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ON THE DECOMPOSITION OF GENERALIZED S-CURVATURE-LIKE TENSOR FIELDS

YASHIRO YOSHIMATSU

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The purpose of this note is to study a decomposition of generalized S-curvature-like tensor fields on a Sasakian manifold, and to get a certain relationship among Ricci tensor, Bianchi identity and contact Bochner curvature tensor.

K. Nomizu [4] studied a decomposition of generalized curvature tensor fields on a Riemannian manifold, and revealed a certain interesting relationship among the tensors and tensor identities named after Codazzi, Ricci, Bianchi and Weyl. Studying its Kaehlerian analogy H. Mori [2] obtained a similar relationship among Bochner tensor (in place of Weyl tensor), Ricci tensor and the other two tensor identities.

In this paper, first, we define a (ϕ, ξ, η)-structure on a vector space with an inner product, and an S-curvature-like tensor on V. As a component of an orthogonal decomposition of an S-curvature-like tensor L, we obtain a contact Bochner tensor associated to L. The same decomposition implies directly a necessary and sufficient condition, obtained by Tagawa [7], in order that the contact Bochner tensor on a Sasakian manifold vanishes. Then we define a generalized S-curvature-like tensor field L on a Sasakian manifold M so that L_p is an S-curvature-like tensor over the tangent space $T_p(M)$ at each point p of M. When we consider the decomposition of a generalized S-curvature-like tensor field, a natural question arises: When are the components of the decomposition proper (i.e., When do they satisfy the second Bianchi identity)? An answer is given by a certain equation to be satisfied by the Ricci tensor field. In view of analogy which exists between an S-curvature-like tensor (resp. a generalized S-curvature-like tensor field) and a K-curvature tensor (resp. a generalized K-curvature tensor field) defined in [3], all our methods can be applied also to the case when V is a vector space with a Hermitian inner product (resp. M is a Kaehler manifold) (cf. [3]). In this case, Tagawa's result mentioned above is reduced to Chen and Yano's result [1] which gives a necessary and sufficient condition in order that the Bochner tensor on M vanishes.

1. Statement of results. Let V be a $(2n+1)$-dimensional real vector space with an inner product denoted by $\langle\ ,\ \rangle$. A tensor L of type (1, 3) over V can be considered as a bilinear mapping

$$(x, y) \in V \times V \mapsto L(x, y) \in \operatorname{Hom}(V, V)\,.$$

Such a tensor L is called a curvature tensor over V if it has the following properties:

$$L(y, x) = -L(x, y)\,; \tag{1.1}$$

$L(x, y)$ is a skew-symmetric endomorphism of V, i.e.,

$$\langle L(x, y)u, v\rangle + \langle u, L(x, y)v\rangle = 0\,; \tag{1.2}$$

$$\sigma(L(x, y)z) = 0 \qquad \text{(the first Bianchi identity)}\,, \tag{1.3}$$

where σ denotes the cyclic sum over x, y, and z.

For a curvature tensor L, the Ricci tensor $K = K_L$ of type (1, 1) is a symmetric endomorphism of V defined by

$$K(x) = \text{trace of the bilinear map:}$$
$$(y, z) \in V \times V \mapsto L(x, y)z \in V\,.$$

The trace of the Ricci tensor K_L is called the scalar curvature of L.

A (ϕ, ξ, η)-structure is defined on V by tensors ϕ, ξ, and η of type (1, 1), (1, 0), and (0, 1), respectively, over V, satisfying the following conditions:

$$\eta(\xi) = 1\,; \tag{1.4}$$

$$\eta(\phi x) = 0\,; \tag{1.5}$$

$$\phi^2(x) = -x + \eta(x)\xi\,; \tag{1.6}$$

$$\langle \xi, \xi\rangle = 1\,; \tag{1.7}$$

$$\eta(x) = \langle \xi, x\rangle\,; \tag{1.8}$$

$$\langle \phi x, \phi y\rangle = \langle x, y\rangle - \eta(x)\eta(y)\,. \tag{1.9}$$

Let V be a $(2n + 1)$-dimensional vector space with a (ϕ, ξ, η)-structure. A curvature tensor L is called an S-curvature tensor over V if it has the following properties:

$$L(x, y)\phi z = \phi(L(x, y)z) + \langle \phi x, z\rangle y - \langle \phi y, z\rangle x - \langle y, z\rangle \phi x + \langle x, z\rangle \phi y\,;$$
$$L(\xi, x)y = \langle y, x\rangle \xi - \eta(y)x\,.$$

A curvature tensor L is called an S-curvature-like tensor over V if it has the following properties:

$$L(x, y) \circ \phi = \phi \circ L(x, y)\,; \tag{1.10}$$

(1.11) $$L(\xi, x) = 0 .$$

We denote by $\mathscr{L}(V)$ the vector space of all S-curvature-like tensors over V.

Let P be a ϕ-invariant 2-plane in V and let x be a unit vector in P. For an S-curvature like tensor L, we set

$$k(P) = \langle L(x, \phi x)\phi x, x\rangle .$$

We call that $k(P)$ is the ϕ-sectional curvature of L for P.

