statement of results. When a foliation is given, the following situation occurs generally. A leaf F_1 coils a leaf F_2 and the leaf F_2 coils another leaf F_3 and so on. The purpose of this paper is to investigate what happens in this situation.

Now, let M be a closed orientable C^r manifold, $0 \leqq r \leqq \infty$. Let $\mathscr{F}$ be a transversely orientable C^r foliation of codimension one on M. We denote by $M/\mathscr{F}$ the set of all leaves of $\mathscr{F}$. Let us consider the relation $\leqq$ on $M/\mathscr{F}$ defined as follows. We say $F_1 \leqq F_2$ if and only if $F_1 \subset \bar{F}_2$. We write $F_1 < F_2$ if and only if $F_1 \leqq F_2$ and $F_1 \neq F_2$. We denote by $d(F)$ the supremum of k such that there are k leaves $F_1, \cdots, F_k$ of $\mathscr{F}$ satisfying $F_1 < \cdots < F_k = F$. Let $d(\mathscr{F})$ be the supremum of $d(F)$ where F runs through the set $M/\mathscr{F}$. We call $d(F)$ or $d(\mathscr{F})$ the *depth* of F or $\mathscr{F}$ respectively. Then these numbers represent some complexity of the leaf F or the foliation $\mathscr{F}$.

A leaf F is called *proper* if the topology of F as a manifold and the topology of F as a subset of M coincide. A leaf F is called *locally dense* if the closure of F contains an interior point. If a leaf F is neither proper nor locally dense, F is called *exceptional*.

The relation $\leqq$ on $M/\mathscr{F}$ is clearly reflexive and transitive, but in many cases $\leqq$ is not asymmetric. We are interested in the case where $(M/\mathscr{F}, \leqq)$ is a partially ordered set. In the following cases $(M/\mathscr{F}, \leqq)$ becomes a partially ordered set.

PROPOSITION 1. *$(M/\mathscr{F}, \leqq)$ is a partially ordered set if $d(\mathscr{F})$ is finite or if all leaves of $\mathscr{F}$ are proper.*

We state a property of a foliation $\mathscr{F}$ satisfying that $(M/\mathscr{F}, \leqq)$ is a partially ordered set. A subset K of M is called *invariant* if K is a union of some family of leaves of $\mathscr{F}$. A *minimal set* K is a non-empty compact invariant subset of M such that if $K' \subset K$ is a non-empty compact invariant subset then $K' = K$.

PROPOSITION 2. (1) *Any non-empty compact invariant subset contains a minimal set.* (2) *If $(M/\mathscr{F}, \leqq)$ is a partially ordered set, then any* *minimal set consists of just one compact leaf.*

Since M is a non-empty compact invariant subset of M, we have the following.

COROLLARY 1. *If* $(M/\mathscr{F}, \leqq)$ *is a partially ordered set,* $\mathscr{F}$ *has a compact leaf.*

We write down some problems which are interesting in our situations.

PROBLEM 1. When does $d(\mathscr{F})$ become finite?

PROBLEM 2. Is a leaf F proper under the assumption that $d(F)$ is finite?

PROBLEM 3. Does there exist a codimension-one foliation $\mathscr{F}$ such that $(M/\mathscr{F}, \leqq)$ is a partially ordered set and $d(\mathscr{F})$ is infinite?

PROBLEM 4. Are all leaves of $\mathscr{F}$ proper under the assumption that $(M/\mathscr{F}, \leqq)$ is a partially ordered set?

PROBLEM 5. How behave the ends of a leaf F when $d(F)$ is finite?

As to Problems 3 and 4, the author knows nothing about them. The main theme of this paper concerns to Problems 1, 2 and 5.

The foliations $\mathscr{F}$ satisfying $d(\mathscr{F}) = 1, 2$ are characterized as follows.

THEOREM 1. (1) $d(\mathscr{F}) = 1$ *if and only if all leaves of* $\mathscr{F}$ *are compact.* (2) *Let* $\mathscr{F}$ *be* C^1. *Then* $d(\mathscr{F}) \leqq 2$ *if and only if* $(M/\mathscr{F}, \leqq)$ *is a partially ordered set and* $\mathscr{F}$ *is almost without holonomy, that is, all holonomy groups of non-compact leaves of* $\mathscr{F}$ *are trivial.*

There are a lot of manifolds which admit codimension-one foliations of depth 2, see the following.

EXAMPLE. The codimension-one foliations constructed in Tamura [6] and Mizutani-Tamura [2] by using spinnable structures are of depth 2.

Now we make preparations for treating Problem 1.

We denote by $P(F)$ the set of all continuous maps $\omega: [0, 1] \to F$. Let $P(\mathscr{F}) = \bigcup \{P(F) \mid F \in M/\mathscr{F}\}$. We denote by $LD_0(R, 0)$ the set of all orientation-preserving C^r diffeomorphisms $f: (D(f), 0) \to (R(f), 0)$ where $D(f)$ and $R(f)$ are open intervals containing 0. Clearly $LD_0(R, 0)$ is a subpseudogroup of the pseudogroup of the local diffeomorphisms of R. Let $\varphi: M \times R \to M$ be a C^r flow transverse to $\mathscr{F}$.

DEFINITION 1. The *global holonomy* of $(\mathscr{F}, \varphi)$ is the map $\Phi: P(\mathscr{F}) \to LD_0(R, 0)$ defined as follows. For each $\omega \in P(\mathscr{F})$, let $\mathscr{F}_\omega$ be the foliation on $[0, 1] \times R$ induced from $\mathscr{F}$ by the composed map

$$[0, 1] \times R \xrightarrow{\omega \times id} M \times R \xrightarrow{\varphi} M .$$

We define $\Phi(\omega)(x) = y$ if and only if there is a leaf of $\mathscr{F}_\omega$ containing the points $(0, 1)$ and $(1, y)$. The domain $D\Phi(\omega)$ of $\Phi(\omega)$ consists of $x \in R$ such that the leaf of $\mathscr{F}_\omega$ containing the point $(0, x)$ intersects $\{1\} \times R$.

For a sequence $\omega_1, \cdots, \omega_n \in P(\mathscr{F})$ such that $\omega_i(1) = \omega_{i+1}(0)$, $i = 1, \cdots, n^{-1}$, we define $\omega_1 \# \cdots \# \omega_n \in P(\mathscr{F})$ by the equation

$$\omega_1 \# \cdots \# \omega_n(t) = \omega_i(nt - i + 1) \quad \text{if} \quad (i-1)/n \leqq t \leqq i/n .$$

PROPOSITION 3. *If $\omega_1, \cdots, \omega_n \in P(\mathscr{F})$ satisfy $\omega_i(1) = \omega_{i+1}(0)$, then*

$$\Phi(\omega_1 \# \cdots \# \omega_n) = \Phi(\omega_n) \circ \cdots \circ \Phi(\omega_1)$$

and

$$D\Phi(\omega_1 \# \cdots \# \omega_n) = \Phi(\omega_1)^{-1}(D\Phi(\omega_2 \# \cdots \# \omega_n)) .$$

DEFINITION 2. The global holonomy Φ of $(\mathscr{F}, \varphi)$ is called *abelian* if, for all $\omega_1, \omega_2 \in P(\mathscr{F})$ satisfying $\omega_1(0) = \omega_1(1) = \omega_2(0) = \omega_2(1)$,

$$\Phi(\omega_1 \# \omega_2)(t) = \Phi(\omega_2 \# \omega_1)(t)$$

for all $t \in D\Phi(\omega_1 \# \omega_2) \cap D\Phi(\omega_2 \# \omega_1)$.

As to the global holonomy, Imanishi [1] proved the following.

THEOREM 2. (1) *Let $\mathscr{F}$ be C^1 and almost without holonomy. Let φ be a transverse flow satisfying that, for any non-compact leaf F of $\mathscr{F}$, φ has a closed orbit intersecting F. Then the global holonomy of $(\mathscr{F}, \varphi)$ is abelian.* (2) *Let $\mathscr{F}$ be C^1 and almost without holonomy. Then all holonomy groups of $\mathscr{F}$ are abelian.*

By using Theorem 2 (2), we obtain the following.

PROPOSITION 4. *If $d(\mathscr{F}) \leqq 2$, then for all transverse flow φ the global holonomy Φ of $(\mathscr{F}, \varphi)$ is abelian.*

Now we can state the result concerning Problem 1.

THEOREM 3. *Let $\mathscr{F}$ be a transversely orientable codimension-one C^r foliation on a closed orientable C^r manifold M. Let d be a positive integer. Suppose that $2 \leqq r \leqq \infty$ and $(M/\mathscr{F}, \leqq)$ is a partially ordered set. Suppose that there is a flow φ transverse to $\mathscr{F}$ satisfying the following conditions* (1) *and* (2).

(1) *The global holonomy of $(\mathscr{F}, \varphi)$ is abelian.*

(2) *For all $x \in M$ there are $s < 0$ and $t > 0$ such that $\varphi(x, s)$ and $\varphi(x, t)$ are on leaves of depth $\leqq d$.*

Then $d(\mathscr{F}) \leqq d+1$.

In Theorem 3 the condition that the global holonomy is abelian is essential. See the following.

THEOREM 4. *Let* S_2 *be the closed surface of genus* 2. *For all positive integer* d *there is a codimension-one* C^∞ *foliation* $\mathscr{F}$ *on* $S_2 \times [0, 1]$ *satisfying the following conditions* (1), (2) *and* (3).

(1) *All leaves of* $\mathscr{F}$ *are proper and transverse to* $\{x\} \times [0, 1]$ *for all* $x \in S_2$. $S_2 \times \{0\}$ *and* $S_2 \times \{1\}$ *are compact leaves.*

(2) $d(\mathscr{F}) = d$.

(3) *All holonomy groups of* $\mathscr{F}$ *are abelian.*

The foliation in the following theorem has the minimal depth as a foliation having a non-abelian holonomy group. See Proposition 4, too.

THEOREM 5. *There is a topological foliation* $\mathscr{F}$ *of codimension one on* $S_2 \times [0, 1]$ *satisfying the following conditions* (1), (2) *and* (3).

(1) *All leaves of* $\mathscr{F}$ *are proper and transverse to* $\{x\} \times [0, 1]$ *for all* $x \in S_2$.

(2) $d(\mathscr{F}) = 3$.

(3) *The holonomy group of the leaf* $S_2 \times \{0\}$ *of* $\mathscr{F}$ *is non-abelian.*

Now we treat Problems 2 and 5. At first we recall the definition of ends.

DEFINITION 4. Let F be an open manifold. A family ε of non-empty connected open subsets of F is called an *end* of F if ε satisfies the following conditions (1)-(4).

(1) $\partial U = U^a - U$ is compact for all $U \in \varepsilon$ where $(\ )^a$ means the closure with respect to the topology of F.

(2) If $U, U' \in \varepsilon$, then there is $U'' \in \varepsilon$ with $U'' \subset U \cap U'$.

(3) $\bigcap \{U | U \in \varepsilon\} = \emptyset$.

(4) ε *is a maximal family with respect to* (1), (2) *and* (3).

DEFINITION 5. Let ε be an end of a non-compact leaf F of $\mathscr{F}$. Let $L_\varepsilon(F) = \bigcap \{\bar{U} | U \in \varepsilon\}$ where $(\overline{\ \ })$ means the closure with respect to the topology of M. We call $L_\varepsilon(F)$ the ε *limit set of* F or the *limit set* of ε.

For the fundamental property of $L_\varepsilon(F)$, see Nishimori [3].

DEFINITION 6. Let F be a non-compact leaf of $\mathscr{F}$. An end ε of F is called a *tame end of depth one* if ε satisfies the following conditions (1), (2) and (3).

(1) ε is *isolated*, that is, there is $U \in \varepsilon$ such that if an end ε' contains U then $\varepsilon' = \varepsilon$.

(2) $L_\varepsilon(F) \cap F = \emptyset$.

(3) *ε approaches $L_\varepsilon(F)$ from one side*, that is, when some transverse flow φ fixed, for all $x \in L_\varepsilon(F)$ there are $U \in \varepsilon$ and $\delta > 0$ such that $\varphi(\{x\} \times [-\delta, 0]) \cap U = \emptyset$.

Now we fix a metric d of M. For a tame end ε of depth 1, we define $a(\varepsilon) = \sup\{d(\partial U, L_\varepsilon(F)) \mid U \in \varepsilon.\ \text{If } U \in \varepsilon', \text{ then } \varepsilon' = \varepsilon\}$.

By using the induction, we define a tame end of depth greater than one as follows.

DEFINITION 7. An end ε of F is called a *tame end of depth d* if ε satisfies the following conditions (1)-(4).

(1) $d(\varepsilon) = d$.

(2) $L_\varepsilon(F) \cap F = \emptyset$.

(3) ε approaches $L_\varepsilon^*(F) = L_\varepsilon(F) - \bigcap_{U \in \varepsilon}(\bigcup_{\varepsilon \neq \varepsilon' \ni U} L_{\varepsilon'}(F))$ from one side.

(4) There are $U \in \varepsilon$ and $a > 0$ such that (i) if an end $\varepsilon' \neq \varepsilon$ contains U then ε' is a tame end of depth $< d$ and $a(\varepsilon') > a$.

For a tame end ε of depth d, we define $a(\varepsilon) = \operatorname{Sup}\{d(\partial U, L_\varepsilon(F)) \mid U \in \varepsilon.$ If $U \in \varepsilon' \neq \varepsilon$ then ε' is a tame end of depth $< d\}$.

Now we can state the result concerning Problems 2 and 5.

THEOREM 6. *Let $\mathscr{F}$ be a transversely orientable C^r foliation of codimension one on a closed orientable C^r manifold M. Suppose that $2 \leqq r \leqq \infty$. Let F be a leaf of $\mathscr{F}$ such that $d = d(F)$ is finite. Suppose that all holonomy groups of leaves $F' \subset \bar{F}$ are abelian. Then F is a proper leaf. Moreover F has a finite number of tame ends of depth $d-1$ and a countable number of tame ends of depth $< d-1$ and satisfies the following conditions.*

(1) *$\bar{F}$ consists of finite leaves of $\mathscr{F}$.*

(2) *For each end ε of F, there is just one leaf $F' < F$ such that $L_\varepsilon(F) = \bar{F}'$ and $d(\varepsilon) = d(F')$.*

(3) *For each leaf $F' < F$, there is an end ε of F such that $L_\varepsilon(F) = \bar{F}'$ and $d(\varepsilon) = d(F')$. If $d(F') < d-1$, there are a countable number of such ε's. If $d(F') = d-1$, such ε is unique.*

The proof of Theorem 6 will clarify how the ends ε of F approach their limit sets $L_\varepsilon(F)$.

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2. The proofs of Propositions 1, 2, 3 and 4. At first we state some properties of limit sets. Let F be a non-compact leaf of $\mathscr{F}$. We denote by $L(F)$ the set of points $y \in M$ such that there is a sequence $x_1, x_2, \cdots \in F$, which are discrete with respect to the topology of F as a manifold, and which converges to y with respect to the topology of M. We call $L(F)$ the *limit set* of F.

LEMMA 1. *A non-compact leaf F is proper if and only if $L(F) \cap F = \emptyset$.*

LEMMA 2. *$L(F)$ and $L_\varepsilon(F)$ are non-empty compact invariant subsets of M.*

We omit the proofs of Lemmas 1 and 2. The relation between $L(F)$ and $L_\varepsilon(F)$ is as follows.

LEMMA 3. *$L(F) = \bigcup \{L_\varepsilon(F) \mid \varepsilon$ is an end of $F\}$.*

PROOF. Clearly $L(F) \supset \bigcup_\varepsilon L_\varepsilon(F)$. From now we will show that $L(F) \subset \bigcup_\varepsilon L_\varepsilon(F)$. Choose a sequence $K_1, K_2, \cdots$ of subsets of F such that

(1) K_i is a compact connected submanifold of same dimension as F,

(2) $F = \bigcup_{i=1}^{\infty} K_i$,

(3) $K_i \subset \operatorname{Int} K_{i+1}$.