For $x, y \in V$, we denote by $x\Lambda y$ and $x\Delta y$ the skew-symmetric endomorphisms of V, respectively, defined by

$$(x\Lambda y)z = \langle z, y\rangle x - \langle z, x\rangle y ,$$
$$(x\Delta y)z = (x\Lambda y)z - \eta(z)(\eta(y)x - \eta(x)y) - (\eta(x)\langle z, y\rangle - \eta(y)\langle z, x\rangle)\xi .$$

REMARK 1. Let L_0 be the S-curvature tensor defined by

$$L_0(x, y) = x\Lambda y .$$

Then L is an S-curvature tensor if and only if $L - L_0$ is an S-curvature-like tensor. If $\tilde{L}$ is the S-curvature-like tensor corresponding to an S-curvature tensor L, that is,

$$\tilde{L} = L - L_0 ,$$

and K and $\tilde{K}$ are the Ricci tensors, respectively, for L and $\tilde{L}$, then

$$\tilde{K} = K - (2n)I ,$$

where I denotes the identity transformation of V.

From now on we shall discuss only S-curvature-like tensors, since they are more convenient than S-curvature tensors for our computing.

The following proposition gives examples of S-curvature-like tensors.

PROPOSITION 1. *Let A and B be two symmetric endomorphisms of V, each of which commutes with ϕ. If we define $L = L_{A,B}$ by*

(1.12) $$L(x, y) = Ax\Delta By + Bx\Delta Ay + \phi Ax\Lambda\phi By + \phi Bx\Lambda Ay + 2\langle Ax, \phi y\rangle\phi B + 2\langle Bx, \phi y\rangle\phi A ,$$

then L is an S-curvature-like tensor.

We define $\mathscr{L}_1(V)$ to be the subspace of $\mathscr{L}(V)$ consisting of all S-curvature-like tensors

$$L = \frac{c}{2}L_{I,I} ,$$

where c is an arbitrary constant, i.e.,

$\mathscr{L}_1(V) = \{L \in \mathscr{L}(V)$ with constant ϕ-sectional curvature$\}$. Let $\mathscr{L}_1^{\perp}(V)$ be the orthogonal complement of $\mathscr{L}_1(V)$ in $\mathscr{L}(V)$. Then we have the following propositions:

PROPOSITION 2.

$$\mathscr{L}_1^{\perp}(V) = \{L \in \mathscr{L}(V) \text{ with vanishing scalar curvature}\}$$

and

$$\mathscr{L}(V) = \mathscr{L}_1(V) \oplus \mathscr{L}_B(V) \oplus \mathscr{L}_2(V) \qquad (\textit{orthogonal}) ,$$

where

$$\mathscr{L}_B(V) = \{L \in \mathscr{L}(V) \text{ with vanishing Ricci tensor}\} ,$$
$$\mathscr{L}_2(V) = \text{orthogonal complement of } \mathscr{L}_B(V) \text{ in } \mathscr{L}_1 + (V) .$$

PROPOSITION 3. *For $L \in \mathscr{L}(V)$, let*

$$L = L_1 + L_B + L_2 ,$$

where $L_1 \in \mathscr{L}_1(V)$, $L_B \in \mathscr{L}_B(V)$, $L_2 \in \mathscr{L}_2(V)$. Then

$$L_1 = \frac{\operatorname{tr} K}{8n(n+1)} L_{I,I} ,$$

$$L_2 = \frac{1}{2(n+2)} L_{K,I} - \frac{\operatorname{tr} K}{4n(n+2)} L_{I,I} ,$$

$$L_B = L - \frac{1}{2(n+2)} L_{K,I} + \frac{\operatorname{tr} K}{8(n+1)(n+2)} L_{I,I} ,$$

where K is the Ricci tensor of L and $L_{A,B}$ is the tensor defined by (1.12).

For each $L \in \mathscr{L}(V)$, the $\mathscr{L}_B(V)$-component L_B is called the contact Bochner tensor associated to L.

COROLLARY 1. *The contact Bochner tensor associated to $L \in \mathscr{L}(V)$ is* 0 *if and only if*

$$L = L_{A,I} \tag{1.13}$$

for a symmetric endomorphism A of V which commutes with ϕ.

From Corollary 1 we get easily the following fact which is stated in [7] in terms of an S-curvature tensor.

COROLLARY 2 (Tagawa [7], cf. Chen and Yano [1]). *In order that the contact Bochner tensor associated to an S-curvature-like tensor vanishes, it is necessary and sufficient that there exists a (unique) symmetric endomorphisms Q of V which commutes with ϕ and satisfies the following: the ϕ-sectional curvature $k(P)$ for a 2-plane P is the trace* *of the restriction Q to P, i.e., $k(P) = trace\ Q/P$, the inner product being also restricted to P.*

A Sasakian structure (ϕ, ξ, η) is defined on a Riemannian manifold (M, g) by tensor fields ϕ, ξ, and η of type (1, 1), (1, 0), and (0, 1) which give (ϕ_p, ξ_p, η_p)-structure on the tangent space $T_p(M)$ with the inner product g_p for each point p of M and satisfy the following conditions:

(1.14) $$(\nabla_X\phi)Y = \eta(Y)X - \langle X, Y\rangle\xi ;$$

(1.15) $\nabla_X\xi = \phi(X)$, (which is equivalent to $(\nabla_X\eta)Y = \langle Y, \phi X\rangle$) ,

where X and Y are any vector fields. Here and in the following, we denote $g(\ ,\ )$ by $\langle\ ,\ \rangle$ for brevity.

A Riemannian manifold with a Sasakian structure is called an Sasakian manifold. A (differentiable) tensor field L of type (1, 3) on a Sasakian manifold is called a generalized S-curvature tensor field (resp. a generalized S-curvature-like tensor field) if for each point p the tensor L_p is an S-curvature tensor (resp. an S-curvature-like tensor) over $T_p(M)$. We shall say that L is proper if it satisfies the second Bianchi identity, that is,

$$\sigma(\nabla_X L)(Y, Z) = 0 .$$

For vector fields X and Y on M, we denote by $X\Lambda Y$ and $X\Delta Y$ the tensor fields of type (1, 1) which map a vector field Z, respectively, into

$$\langle Z, Y\rangle X - \langle Z, X\rangle Y$$

and

$$\langle Z, Y\rangle X - \langle Z, X\rangle Y - \eta(Z)(\eta(Y)X - \eta(X)Y) \\ - (\eta(X)\langle Z, Y\rangle - \eta(Y)\langle Z, X\rangle)\xi .$$

REMARK 2. Let L_0 be the proper generalized S-curvature tensor field defined by

$$L_0(X, Y) = X\Lambda Y .$$

Then L is a (proper) generalized S-curvature tensor field if and only if $L - L_0$ is a (proper) generalized S-curvature-like tensor field.

From now on we shall discuss only generalized S-curvature-like tensor fields, since they are more advantageous than generalized S-curvature tensor fields for our computing.

We see the following fact corresponding to Proposition 1: Let A and B be two tensor fields of type (1, 1) which are symmetric as endomorphisms of the tangent space and commute with ϕ. Then

$$L_{A,B}(X, Y) = AX\varDelta BY + BX\varDelta AY + \phi AX\varLambda\phi BY + \phi BX\varLambda\phi AY + 2\langle AX, \phi Y\rangle\phi B + 2\langle BX, \phi Y\rangle\phi A$$

defines a generalized S-curvature-like tensor field.

If L is a generalized S-curvature-like tensor field on a Sasakian manifold M, then applying the decomposition in Proposition 3 at each point p of M we obtain

$$L = L_1 + L_B + L_2 ,$$

where L_1, L_B, and L_2 are generalized S-curvature-like tensor fields which, at each point p, belong to $\mathscr{L}_1$, $\mathscr{L}_B$, and $\mathscr{L}_2$ over $T_p(M)$, respectively.

THEOREM 1. *On a $(2n+1)$-dimensional Sasakian manifold M, let*

$$L = L_1 + L_B + L_2$$

be the natural decomposition of a proper generalized S-curvature-like tensor field L. If the Ricci tensor field K of L satisfies the following equation:

$$(\nabla_X K)Y = -\eta(Y)K\phi X - \langle K\phi X, Y\rangle\xi , \tag{1.16}$$

then L_1, L_B and L_2 are proper. Conversely, if L_1, L_B, and L_2 are proper and if $n \geqq 2$, then K satisfies the equation (1.16).

COROLLARY 3. *On a Sasakian manifold M of dimension $\geqq 5$ let L be a proper generalized S-curvature-like tensor field whose scalar curvature is constant. Then the associated contact Bochner tensor field L_B is proper if and only if the Ricci tensor field K of L satisfies the equation* (1.16).

We get Theorem 1 by the help of the following propositions.

PROPOSITION 4. *Let L be a proper generalized S-curvature-like tensor field on a Sasakian manifold M, and let K be the Ricci tensor field of L. Then* (1.16) *is equivalent to the following formula:*

$$\langle(\nabla_Y K)X - (\nabla_X K)Y, Z\rangle = \eta(Y)\langle\phi KX, Z\rangle - \eta(X)\langle\phi KY, Z\rangle + 2\eta(Z)\langle Y, \phi KX\rangle . \tag{1.17}$$

PROPOSITION 5. *The assumptions and notation being as in Proposition 4, suppose that K satisfies the equation* (1.16). *Then* $\operatorname{tr} K$ *is constant on M.*