Let $y \in L(F)$. Then there is a sequence $x_1, x_2, \cdots \in F$ such that $\{x_i\}_{i=1}^{\infty}$ is discrete in F and $\{x_i\}_{i=1}^{\infty}$ converges to y in M.

We remark that, for each i, $F - K_i$ consists of a finite number of connected components. We remark also that K_i contains at most a finite number of x_i's. By induction we choose connected components V_i of $F - K_i$ as follows.

(i) Let V_1 be one of the connected components of $F - K_1$ containing an infinite number of x_i's.

(ii) Suppose that $V_1, \cdots, V_i$ have been already chosen. Let V_{i+1} be one of the connected components of $V_i - K_{i+1}$ containing an infinite number of x_i's.

Let ε be the family of non-empty connected open subsets V of F such that V contains some V_i and $V^a - V$ is compact where $(\ )^a$ means the closure with respect to the topology of F as a manifold. It is easy to see that ε is an end of F and $L_\varepsilon(F)$ contains y. Thus we obtain $L(F) \subset \bigcup_\varepsilon L_\varepsilon(F)$, which completes the proof of Lemma 3.

LEMMA 4. *For any non-compact leaf F, $L(F)$ contains another leaf F'.*

COROLLARY 2. *For any non-compact leaf F, there is a leaf F' satisfying $F' < F$.*

In order to prove Lemma 4, we need the following known lemma. We give a proof for self-containedness.

LEMMA 5. *A non-empty perfect subset E of R is an uncountable set.*

PROOF OF LEMMA 5. Since E is perfect, E is closed and, for all $x \in E$, $\overline{E - \{x\}}$ contains x. If the Lebesgue measure of E is positive, clearly E is uncountable. Suppose that E is measure zero. Then $R - E$ is dense. Since $R - E$ is open, $R - E$ is a union of a countable number of open intervals $\{I_\lambda\}$; $R - E = \bigcup_\lambda I_\lambda$. Let (a_1, b_1) and (a_2, b_2) be two of the intervals I_λ such that $b_1 < a_2$. We number the connected components of $[b_1, a_2] - E$ and we obtain a sequence $I_1, I_2, \cdots$ of open intervals.

The perfectness of E implies that between any two intervals I_i and I_j there is an interval I_k which is different from I_i and I_j.

Now we construct a locally constant monotonely increasing map $\psi: [b_1, a_2] - E \to \{m/2^n \mid m \text{ and } n \text{ are integers}\}$. Let $\psi(x) = 1/2$ for $x \in I_1$. Let $i_1 = \mathrm{Min}\,\{i \mid I_i \text{ is between } (a_1, b_1) \text{ and } I_1\}$ and $i_2 = \mathrm{Min}\,\{i \mid I_i \text{ is between } I_1 \text{ and } (a_2, b_2)\}$. Let $\psi(x) = 1/2^2$ for $x \in I_{i_1}$ and $\psi(x) = 3/2^2$ for $x \in I_{i_2}$. By continuing this process we can define ψ uniquely. Since $\bigcup_{i=1}^\infty I_i$ is dense in $[b_1, a_2]$, ψ is naturally extended to a continuous map $\bar{\psi}: [b_1, a_2] \to [0, 1]$. Let $A = \{\mathrm{Min}\, I_i \mid i = 1, 2, \cdots\}$. Then we see that A is a countable set and $\bar{\psi}: ([b_1, a_2] \cap E) - A \to [0, 1]$ is a bijection. Thus E is uncountable.

PROOF OF LEMMA 4. Suppose that $L(F)$ does not contain another leaf. Since $L(F)$ is non-empty and invariant, $L(F) = F$. Let C be a line segment transverse to $\mathscr{F}$ and intersecting F. Then $L(F) \cap C = F \cap C$ is a non-empty perfect set. By Lemma 5, $F \cap C$ is an uncountable set. On the other hand the standard arguments of foliation theory tell us that, for any leaf F, $F \cap C$ is a countable set. This is a contradiction.

By the same way, we can prove the following.

LEMMA 6. *If F is not proper, $\bar{F}$ contains an uncountable number of leaves.*

PROOF. Suppose that F is not proper and $\bar{F}$ contains at most a countable number of leaves. Let C be a line segment transverse to $\mathscr{F}$ and intersecting F. Since F is not proper, $\bar{F} \cap C$ is a perfect set. By Lemma 5, $\bar{F} \cap C$ is an uncountable set. On the other hand, since $F_\lambda \cap C$ is a countable set for any leaf F_λ and $\bar{F}$ consists of at most a countable number of leaves, $\bar{F} \cap C$ is at most a countable set. This is a contradiction, which completes the proof of Lemma 6.

As to the following lemma, recall that M is a compact manifold.

LEMMA 7. *Let F be a non-compact leaf. Then $\bar{F}$ contains at most a finite number of compact leaves.*

Since $L(F)$ and $L_\varepsilon(F)$ are contained by $\bar{F}$, we have the following.

COROLLARY 3. *$L(F)$ and $L_\varepsilon(F)$ contain at most a finite number of compact leaves.*

PROOF OF LEMMA 7. Suppose that $\bar{F}$ contains an infinite number of compact leaves, from which we take an infinite sequence $F_1, F_2, \cdots$. We will bring out a contradiction.

Since M is compact, $A = \bigcap_{n=1}^{\infty}(\overline{\bigcup_{i=n}^{\infty} F_i})$ is not empty. Let $x_0 \in A$ and let C be a line segment transverse to $\mathscr{F}$ and containing x_0. Then C intersects an infinite number of compact leaves contained by $\bar{F}$. We can choose infinite sequences $a_1, a_2, \cdots, b_1, b_2, \cdots \in C$ satisfying the following conditions (1)–(4).

(1) a_i belongs to F.

(2) b_i belongs to some compact leaf contained by $\bar{F}$.

(3) b_i is between a_i and a_{i+1}, and between b_{i-1} and b_{i+1}.

(4) a_{i+1} is between b_i and b_{i+1}, and between a_i and a_{i+2}.

Let C_i be the circle consisting of a path from a_i to a_{i+1} in F and the subset $\{x \in C \mid x \text{ is between } a_i \text{ and } a_{i+1}\}$. We renumber the compact leaves so that F_i contains b_i. We may suppose that $F_i \neq F_j$ if $i \neq j$. As to the intersection numbers of C_i and F_j, we have that $C_i \cdot F_i = \pm 1$ and $C_i \cdot F_j = 0$ if $i \neq j$. Therefore the homology classes $[C_i] \in H_1(M, R)$ are linearly independent and $H_1(M, R)$ is not finitely generated, which is a contradiction since M is a compact manifold.

Now we prove the propositions.

PROOF OF PROPOSITION 1. Suppose that $(M/\mathscr{F}, \leqq)$ is not a partially ordered set. Then we have two leaves F_1, F_2 satisfying $F_1 < F_2$ and $F_2 < F_1$. Since we have an infinite sequence $F_1 < F_2 < F_1 < F_2 < F_1 < \cdots$, $d(\mathscr{F}) = \infty$. $F_1 < F_2$ means that $F_1 \subset L(F_2)$. Since $L(F_2)$ is closed and $F_2 < F_1$, $F_2 \subset \bar{F}_1 \subset L(F_2)$. Therefore F_2 is not proper. Thus we completes the proof.

PROOF OF PROPOSITION 2. (1) Let K be a non-empty compact invariant subset of M. Let $\mathscr{K}$ be the family of all non-empty compact invariant subsets $K' \subset K$. It is easy to see that $\mathscr{K}$ is inductively ordered by the relation $\subset$. By Zorn's lemma, $\mathscr{K}$ has a minimal element, which is a minimal set.