PROPOSITION 6. *On a Sasakian manifold M let A be a tensor field of type* (1, 1) *which is symmetric at each point and satisfies*

$$A\phi = \phi A \quad \textit{and} \quad A\xi = 0 .$$

Let L be a generalized S-curvature-like tensor field defined by

$$L = L_{A,I} .$$

If L is proper and if tr A *is constant, then A satisfies the following equation*:

$$(1.18) \qquad \langle (\nabla_Y A)X - (\nabla_X A)Y, Z\rangle = \eta(Y)\langle \phi AX, Z\rangle - \eta(X)\langle \phi AY, Z\rangle + 2\eta(Z)\langle Y, \phi AX\rangle .$$

PROPOSITION 7. *On a Sasakian manifold M let A be a tensor field of type* (1, 1) *which is symmetric at each point and satisfies*

$$A\phi = \phi A \quad and \quad A\xi = 0 .$$

Let L be a generalized S-curvature-like tensor field defined by

$$L = L_{A,I} .$$

If A satisfies the following equation:

$$(1.19) \qquad (\nabla_X A)Y = -\eta(Y)\phi AX - \langle A\phi X, Y\rangle\xi ,$$

then L is proper.

Now let $\mathfrak{A}(M)$ be the vector space of all tensor fields A of type (1, 1) on a Sasakian manifold which satisfy the following conditions:

i) A is symmetric at each point;
ii) A commutes with ϕ;
iii) $A\xi = 0$;
iv) A satisfies the equation (1.19);
v) $\mathrm{tr}\, A$ is constant.

Let $\mathscr{L}(M)$ denote the vector space of all proper generalized S-curvature-like tensor fields whose Ricci tensor fields satisfy the equation (1.16). We assume that $\dim M \geqq 5$.

We have a linear mapping $A \in \mathfrak{A}(M) \mapsto L_A \in \mathscr{L}(M)$ given by

$$(1.20) \qquad L_A = \frac{1}{2(n+2)} L_{A,I} - \frac{\mathrm{tr}\, A}{8(n+1)(n+2)} L_{I,I} .$$

We get the following theorem:

THEOREM 2. *If* $\dim M \geqq 5$, $A \mapsto L_A$ *is a linear isomorphism of* $\mathfrak{A}(M)$ *onto the subspace*

$$\{L \in \mathscr{L}(M) \mid L_B = 0\} .$$

2. Proof of propositions.

PROOF OF PROPOSITION 1. It follows from (1.5), (1.6), and (1.9) that

ϕ is skew-symmetric. Making use of this fact, we can show easily that L has the properties (1.1), (1.2), and (1.3). We prove that L has the properties (1.10) and (1.11). We see that

$$\phi((x\varDelta y)z) = (\phi x\varLambda\phi y)\phi z \tag{2.1}$$

holds for x, y and $z \in V$. Since

$$(\xi\varDelta y)z = 0 , \tag{2.2}$$

we get, making use of (1.6) and (2.1),

$$\begin{aligned}\phi((\phi x\varLambda\phi y)z) &= \phi((\phi x\varDelta\phi y)z) = (\phi^2 x\varLambda\phi^2 y)\phi z \\ &= (\phi^2 x\varDelta\phi^2 y)\phi z \\ &= ((-x + \eta(x)\xi)\varDelta(-y + \eta(y)\xi))\phi z \\ &= (x\varDelta y)\phi z .\end{aligned} \tag{2.3}$$

By (2.1) and (2.3) we see that L has the property (1.10).

From (1.4), (1.7), and (1.9) we get

$$\phi\xi = 0 . \tag{2.4}$$

Since A commutes with ϕ, we have, making use of (1.6), (1.8), and (2.4),

$$\langle A\xi, x\rangle = \langle A\xi, \eta(x)\xi - \phi^2 x\rangle = \eta(x)\langle A\xi, \xi\rangle = \langle\langle A\xi, \xi\rangle\xi, x\rangle ,$$

for all $x \in V$, and therefore $A\xi = \langle A\xi, \xi\rangle\xi$. From this equality and (2.2) we get $A\xi\varDelta By = 0$. We also get $B\xi\varDelta Ay = 0$. It follows from (2.4) that $\phi A\xi\varLambda\phi By = 0$, $\phi B\xi\varLambda\phi Ay = 0$, $\langle A\xi, \phi y\rangle = 0$ and $\langle B\xi, \phi y\rangle = 0$ hold. Thus we get $L(\xi, y) = 0$, which completes the proof of Proposition 1.

Let L be an S-curvature-like tensor defined by (1.12). Then the Ricci tensor K of L is given by

$$\begin{aligned}Kx = &(\operatorname{tr} B - b)Ax + (\operatorname{tr} A - a)Bx + 2(BAx + ABx) \\ &- a(\operatorname{tr} B)\eta(x)\xi - b(\operatorname{tr} A)\eta(x)\xi - 2ab\eta(x)\xi ,\end{aligned} \tag{2.5}$$

and the scalar curvature of L is given by

$$\operatorname{tr} K = 2 \operatorname{tr} A \operatorname{tr} B + 4 \operatorname{tr}(AB) - 2(b \operatorname{tr} A + a \operatorname{tr} B) - 2ab , \tag{2.6}$$

where a and b are constants defined by $a = \langle\xi, A\xi\rangle$ and $b = \langle\xi, B\xi\rangle$. As special cases of Proposition 1, we obtain the following examples:

EXAMPLE 1. Take $A = (c/2)I$, $B = I$, where c is a constant. Then L is given by

$$L(x, y) = c\{x\varDelta y + \phi x\varLambda\phi y + 2\langle x, \phi y\rangle\phi\} .$$

The Ricci tensor and the scalar curvature are

$$Kx = 2(n+1)c\{x - \eta(x)\xi\}\ , \quad \operatorname{tr} K = 4n(n+1)c\ .$$

And the ϕ-sectional curvature $k(P)$ for all ϕ-invariant planes P in V is identically equal to $4c$. Conversely, if $L \in \mathscr{L}(V)$ has constant ϕ-sectional curvature, say, $4c$, then it is of the above form (Ogiue [5]).

EXAMPLE 2. Take $B = I$ and a symmetric endomorphism A which commutes with ϕ. Then L is given by

$$\begin{aligned} L(x, y) = {} & Ax\Lambda y + x\Lambda Ay + \phi Ax\Lambda\phi y + \phi x\Lambda\phi Ay \\ & + 2\langle Ax, \phi y\rangle\phi + 2\langle x, \phi y\rangle\phi A\ . \end{aligned}$$

The Ricci tensor K and the scalar curvature are

$$\begin{aligned} & Kx = 2(n+2)(Ax - a\eta(x)\xi) + (\operatorname{tr} A - a)(x - \eta(x)\xi)\ , \\ & \operatorname{tr} K = 4(n+1)(\operatorname{tr} A - a)\ . \end{aligned}$$

LEMMA 1. *Let L be an S-curvature-like tensor, and let K be the Ricci tensor of L, then we have the following identities*:

$$\langle L(x, y)z, w\rangle = 0 \tag{2.7}$$

if at least one of x, y, z, and w is equal to ξ;

$$K\xi = 0\ ; \tag{2.8}$$

$$K\phi = \phi K\ . \tag{2.9}$$

And if $\{e_1, \cdots, e_{2n+1}\}$ is an orthonormal basis of V, then

$$\begin{aligned} 2\langle L(x, y)v, u\rangle & = \sum_{i,j} \langle L(x, y)e_j, e_i\rangle\langle(u\Lambda v)e_j, e_i\rangle \\ & = \sum_{i,j} \langle L(x, y)e_j, e_i\rangle\langle(\phi u\Lambda\phi v)e_j, e_i\rangle\ . \end{aligned} \tag{2.10}$$

We make use of these formulas for the proof of Propositions 2 and 3.

PROOF OF PROPOSITION 2. It is sufficient to show that $\mathscr{L}_1^{\perp}(V)$ consists of all $L \in \mathscr{L}(V)$ whose scalar curvature is 0. $L' \in \mathscr{L}_1(V)$ can be expressed by definition of $\mathscr{L}_1(V)$ as follows

$$\begin{aligned} L'(x, y)z = {} & c\{(x\Lambda y)z - \eta(z)(\eta(y)x - \eta(x)y) \\ & - \xi(\langle z, y\rangle\eta(x) - \langle z, x\rangle\eta(y)) + (\phi x\Lambda\phi y)z + 2\langle x, \phi y\rangle\phi z\}\ . \end{aligned}$$

Let $\{e_1, \cdots, e_{2n+1}\}$ be an orthonormal basis of V, then

$$\begin{aligned} \langle L'(e_k, e_m)e_j, e_i\rangle = {} & c\{\langle(e_k\Lambda e_m)e_j, e_i\rangle \\ & - \langle\eta(e_j)(\eta(e_m)e_k - \eta(e_k)e_m), e_i\rangle \\ & - \langle\xi, e_i\rangle(\langle e_j, e_m\rangle\eta(e_k) - \langle e_j, e_k\rangle\eta(e_m)) \\ & + \langle(\phi e_k\Lambda\phi e_m)e_j, e_i\rangle + 2\langle e_k, \phi e_m\rangle\langle\phi e_j, e_i\rangle\}\ . \end{aligned} \tag{2.11}$$

Let L be an S-curvature-like tensor.
From (2.7) follows

$$\sum_{i,j,k,m} \langle L(e_k, e_m)e_j, e_i\rangle\{\langle\eta(e_j)(\eta(e_m)e_k - \eta(e_k)e_m), e_i\rangle + \langle\xi, e_i\rangle(\langle e_j, e_m\rangle\eta(e_k) - \langle e_j, e_k\rangle\eta(e_m))\} = 0 . \tag{2.12}$$

From (2.10) follows

$$\sum_{i,j,k,m} \langle L(e_k, e_m)e_j, e_i\rangle\{\langle(e_k\Lambda e_m)e_j, e_i\rangle + \langle(\phi e_k\Lambda\phi e_m)e_j, e_i\rangle\} = 4\sum_{k,m}\langle L(e_k, e_m)e_m, e_k\rangle = 4 \text{ (scalar curvature of } L) . \tag{2.13}$$