(2) Suppose that $(M/\mathscr{F}, \leqq)$ is a partially ordered set. Let K be a minimal set. At first suppose that K contains a non-compact leaf F_1. By Lemma 4, $\bar{F}_1$ contains another leaf F_2. Since $(M/\mathscr{F}, \leqq)$ is a partially ordered set, $F_1 \cap \bar{F}_2 = \varnothing$. Then $\bar{F}_2 \subset \bar{F}_1 - F_1$. Therefore $\bar{F}_2 \subsetneqq K$. Since $\bar{F}_2$ is a non-empty compact invariant subset, K is not minimal. This is

a contradiction. Thus we saw that K does not contain any non-compact leaf. Secondly let F be a compact leaf contained by K. Since F is a non-empty compact invariant subset and K is minimal, $F = K$. This completes the proof.

PROOF OF PROPOSITION 3. It is clear.

PROOF OF PROPOSITION 4. We suppose Theorem 1. Suppose that the global holonomy Φ of $(\mathscr{F}, \varphi)$ is non-abelian and $d(\mathscr{F}) \leqq 2$. Then there are $\omega_1, \omega_2 \in P(\mathscr{F})$ such that $\omega_1(0) = \omega_1(1) = \omega_2(0) = \omega_2(1)$ and $\Phi(\omega_1 \# \omega_2)(t) < \Phi(\omega_2 \# \omega_1)(t)$ for some $t \in D\Phi(\omega_1 \# \omega_2) \cap D\Phi(\omega_2 \# \omega_1)$. We may suppose that $t > 0$. According to Imanishi [1], all holonomy groups of compact leaves of $\mathscr{F}$ are abelian if $\mathscr{F}$ is almost without holonomy. Let

$$t_1 = \lim_{n\to\infty} (\Phi(\omega_2 \# \omega_1)^{-1} \circ \Phi(\omega_1 \# \omega_2))^n(t) .$$

Then the leaf F_1 containing $\varphi(\omega_1(0), t_1)$ is not a compact leaf. In fact, if F_1 is a compact leaf then we see that the holonomy group of F_1 is non-abelian, which contradicts to the result of Imanishi. By Lemma 4, $\bar{F}_1$ contains another leaf F_2. Then $F_2 < F_1$. Let F_0 be the leaf containing $\varphi(\omega_1(0), t)$.

In the case where $F_0 \neq F_1$, $F_1 < F_0$ and $d(\mathscr{F}) \geqq 3$, which is a contradiction.

In the case where $F_0 = F_1$, F_1 is not proper. By Lemma 6, $\bar{F}_1$ contains an uncountable number of leaves. By Lemma 7, $\bar{F}_1$ contains at most a finite number of compact leaves. Therefore $\bar{F}_1$ contains a non-compact leaf F_3. By Lemma 4, $\bar{F}_3$ contains another leaf F_4. Then $F_1 > F_3 > F_4$ and $d(\mathscr{F}) \geqq 3$, which is a contradiction.

Thus we completed the proof of Proposition 4, by supposing Theorem 1.

3. The proof of Theorem 1. At first we prepare some lemmas. The following lemma concerns Problem 2.

LEMMA 8. *If $d(F) \leqq 2$, then F is proper.*

PROOF. Suppose that F is not proper. As in the proof of Proposition 4, $\bar{F}$ contains another non-compact leaf F_1. By Lemma 4, $\bar{F}_1$ contains another leaf F_2. Therefore $F > F_1 > F_2$ and $d(F) \geqq 3$, which completes the proof.

LEMMA 9. *Let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a family of compact leaves. If $(M/\mathscr{F}, \leqq)$ is a partially ordered set, then $\overline{\bigcup_{\lambda\in\Lambda} F_\lambda}$ consists of only compact leaves.*

PROOF. It is easy to see that $\overline{\bigcup_{\lambda\in\Lambda} F_\lambda}$ is invariant. Let F be a leaf satisfying $F \subset \overline{\bigcup_{\lambda\in\Lambda} F_\lambda}$, $F \notin \{F_\lambda \mid \lambda \in \Lambda\}$. Suppose that F is not compact, from which we will bring out a contradiction.

Let $x_0 \in F$. Let C be a line segment transverse to $\mathscr{F}$ and containing x_0. There is a sequence $x_1, x_2, \cdots \in C \cap (\bigcup_{\lambda \in \Lambda} F_\lambda)$ converging to x_0. We denote by F_i the compact leaf containing x_i. Then $F \subset \bigcap_{n=1}^{\infty} \overline{(\bigcup_{i \geqq n} F_i)}$.

By Proposition 2, $\bar{F}$ contains a compact leaf F_0 since $\bar{F}$ is a non-empty compact invariant set. Let U be a closed tubular neighborhood of F_0. We may suppose that there is a locally trivial fibration $p: U \to F_0$ such that all fibres of p are transverse to $\mathscr{F}$. Let U_+ be the union of F_0 and a connected component of $U - F_0$. We may suppose that $\overline{F \cap U_+}$ contains F_0.

Let $y_0 \in F_0$ and $D = (p|U_+)^{-1}(y_0)$. Since $F_0 \subset \overline{F \cap U_+} \subset \bigcap_{n=1}^{\infty} \overline{(\bigcup_{i \geqq n} F_i)}$, there is a sequence $z_1, z_2, \cdots \in D \cap (\bigcup_{i=1}^{\infty} F_i)$ converging to y_0. We denote by G_i the compact leaf containing z_i. Then for all $y \in F_0$ there are a neighborhood $V(y)$ of y in F_0 and a positive integer $n(y)$ such that $(p|U_+)^{-1}(y') \cap G_i \neq \emptyset$ for all $y' \in V(y)$ and all $i \geqq n(y)$. Since $F_0 = \bigcup_y V(y)$ and F_0 is compact, there are $y_1, \cdots, y_k \in F_0$ such that $F_0 = V(y_1) \cup \cdots \cup V(y_k)$. Let n_0 be the maximum of $\{n(y_1), \cdots, n(y_k)\}$.

Let us take $\bar{x} \in D \cap F$ and fix it. Then there exists $i \geqq n_0$ such that $G_i \cap \{x \in D | x \text{ is between } y_0 \text{ and } \bar{x}\} \neq \emptyset$. Since $i \geqq n_0$, $G_i \cap (p|U_+)^{-1}(y) \neq \emptyset$ for all $y \in F_0$. Since G_i is a compact leaf, $G_i \cap (p|U_+)^{-1}(y)$ is a finite set. Therefore we can consider the nearest point $a(y)$ of $G_i \cap (p|U_+)^{-1}(y)$ to y. It is easy to see that $\{a(y) | y \in F_0\}$ is a leaf of $\mathscr{F}$. Consequently $G_i = \{a(y) | y \in F_0\}$ and $a: F_0 \to G_i$ is a C^r diffeomorphism. Let $A = \bigcup_y \{x \in (p|U_+)^{-1}(y) | x \text{ is between } y \text{ and } a(y)\}$. Then A is C^r diffeomorphic to $F_0 \times [0, 1]$. Since $\overline{F \cap U_+}$ contains F_0, there is $\bar{x}' \in (\operatorname{Int} A) \cap F$. Then $\bar{x}$ and $\bar{x}'$ belong to different connected components of $M - F_0 - G_i$. This is a contradiction, which completes the proof of Lemma 9.

The proof of Lemma 9 suggests the following.

COROLLARY 4. *If $(M/\mathscr{F}, \leqq)$ is a partially ordered set, all compact leaves are classified into a finite number of C^r diffeomorphic classes of manifolds.*

PROOF. At first we claim that for all $x \in M$ there is a neighborhood $U(x)$ of x such that the compact leaves of $\mathscr{F}$ intersecting $U(x)$ are C^r diffeomorphic. In fact for x belonging to non-compact leaves, we can find a neighborhood $U(x)$ which does not intersect any compact leaf by Lemma 9. For x belonging to isolated compact leaves, there is a neighborhood $U(x)$ such that if a compact leaf F' intersects $U(x)$ then $F' = F$. For x belonging to non-isolated compact leaves, by the proof of Lemma 9 we obtain a neighborhood $U(x)$ such that the compact leaves intersecting $U(x)$ is C^r diffeomorphic to the leaf containing x.

Since $M = \bigcup_x U(x)$ and M is compact, there are $x_1, \cdots, x_k \in M$ such that $M = U(x_1) \cup \cdots \cup U(x_k)$. Any compact leaf intersects some of $U(x_1), \cdots, U(x_k)$, so we have a finite number of C^r diffeomorphic classes.

Now we introduce some notations. Let A be a compact C^r manifold with boundary or without boundary, and B a closed transversely oriented codimension-one submanifold of A. We denote by $C(A, B)$ the compact manifold obtained from $A - B$ by attaching two copies B_1, B_2 of B, where the suffixes 1, 2 depend the transverse orientation of B. Let $f: [0, \varepsilon] \to [0, \delta]$ be a C^r diffeomorphism such that $f(0) = 0$ and $\delta < \varepsilon$. We denote by $X(A, B, f)$ the quotient space of $C(A, B) \times [0, \varepsilon]$ by the equivalence relation $\sim$ defined by $(x_1, t) \sim (x_2, f(t))$ for $t \in [0, \varepsilon]$ and $x_1 \in B_1$, $x_2 \in B_2$ such that $x_1 = x_2$ as elements of B. Then $X(A, B, f)$ is a compact manifold with corner on the boundary. We denote by $\mathscr{F}(A, B, f)$ the foliation on $X(A, B, f)$ induced from the foliation on $C(A, B) \times [0, \varepsilon]$ whose leaves are $C(A, B) \times \{t\}$, $t \in [0, \varepsilon]$.

Now we prove Theorem 1.

PROOF OF THEOREM 1. (1) By Corollary 2, it is clear.

(2) Suppose that $d(\mathscr{F}) \leqq 2$. By Proposition 1, $(M/\mathscr{F}, \leqq)$ is a partially ordered set. Suppose that the holonomy group of some non-compact leaf F is not trivial. We will bring out a contradiction.

By assumption, there is an immersion $f: [0, 1] \times [0, \varepsilon] \to M$ satisfying the following conditions (a)-(e).

(a) $f([0, 1] \times \{0\})$ is contained in F.

(b) $f(0, t) = f(1, t)$ for all $t \in [0, \varepsilon]$.

(c) $f|[0, 1) \times [0, \varepsilon]$ and $f|(0, 1] \times [0, \varepsilon]$ are embeddings.

(d) $f|\{s\} \times [0, \varepsilon]$ is transverse to $\mathscr{F}$ for all $s \in [0, 1]$.

(e) We define a map $g: (U, 0) \to (V, 0)$, where U and V are some neighborhoods of 0 in $[0, \varepsilon]$, as follows. $g(s) = t$ if and only if there is a leaf, of the induced foliation $f^*\mathscr{F}$ on $[0, 1] \times [0, \varepsilon]$, containing the points $(0, s)$ and $(1, t)$. Then there is a sequence $t_1, t_2, \cdots$ such that $g(t_i) < t_i$ and $\lim_{i\to\infty} t_i = 0$.

In (e), since the sequence $t_i, g(t_i), g^2(t_i), \cdots$ is monotonely decreasing, it has the limit. Let $s_i = \lim_{n\to\infty} g^n(t_i)$. Then the leaf F_i of $\mathscr{F}$ containing $f(0, t_i)$ is different from the leaf E_i of $\mathscr{F}$ containing $f(0, s_i)$. In fact if $F_i = E_i$, then we see that the leaf F_i is not proper. On the other hand, by Lemma 8 F_i is proper since $d(F_i) \leqq 2$. This is a contradiction.

At most a finite number of the leaves $\{E_1, E_2, \cdots\}$ are compact. In fact if an infinite number of $\{E_1, E_2, \cdots\}$ are compact, then F is contained by the closure of the union of the family of compact leaves. By Lemma

9, F is a compact leaf, which is a contradiction.

Let E_{i_0} be a non-compact leaf. By Corollary 2, there is a leaf F' satisfying $F' < E_{i_0}$. Clearly $E_{i_0} < F_{i_0}$. Then $d(\mathscr{F}) \geqq 3$, which contradicts the assumption. Therefore $\mathscr{F}$ is almost without holonomy, which completes the half of the proof of Theorem 1 (2).

Conversely suppose that $(M/\mathscr{F}, \leqq)$ is a partially ordered set and $\mathscr{F}$ is almost without holonomy. By Proposition 2, $\mathscr{F}$ has a compact leaf. It is sufficient to consider the case where $\mathscr{F}$ has a non-compact leaf. Let Ω be a connected component of

$$M - \bigcup \{F | F \text{ is a compact leaf of } \mathscr{F}\} .$$

By the same way of the proof of Lemma 7, we can prove that the closure of Ω contains only a finite number of compact leaves. Therefore Ω has a finite number of ends and we can attach the boundary. Let $\bar{\Omega}$ be the compact manifold obtained from Ω by attaching the boundary. Then $\bar{\Omega}$ is naturally immersed in M. Let F be a connected component of the boundary of $\bar{\Omega}$. Then F is a compact leaf of $\mathscr{F}|\bar{\Omega}$ where $\mathscr{F}|\bar{\Omega}$ means the foliation on $\bar{\Omega}$ induced from $\mathscr{F}$ by the immersion $\bar{\Omega} \to M$. By Imanishi [1], the holonomy group of F is a free abelian group.

We claim that the holonomy group $\Phi(F)$ of F is an infinite cyclic group. In fact suppose that the rank of $\Phi(F)$ is greater than one. We will bring out a contradiction. There are homeomorphisms $f: [0, \varepsilon_1) \to [0, \delta_1)$ and $g: [0, \varepsilon_2) \to [0, \delta_2)$ such that the germes of f and g at 0 are linearly independent elements of $\Phi(F)$.

Remark that f and g have no fixed point except 0. In fact, if otherwise, some leaf $F_1 \subset \Omega$ has non-trivial holonomy group. Since Ω consists of only non-compact leaves, F_1 is non-compact. This contradicts the assumption that $\mathscr{F}$ is almost without holonomy.

Consider the equivalence relation $\sim$ on $(0, \varepsilon_1)$ defined by $t \sim f(t)$, $t \in (0, \varepsilon_1)$. The quotient space $S = (0, \varepsilon_1)/\sim$ is a circle. By the commutativity of f and g, we can define an orientation-preserving homeomorphism $\bar{g}: S \to S$ as follows. For $t \in (0, \varepsilon_1)$, let $\bar{g}([t]) = [g(t)]$ where $[t]$ means the equivalence class of t. By the way of choosing f and g, g^n is not the identity for all non-zero integer n. If g has a periodic point, there is a leaf $F_2 \subset \Omega$ whose holonomy group is non-trivial, which contradicts the assumption that $\mathscr{F}$ is almost without holonomy. Therefore the rotation number of g is irrational.

We need the following theorem, see Nitecki [5] p. 40.

THEOREM 7. *Let $g: S^1 \to S^1$ be an orientation-preserving homeomorphism whose rotation number τ is irrational. Then there is a surjective* *continuous map* $h: S^1 \to S^1$ *such that* $h \circ g = r(\tau) \circ h$ *and* $h^{-1}(x)$ *is a point or closed interval for all* $x \in S^1$ *where* $r(\tau): S^1 \to S^1$ *is the rotation map whose rotation number is* τ.

By Theorem 7, we see that $(M/\mathscr{F}, \leqq)$ is not a partially ordered set, which is a contradiction. Therefore the holonomy group $\Phi(F)$ is infinite cyclic.

As in Nishimori [4], from the holonomy homomorphism, we have an isomorphism $u: H_1(F, Z) \to Z$. Consider the homology class $d \circ \alpha(u) \in H_{n-2}(F^{n-1}, Z)$ where $\alpha: \mathrm{Hom}(H_1(F, Z), Z) \to H^1(F, Z)$ is the canonical isomorphism and $d: H^1(F, Z) \to H_{n-2}(F, Z)$ is the Poincaré duality. There is a closed C^r submanifold N of F such that $[N] = d \circ \alpha(u) \in H_{n-2}(F, Z)$. For some C^r diffeomorphism $f: [0, \varepsilon] \to [0, \delta]$ there is a C^r embedding $h: X(F, N, f) \to \bar{\Omega}$ such that $h^*\mathscr{F} = \mathscr{F}(F, N, f)$. We do the same for all connected components of the boundary of $\bar{\Omega}$. By subtracting $h(X(F, N, f))$'s from $\bar{\Omega}$ and taking the closure, we obtain Ω_0. Remark that all holonomy groups of $\mathscr{F}|\Omega_0$ are trivial. By Proposition 2 in a somewhat generalized formulation, $\mathscr{F}|\Omega_0$ has a compact leaf. Therefore all leaves of $\mathscr{F}|\Omega_0$ are compact. Thus we have $d(\mathscr{F}) \leqq 2$. This completes the proof of Theorem 1.