On the other hand,

$$\begin{aligned} &\sum_{i,j,k,m} \langle L(e_k, e_m)e_j, e_i\rangle\langle e_k, \phi e_m\rangle\langle\phi e_j, e_i\rangle \\ &= \sum_{m,j}\langle L(\phi e_m, e_m)e_j, \phi e_j\rangle \\ &= -\sum_{m,j}(\langle L(e_m, e_j)\phi e_m, \phi e_j\rangle + \langle L(e_j, \phi e_m)e_m, \phi e_j\rangle) \\ &= \sum_{m,j}(\langle L(e_j, e_m)\phi e_m, \phi e_j\rangle + \langle L(e_j, \phi e_m)\phi e_m, e_j\rangle) \\ &= \sum_{m,j}\langle L(e_j, e_m)e_m, e_j\rangle + \sum_m\langle K\phi e_m, \phi e_m\rangle \\ &= \sum_{m,j}\langle L(e_j, e_m)e_m, e_j\rangle + \sum_m\langle Ke_m, e_m\rangle \\ &= 2 \text{ (scalar curvature of } L) . \end{aligned} \tag{2.14}$$

From (2.11), (2.12), (2.13), and (2.14) we get

$$\begin{aligned} \langle L, L'\rangle &= \sum_{i,j,k,m}\langle L(e_k, e_m)e_j, e_i\rangle\langle L'(e_k, e_m)e_j, e_i\rangle \\ &= 8c \text{ (scalar curvature of } L) . \end{aligned}$$

This proves our assertion.

Proof of Proposition 3. By Examples 1 and 2 we can show easily that tensors L_1, L_B, and L_2 belong, respectively, to $\mathscr{L}_1(V)$, $\mathscr{L}_B(V)$, and $\mathscr{L}_1^\perp(V)$. So it is sufficient to show that tensor L_2 is orthogonal to $\mathscr{L}_B(V)$. Since $L_{I,I}$ is orthogonal to $\mathscr{L}_B(V)$, we have only to show that $L_{K,I}$ is orthogonal to $\mathscr{L}_B(V)$. Let L' be a tensor which belongs to $\mathscr{L}_B(V)$. Making use of (2.7) and (2.10), we get

$$\begin{aligned} &\sum_{k,m,j,i}\langle L'(e_k, e_m)e_j, e_i\rangle\{\langle(Ke_k\Delta e_m)e_j, e_i\rangle + \langle(\phi Ke_k\Lambda\phi e_m)e_j, e_i\rangle\} \\ &= \sum_{k,m,j,i}\langle L'(e_k, e_m)e_j, e_i\rangle\{\langle(Ke_k\Lambda e_m)e_j, e_i\rangle + \langle(\phi Ke_k\Lambda\phi e_m)e_j, e_i\rangle\} \\ &= 4\sum_{k,m}\langle L'(e_k, e_m)e_m, Ke_k\rangle = 0 ; \end{aligned} \tag{2.15}$$

$$(2.16)\qquad \sum_{k,m,j,i} \langle L'(e_k, e_m)e_j, e_i\rangle\{\langle(e_k \varLambda K e_m)e_j, e_i\rangle + \langle(\phi e_k \varLambda \phi K e_m)e_j, e_i\rangle\} = 0 .$$

On the other hand,

$$(2.17)\qquad \begin{aligned} &\sum_{k,m,j,i} \langle L'(e_k, e_m)e_j, e_i\rangle\langle K e_k, \phi e_m\rangle\langle \phi e_j, e_i\rangle \\ &= \sum_{m,j} \langle L'(K\phi e_m, e_m)e_j, \phi e_j\rangle \\ &= -\sum_{m,j} (\langle L'(e_m, e_j)K\phi e_m, \phi e_j\rangle + \langle L'(e_j, K\phi e_m)e_m, \phi e_j\rangle) \\ &= -\sum_{m,j} (\langle L'(e_m, e_j)K e_m, e_j\rangle - \langle L'(e_j, K\phi e_m)\phi e_m, e_j\rangle) \\ &= \sum_{m,j} (\langle K e_m, L'(e_m, e_j)e_j\rangle + \langle \phi e_m, L'(K\phi e_m, e_j)e_j\rangle) = 0 ; \end{aligned}$$

$$(2.18)\qquad \sum_{k,m,j,i} \langle L'(e_k, e_m)e_j, e_i\rangle\langle e_k, \phi e_m\rangle\langle \phi K e_j, e_i\rangle = 0 .$$

From (2.15), (2.16), (2.17), and (2.18) we get

$$\langle L', L_{K,I}\rangle = 0 .$$

This proves our assertion.

EXAMPLE 3. Corresponding to Example 1 we consider

$$L = f L_{I,I} ,$$

where f is a (differentiable) function. If $\dim M \geqq 5$, L is proper if and only if f is a constant function.