4. **The proof of Theorem 3.** In order to prove Theorem 3, we need the following.

THEOREM 8 (Nishimori [4]). *Let* $\mathscr{F}$ *be a transversely orientable codimension-one* C^r *foliation on an orientable* C^r *manifold and* F *a compact leaf of* $\mathscr{F}$. *Suppose that* $2 \leqq r \leqq \infty$. *Let* T *be a tubular neighborhood of* F *and let* U_+ *be the union of* F *and a connected component of* $T - F$. *Suppose that the one-sided holonomy group of* F *defined by* $\mathscr{F}|U_+$ *is abelian. Then only one of the following three cases occurs.*

(1) *For all neighborhoods* V *of* F, *the restricted foliation* $\mathscr{F}|V \cap U_+$ *has a compact leaf which is not* F.

(2) *For all neighborhoods* V *of* F, *there is a neighborhood* V_1 *of* F *such that* $V_1 \subset V$ *and all leaves of* $\mathscr{F}|V_1 \cap U_+$ *except* F *are dense in* $V_1 \cap U_+$.

(3) *There are a transversely orientable codimension-one* C^r *submanifold* N *of* F, *a* C^r *diffeomorphism* $f: [0, \varepsilon] \to [0, \delta]$ *satisfying that* $f(t) < t$ *for all* $t \in (0, \varepsilon]$, *and a* C^r *embedding* $h: X(F, N, f) \to U_+$ *such that* $h(C(F, N) \times \{0\}) = F$ *and* $h^*\mathscr{F} = \mathscr{F}(F, N, f)$.

Now let us consider a foliation $\mathscr{F}$ on a manifold M and a transverse flow φ satisfying the conditions of Theorem 3.

By Proposition 2, $\mathscr{F}$ has a compact leaf. If all leaves of $\mathscr{F}$ are compact, then $d(\mathscr{F})=1=d$ and the proof is finished. Suppose that $\mathscr{F}$ has a non-compact leaf. Let Ω^1 be a connected component of

$$M-\bigcup\{F|F \text{ is a compact leaf of } \mathscr{F}\}.$$

It is sufficient to prove that $d(\mathscr{F}|\Omega^1)<d+1$ for all connected components Ω^1.

As Lemma 7, we see that the closure of Ω^1 contains a finite number of compact leaves. Therefore Ω^1 has a finite number of ends and we can attach the boundary. Let $\bar{\Omega}^1$ be the compact manifold obtained from Ω^1 by attaching the boundary. $\bar{\Omega}^1$ is naturally immersed in M. We denote by $\mathscr{F}|\bar{\Omega}^1$ the foliation induced from $\mathscr{F}$ by the immersion.

Let $F_1, \cdots, F_k$ be the compact leaves of $\mathscr{F}|\bar{\Omega}^1$, that is, connected components of $\partial\bar{\Omega}^1$. For each F_i, the conditions of Theorem 8 are satisfied. By the way of defining $\bar{\Omega}^1$, (1) does not occur. If (2) occurs, then $(M/\mathscr{F}, \leqq)$ is not a partially ordered set. Therefore (3) of Theorem 8 occurs and there are $N_i, f_i\colon [0, \varepsilon_i]\to[0, \delta_i]$ and $h_i\colon X(F_i, N_i, f_i)\to\bar{\Omega}^1$ for $i=1, \cdots, k$ as in Theorem 8. We may suppose that, for each $x\in F_i$, $h_i(\{x\}\times[0, \varepsilon_i])$ is contained by some orbit of φ. Let $\Omega^1_0=\bar{\Omega}^1-\bigcup_{i=1}^k h_i(\operatorname{Int} X(F_i, N_i, f_i))-\bigcup_{i=1}^k F_i$. Then Ω^1_0 is a compact manifold with corner.

Clearly $(\Omega^1_0/(\mathscr{F}|\Omega^1_0), \leqq)$ is a partially ordered set. We can generalize Proposition 2 so that we can apply it in this case. Therefore $\mathscr{F}|\Omega^1_0$ has a compact leaf. If all leaves of $\mathscr{F}|\Omega^1_0$ are compact, then $d(\mathscr{F}|\Omega^1)=1$ and the proof is finished. Suppose that $\mathscr{F}|\Omega^1_0$ has a non-compact leaf. We can also generalize Theorem 8 so that we can repeat the same process.

We denote by Ω^{i+1} a connected component of

$$\Omega^i_0-\bigcup\{F|F \text{ is a compact leaf of } \mathscr{F}|\Omega^i_0\}$$

for some Ω^i. If the process ends before obtaining Ω^d, then $d(\mathscr{F})\leqq d$. Thus it is sufficient to consider the case where we have Ω^d.

Let $F_1, \cdots, F_k$ be the compact leaves of $\mathscr{F}|\bar{\Omega}^d$. Remark that the leaf of $\mathscr{F}$ containing F_i has depth d and all leaves of $\mathscr{F}$ passing Ω^d have depth greater than d. By reversing the direction of φ if necessary, we may suppose that for each $x\in F_1$ there is $\varepsilon>0$ such that $\varphi(\{x\}\times(0, \varepsilon))\subset\Omega^d$. For $i=2, \cdots, k$, let

$$K_i=\left\{x\in F_1\,\middle|\,\begin{array}{l}\text{There is } t>0 \text{ such that}\\ \varphi(\{x\}\times(0, t))\subset\Omega^d \text{ and } \varphi(x, t)\in F_i\,.\end{array}\right\}$$

Here we use confused notations since $\bar{\Omega}^d$ and F_i's are not subsets of M. It is easy to see that K_i is an open subset of F_1. By the condition (2) of Theorem 3, for all $x\in F_1$ there is $t>0$ such that $\varphi(x, t)\notin\Omega^d$. Therefore

$F_1 = \bigcup_{i=2}^k K_i$. Clearly $K_i \cap K_j = \emptyset$ if $i \neq j$. Since F_1 is connected, only one of K_i's is non-empty. By renumbering F_i's, we may suppose that $K_2 = F_1$.

Let $\Omega = \{\varphi(x, t) \in \bar{\Omega}^d \mid x \in F_1, t \geqq 0, \varphi(\{x\} \times (0, t)) \subset \Omega^d\}$. Then we see that Ω is a non-empty open closed subset of $\bar{\Omega}^d$. By the connectivity of $\bar{\Omega}^d$, we have $\Omega = \bar{\Omega}^d$. Therefore there is a C^r diffeomorphism $h: F_1 \times [0, 1] \to \bar{\Omega}^d$. The induced foliation $h^*\mathscr{F}$ on $F_1 \times [0, 1]$ satisfying the following.

(1) $F_1 \times \{0\}$ and $F_1 \times \{1\}$ are compact leaves of $h^*\mathscr{F}$.

(2) The other leaves of $h^*\mathscr{F}$ are non-compact.

(3) For all $x \in F_1$, $\{x\} \times [0, 1]$ is transverse to $h^*\mathscr{F}$.

Such foliation was studied in Nishimori [4]. Since the global holonomy of $(\mathscr{F}, \varphi)$ is abelian and $r \geqq 2$, we can apply Theorem 1* of Nishimori [4] to $h^*\mathscr{F}$. Thus we see that $d(\mathscr{F} \mid \bar{\Omega}^d) = 2$ and $d(\mathscr{F}) = d + 1$. This completes the proof of Theorem 3.