LEMMA 2. *Let L be a proper generalized S-curvature-like tensor field on M and let K be its Ricci tensor field. Then we have the following formulas*:

$$(2.19)\qquad \begin{aligned} \langle(\nabla_{\phi X}K)\phi Y, Z\rangle = &-\langle(\nabla_Y K)Z, X\rangle + \langle(\nabla_Z K)X, Y\rangle \\ &- \eta(Y)\langle K\phi X, Z\rangle + 2\eta(X)\langle K\phi Z, Y\rangle ; \end{aligned}$$

$$(2.20)\qquad \nabla_\xi K = 0 ;$$

$$(2.21)\qquad \textit{trace of } \{X \mapsto (\nabla_X K)Y\} = \frac{1}{2} Y(\operatorname{tr} K) .$$

PROOF. (2.20) follows directly from (2.19): If we put $Y = \xi$ in (2.19), then we have

$$-\langle(\nabla_\xi K)Z, X\rangle + \langle X, (\nabla_Z K)\xi\rangle + \langle X, K\phi Z\rangle = 0 .$$

From $K\xi = 0$ follows

$$(2.22)\qquad (\nabla_Z K)\xi = -K(\nabla_Z \xi) = -K\phi Z .$$

Therefore we get $\nabla_\xi K = 0$. (2.21) is proved in [4]. We shall prove (2.19). Let $\{e_1, \cdots, e_{2n+1}\}$ be an orthonormal basis of the tangent space $T_p(M)$ at a point $p \in M$. We see

$$\begin{aligned}\langle KY, Z\rangle &= \sum_i \langle L(Y, e_i)e_i, Z\rangle = \sum_i \langle L(\phi Y, \phi e_i)e_i, Z\rangle \\ &= -\sum_i \langle L(\phi Y, \phi e_i)Z, e_i\rangle \\ &= -\sum_i \langle (Le_i, \phi Z)\phi Y, e_i\rangle + \sum_i \langle \phi L(Z, \phi Y)e_i, e_i\rangle \\ &= -\langle K\phi Z, \phi Y\rangle + \sum_i \langle \phi L(Z, \phi Y)e_i, e_i\rangle \\ &= -\langle KZ, Y\rangle + \sum_i \langle \phi L(Z, \phi Y)e_i, e_i\rangle .\end{aligned}$$

Thus we get

$$\langle KY, Z\rangle = \frac{1}{2}\sum_i \langle \phi L(Z, \phi Y)e_i, e_i\rangle . \tag{2.23}$$

From this equation follows

$$\begin{aligned}\langle (\nabla_X K)Y, Z\rangle &= \frac{1}{2}\sum_i \{\langle (\nabla_X\phi)(L(Z, \phi Y)e_i), e_i\rangle + \langle \phi(\nabla_X L)(Z, \phi Y)e_i, e_i\rangle \\ &\quad + \langle \phi L(Z, (\nabla_X\phi)Y)e_i, e_i\rangle\} \\ &= \frac{1}{2}\sum_i \langle \phi(\nabla_X L)(Z, \phi Y)e_i, e_i\rangle + \frac{1}{2}\eta(Y)\sum_i \langle \phi L(Z, X)e_i, e_i\rangle .\end{aligned}$$

Replacing Y by ϕY in this, we get

$$\begin{aligned}\langle (\nabla_X K)\phi Y, Z\rangle = &-\frac{1}{2}\sum_i \langle \phi(\nabla_X L)(Z, Y)e_i, e_i\rangle \\ &-\frac{1}{2}\sum_i \langle \phi(\nabla_Z L)(Y, \xi)e_i, e_i\rangle .\end{aligned} \tag{2.24}$$

From (2.7) follows

$$(\nabla_Z L)(Y, \xi) = -L(Y, \nabla_Z\xi) = -L(Y, \phi Z) .$$

Putting this into (2.24) and making use of (2.23), we get

$$\langle (\nabla_X K)\phi Y, Z\rangle = -\frac{1}{2}\sum \langle \phi(\nabla_X L)(Z, Y)e_i, e_i\rangle - \langle KX, Z\rangle\eta(Y) .$$

Since L is proper, we get

$$\begin{aligned}&\langle (\nabla_X K)\phi Y, Z\rangle + \langle (\nabla_Y K)\phi Z, X\rangle + \langle (\nabla_Z K)\phi X, Y\rangle \\ &\quad = -(\eta(Y)\langle KX, Z\rangle + \eta(Z)\langle KY, X\rangle + \eta(X)\langle KZ, Y\rangle) .\end{aligned}$$

Replacing X by ϕX in this, we get

(2.25) $$\langle(\nabla_{\phi X}K)\phi Y, Z\rangle + \langle(\nabla_Y K)\phi Z, \phi X\rangle + \langle(\nabla_Z K)(-X+\eta(X)\xi), Y\rangle = -(\eta(Y)\langle K\phi X, Z\rangle + \eta(Z)\langle KY, \phi X\rangle) .$$