5. The proofs of Theorems 4 and 5. At first we give a method to construct a C^r foliation $\mathscr{F}$ on $S_2 \times [0, 1]$, whose leaves are all C^∞ submanifolds and transverse to $\{x\} \times [0, 1]$ for all $x \in S_2$, from given orientation-preserving C^r diffeomorphisms $f_1, f_2: [0, 1] \to [0, 1]$, $0 \leqq r \leqq \infty$.

Take disjoint circles C_1, C_2 in S_2 such that $S_2 - C_1 \cup C_2$ is connected. Let U_1, U_2 be disjoint closed tubular neighborhoods of C_1, C_2. There are C^∞ diffeomorphisms $h_i: C_i \times [-1, 1] \to U_i$, $i = 1, 2$. Let $\alpha: [-1, 1] \to [0, 1]$ be a C^∞ map such that

(1) $\alpha(t) = 0$ in a neighborhood of -1,

(2) $\alpha(t) = 1$ in a neighborhood of 1.

Let $\mathscr{F}_i$ be the foliation on $U \times [0, 1]$ whose leaves are

$$\{(h_i(x, s), t) \mid x \in C_i, s \in [-1, 1], t = \alpha(s)t_0 + (1 - \alpha(s))f_i(t_0)\}, t_0 \in [0, 1] .$$

Let $\mathscr{F}_0$ be the foliation on $(S_2 - \mathrm{Int}(U_1 \cup U_2)) \times [0, 1]$ whose leaves are $(S_2 - \mathrm{Int}(U_1 \cup U_2)) \times \{t\}$, $t \in [0, 1]$. By connecting $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{F}_2$ together, we have a foliation $\mathscr{F}$ on $S_2 \times [0, 1]$ which clearly has the desired property.

Let $f: [0, 1] \to [0, 1]$ be a C^∞ diffeomorphism such that $f(t) < t$ for all $t \in (0, 1)$ and $h: (0, 1) \to R$ a C^∞ diffeomorphism such that $h(f^n(x)) = h(x) - n$ for all integer n. We will construct C^∞ diffeomorphisms $g_4, g_5: R \to R$. Let $\bar{g}_i: [0, 1] \to [0, 1]$ be a map defined by

$$\bar{g}_i(0) = 0, \bar{g}_i(1) = 1, \bar{g}_i \mid (0, 1) = h^{-1} \circ g_i \circ h .$$

Then $\bar{g}_4$ will be a C^∞ diffeomorphism and $\bar{g}_5$ a homeomorphism. The foliation $\mathscr{F}_i$ obtained by using f and $\bar{g}_i$ will be the example of Theorem i, $i = 4, 5$.

Now choose a sequence $a_1, a_2, \cdots, a_{d-1}, b_1, b_2, \cdots, b_{d-1} \in R$ such that $0 = a_1 < a_2 < \cdots < a_{d-1} < b_{d-1} < \cdots < b_2 < b_1 = 1$. Take $g_4: R \to R$ such that

(1) $g_4(x) = x$ if $x \leqq 1 + d_1$ or if $d - 1 + b_{d-1} \leqq x$.

(2) For $n = 1, \cdots, d-2$,

$$g_4(x) = x \quad \text{if} \quad n + b_n \leqq x \leqq n + 1 + a_{n+1} .$$

(3) For $n = 1, \cdots, d-1$,

$$g_4(x) < x \quad \text{if} \quad n + a_n < x < n + b_n .$$

(4) For $n = 2, \cdots, d-1$,

$$g_4(n - 1 + b_n) = n - 1 + a_n .$$

Then clearly g_4 is C^∞ diffeomorphism. The holonomy groups of $S_2 \times \{0\}$, $S_2 \times \{1\}$ and the leaves F_i of $\mathscr{F}_4$ containing $(S_2 - \operatorname{Int}(U_1 \cup U_2)) \times \{h^{-1}(a_i)\}$, $i = 1, \cdots, d-2$ are infinite cyclic groups. The other leaves of $\mathscr{F}_4$ have trivial holonomy groups. The relation $\leqq$ is as follows.

$$S_2 \times \{0\}, S_2 \times \{1\} < F_1 < F_2 < \cdots < F_{d-2} < F$$

where F is an arbitrary leaf different from $S_2 \times \{0\}$, $S_2 \times \{1\}$, $F_1, \cdots, F_{d-2}$. Therefore $d(\mathscr{F}_4) = d$, so $\mathscr{F}_4$ has the desired property.

Let $g: R \to R$ be a C^∞ diffeomorphism such that $g|(-\infty, 0] \cup [1, \infty)$ is the identity and $g(x) < x$ for all $x \in (0, 1)$. Take $g_5: R \to R$ such that for all integer n

$$g_5(x) = x \quad \text{if} \quad 2n \leqq x \leqq 2n + 1 ,$$
$$g_5(x) = 2n + 1 + g(x - 2n - 1) \quad \text{if} \quad 2n + 1 < x < 2n + 2 .$$

Then the holonomy group of the compact leaves $S_2 \times \{0\}$, $S_2 \times \{1\}$ are not abelian. The relation $\leqq$ is as follows.

$$S_2 \times \{0\}, S_2 \times \{1\} < F_0 < F$$

where F_0 is the leaf of $\mathscr{F}_5$ containing $(S_2 - \operatorname{Int}(U_1 \cup U_2)) \times \{h^{-1}(0)\}$ and F is an arbitrary leaf of $\mathscr{F}_5$ different from $S_2 \times \{0\}$, $S_2 \times \{1\}$ and F_0. Therefore $d(\mathscr{F}_5) = 3$ and $\mathscr{F}_5$ has the desired property.

6. The proof of Theorem 6. Let F be a leaf of $\mathscr{F}$ whose depth $d = d(F)$ is finite. If $d = 1$, F is a compact leaf by Corollary 2 and the proof is finished. Suppose that $d \geqq 2$. There is a sequence $F_1, \cdots, F_d$ of leaves such that $F_1 < F_2 < \cdots < F_d = F$. Then we see that $d(F_i) = i$ and F_1 is a compact leaf. Thus $\bar{F}$ contains a compact leaf and $\mathscr{F}$ has a compact leaf. Let Ω^1 be the connected component of

$$M - \bigcup \{F' | F' \text{ is a compact leaf}\}$$

containing F. As in the proof of Theorem 3, the closure of Ω^1 contains a finite number of compact leaves and we can consider the compact manifold $\bar{\Omega}^1$ obtained from Ω^1 by attaching the boundary. $\bar{\Omega}^1$ is naturally immersed in M. Then F can be considered a leaf of the foliation $\mathscr{F}|\bar{\Omega}^1$ on $\bar{\Omega}^1$ induced from $\mathscr{F}$ by the immersion. Let $F'_1, \cdots, F'_k$ be the compact leaves of $\mathscr{F}|\bar{\Omega}^1$ contained by $\bar{F}$. Since the conditions of Theorem 8 are satisfied for $F'_1, \cdots, F'_k$ and the condition (3) of Theorem 8 occurs, we have codimension-one submanifolds N_i of F'_i, C^r diffeomorphisms $f_i: [0, \varepsilon_i] \to [0, \delta_i]$ satisfying $f_i(t) < t$ for all $t \in (0, \varepsilon_i]$, and C^r immersions $h_i: X(F'_i, N_i, f_i) \to \bar{\Omega}^1$ such that $h_i(C(F'_i, N_i) \times \{0\}) = F'_i$ and $h_i^* \mathscr{F} = \mathscr{F}(F'_i, N_i, f_i)$, for $i = 1, \cdots, k$. Let

$$\Omega_0^1 = \bar{\Omega}^1 - \bigcup_{i=1}^{k} h_i(\operatorname{Int} X(F_i, N_i, f_i)) - \bigcup_{i=1}^{k} F_i .$$

Then Ω_0^1 is a compact manifold with corner on the boundary. Remark that Ω_0^1 contains still compact leaves of $\mathscr{F}|\bar{\Omega}^1$ which are not contained by $\bar{F}$. By the way of constructing Ω_0^1, the intersection of Ω_0^1 and a leaf of $\mathscr{F}|\bar{\Omega}^1$ is empty or a leaf of the restricted foliation $\mathscr{F}|\Omega_0^1$.

Let G be a non-compact leaf of $\mathscr{F}|\bar{\Omega}^1$ contained by $\bar{F}$. Then $d(G) \leqq d(F) < \infty$ and $\bar{G}$ contains a compact leaf of $\mathscr{F}|\bar{\Omega}^1$ as above. We claim $d(G \cap \Omega_0^1) = d(G) - 1$. In fact, let $G_1 < G_2 < \cdots < G_{d'} = G$ where $d' = d(G)$. Since G_1 is a compact leaf of $\mathscr{F}|\bar{\Omega}^1$ and $G_1 \subset \bar{G} \subset \bar{F}$, $G_1 \cap \Omega_0^1 = \varnothing$. We see also that $G_2 \cap \Omega_0^1$ is a compact leaf of $\mathscr{F}|\Omega_0^1$ and that $G_2 \cap \Omega_0^1 < G_3 \cap \Omega_0^1 < \cdots < G_{d'} \cap \Omega_0^1 = G \cap \Omega_0^1$. Therefore $d(G \cap \Omega_0^1) \geqq d' - 1$. Now suppose that $d(G \cap \Omega_0^1) > d' - 1$. Then there is a sequence $H_1, \cdots, H_{d'}$ of leaves of $\mathscr{F}|\Omega_0^1$ such that $H_1 < H_2 < \cdots < H_{d'} = G \cap \Omega_0^1$. Let G'_i be the leaf of $\mathscr{F}|\bar{\Omega}^1$ containing H_i. Then $G'_1 < \cdots < G'_{d'} = G$ and G'_1 contains another leaf G'_0 of $\mathscr{F}|\bar{\Omega}^1$ since G'_1 is a non-compact leaf of $\mathscr{F}|\bar{\Omega}^1$. Thus we have $d(G) > d'$, which is a contradiction. Therefore $d(G \cap \Omega_0^1) = d' - 1$.

Let U be a connected component of $h_i(\operatorname{Int} X(F'_i, N_i, f_i)) \cap G$ for some i. It is easy to see that there is uniquely an end ε of G containing U and that ε is isolated. Furthermore $L_\varepsilon(G) = F'_i$ and clearly ε approaches to $L_\varepsilon(G)$ from one side. Therefore ε is a tame end of depth 1.

Since $d(F \cap \Omega_0^1) = d(F) - 1 < \infty$, $\overline{F \cap \Omega_0^1}$ contains a compact leaf of $\mathscr{F}|\Omega_0^1$. Any compact leaf of $\mathscr{F}|\Omega_0^1$ contained by $\overline{F \cap \Omega_0^1}$ has the form $G \cap \Omega_0^1$ where G is a leaf $\mathscr{F}$ contained by $\bar{F}$ and $d(G) = 2$. Let Ω^2 be the connected component of

$$\Omega_0^1 - \bigcup \{F' \mid F' \text{ is a compact leaf of } \mathscr{F} \mid \Omega_0^1\}$$

containing $F \cap \Omega_0^1$. As before the closure of Ω^2 contains a finite number of compact leaves of $\mathscr{F} \mid \Omega_0^1$ and we can consider the compact manifold $\bar{\Omega}^2$ obtained from Ω^2 by attaching the boundary and $\mathscr{F} \mid \bar{\Omega}^2$. We can apply Theorem 8 to the compact leaves $G_1', \cdots, G_l'$ of $\mathscr{F} \mid \bar{\Omega}^2$. Thus we obtain $N_i' \subset G_i'$, $f_i' \colon [0, \varepsilon_i'] \to [0, \delta_i']$ and $h_i' \colon X(G_i', N_i', f_i') \to \bar{\Omega}^2$ such that $h_i'(C(G_i', N_i') \times \{0\}) = G_i'$ and $(h_i')^* \mathscr{F} = \mathscr{F}(G_i', N_i', f_i')$. Let

$$\Omega_0^2 = \Omega^2 - \bigcup_{i=1}^{l} h_i'(\operatorname{Int} X(G_i', N_i', f_i')) - \bigcup_{i=1}^{l} G_i' .$$

Then Ω_0^2 be the compact manifold with corner. Let G' be a leaf of $\mathscr{F}$ contained by $\bar{F}$ such that $G' \cap \bar{\Omega}^2$ is a non-compact leaf of $\mathscr{F} \mid \bar{\Omega}^2$. We can prove that a connected component U' of $G' \cap (M - \Omega_0^2)$ belongs to unique end ε' of G and ε' is a tame end of depth 2. As to the condition (1) of Definition 7, take U' as U and, as a,

$$\operatorname{Min} \{d(\partial(U' \cap (M - \Omega_0^1)), F_1' \cup \cdots \cup F_k'), d(\partial U', G_1' \cup \cdots \cup G_l')\} .$$

Furthermore $d(G' \cap \Omega_0^1) = d(G') - 2$.

We can repeat this process and we obtain sequences $\Omega^1, \cdots, \Omega^{d-1}$ and $\Omega_0^1, \cdots, \Omega_0^{d-1}$ of submanifolds of M such that

(1) $\Omega^1 \supset \Omega_0^1 \supset \Omega^2 \supset \Omega_0^2 \supset \cdots \supset \Omega^{d-1} \supset \Omega_0^{d-1}$,

(2) $F \cap \Omega_0^{d-1} \neq \emptyset$.

Since $d(F) = d$, $F \cap \Omega_0^{d-1}$ is a compact leaf of $\mathscr{F} \mid \Omega_0^{d-1}$ and our process is finished.

F is a proper leaf. In fact, if F is not proper then we can show that $F \cap \Omega_0^{d-1}$ is a non-compact leaf of $\mathscr{F} \mid \Omega_0^{d-1}$, which is a contradiction.

Since $F \cap \Omega_0^{d-1}$ is a compact leaf of $\mathscr{F} \mid \Omega_0^{d-1}$, $F \cap (M - \Omega_0^{d-1})$ consists of a finite number of connected components. Let V be a connected component of $F \cap (M - \Omega_0^{d-1})$. Then we see that V belongs to unique end ε^* of F and ε^* is a tame end of depth $d-1$. As to the condition (1) of Definition 7, it is sufficient to take V as U and, as a,

$$\operatorname{Min} \{d(\partial(V \cap (M - \Omega_0^1)), \partial\Omega^1), d(\partial(V \cap (M - \Omega_0^2)), \partial\bar{\Omega}^2), \cdots, d(\partial(V \cap (M - \Omega_0^{d-1})), \partial\Omega^{d-1})\} .$$

Therefore F has a finite number of tame ends of depth $d-1$. In the same way, a connected component of $F \cap (M - \Omega_0^i)$ corresponds to a tame end of depth i for $i = 1, \cdots, d-2$. For $i < d-1$, clearly there are a countable number of such connected components.

Now we show (1) of Theorem 6. Remark that a leaf F' of $\mathscr{F}$ contained by $\bar{F}$ contains a connected component of the image of $\partial\bar{\Omega}^i$ by the natural immersion: $\bar{\Omega}^i \longrightarrow M$ for some i. Since there are only a finite number of such connected components, $\bar{F}$ contains only a finite number of leaves of $\mathscr{F}$.

By the above arguments, (2) and (3) of Theorem 6 are clear, which completes the proof of Theorem 6.

REFERENCES

[1] H. IMANISHI, Structure of codimension one foliations which are almost without holonomy, to appear.

[2] T. MIZUTANI AND I. TAMURA, Foliations of even dimensional manifolds, Proceedings of the International Conference on Manifolds and Related Topics in Topology, Tokyo, 1973, 189–194.

[3] T. NISHIMORI, Isolated ends of open leaves of codimension-one foliations, Quart. J. Math., Oxford (3), 26 (1975), 159–167.

[4] T. NISHIMORI, Compact leaves with abelian holonomy, Tôhoku Math. J., 27 (1975), 259–272.

[5] Z. NITECKI, Differentiable dynamics, The M. I. T. Press, 1971.

[6] I. TAMURA, Specially spinnable manifolds, Proceedings of the International Conference on Manifolds and Related Topics in Topology, 1973, 181–187.

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