From $K\phi = \phi K$ follows

(2.26) $$\begin{aligned}(\nabla_Y K)\phi Z &= -K(\nabla_Y\phi)Z + (\nabla_Y\phi)KZ + \phi(\nabla_Y K)Z \\ &= -\eta(Z)KY - \langle Y, KZ\rangle\xi + \phi(\nabla_Y K)Z .\end{aligned}$$

Putting this and (2.22) into (2.25), we get

$$\langle(\nabla_{\phi X}K)\phi Y, Z\rangle - \eta(Z)\langle KY, \phi X\rangle + \langle\phi(\nabla_Y K)Z, \phi X\rangle - \langle(\nabla_Z K)X, Y\rangle - \eta(X)\langle K\phi Z, Y\rangle = -(\eta(Y)\langle K\phi X, Z\rangle + \eta(Z)\langle KY, \phi X\rangle) .$$

Since $\langle\phi(\nabla_Y K)Z, \phi X\rangle = \langle(\nabla_Y K)Z, X\rangle + \eta(X)\langle K\phi Y, Z\rangle$, we get (2.19).

Now we can prove Propositions 4 and 5.

PROOF OF PROPOSITION 4. It is now easy to show that (1.17) follows from (1.16). So we shall prove (1.16) under the assumption that (1.17) holds. Interchanging X and Z in (1.17), we have

$$\langle(\nabla_Y K)Z - (\nabla_Z K)Y, X\rangle = \eta(Y)\langle\phi KZ, X\rangle - \eta(Z)\langle\phi KY, X\rangle + 2\eta(X)\langle Y, \phi KZ\rangle .$$

Putting this into (2.19), we get

$$\langle(\nabla_{\phi X}K)\phi Y, Z\rangle = \eta(Z)\langle\phi KY, X\rangle .$$

Replacing X and Y in this, respectively, by $-\phi X$ and $-\phi Y$, and making use of (2.20) and (2.22), we get (1.16).

PROOF OF PROPOSITION 5. From (1.16) we can easily get

$$\text{trace of } \{X \mapsto (\nabla_X K)Y\} = 0 .$$

In view of (2.21), we see

$$Y(\operatorname{tr} K) = 0 ,$$

which proves our assertion.

LEMMA 3. *Under the same assumptions as in Proposition* 6, *we have the following formulas*:

(2.27) $$\text{trace of } \{X \mapsto (\nabla_X A)Y\} = 0 ;$$

(2.28) $$(\nabla_Z A)\phi X + \eta(X)AZ = -\langle Z, AX\rangle\xi + \phi((\nabla_Z A)X) ;$$

(2.29) $$(\nabla_Z A)\xi = -A\phi Z ;$$

(2.30) $$(\nabla_{\phi Y}A)\phi X - (\nabla_{\phi X}A)\phi Y = (\nabla_Y A)X - (\nabla_X A)Y + (A\phi Y)\eta X - (A\phi X)\eta Y ;$$

(2.31) $$\nabla_\xi A = 0\ ;$$

(2.32) $$\textit{trace of } \{Z \mapsto \phi(\nabla_Z A)X\} = (\mathrm{tr}\, A)\eta(X)\ ;$$

(2.33) $$\sum_i \phi((\nabla_{e_i} A)e_i) = 0\ ;$$

(2.34) $$\mathrm{tr}\,(\phi(\nabla_Z A)) = 0\ .$$

PROOF. Let K be the Ricci tensor of L, then from Example 2 we get

(2.35) $$KX = 2(n+2)AX + \mathrm{tr}\, A(X - \eta(X)\xi)\ ;$$

(2.36) $$\mathrm{tr}\, K = 4(n+1)\,\mathrm{tr}\, A\ .$$

From (2.35) follows

(2.37) $$\begin{aligned}(\nabla_X K)Y &= 2(n+2)(\nabla_X A)Y - \mathrm{tr}\, A((\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi) \\ &= 2(n+2)(\nabla_X A)Y - \mathrm{tr}\, A(\langle Y, \phi X\rangle \xi + \eta(Y)\phi X)\ .\end{aligned}$$

Therefore

$$\begin{aligned}\text{trace of } \{X \mapsto (\nabla_X A)Y\} &= \frac{1}{2(n+2)} \text{ trace of } \{X \mapsto (\nabla_X K)Y\} \\ &= \frac{1}{4(n+2)} Y(\mathrm{tr}\, K)\ ,\end{aligned}$$

the second identity of which comes from (2.21). We see by (2.36) that $\mathrm{tr}\, K$ is constant. So we get (2.27). Making use of (2.35) and (2.37), we can rewrite the formula (2.19) into the following:

$$\begin{aligned}\langle (\nabla_{\phi X} A)\phi Y, Z\rangle = &-\langle (\nabla_Y A)Z, X\rangle + \langle (\nabla_Z A)X, Y\rangle \\ &- \eta(Y)\langle A\phi X, Z\rangle + 2\eta(X)\langle A\phi Z, Y\rangle\ .\end{aligned}$$

From this (2.30) follows directly. (2.28), (2.29), and (2.31) can be proved in the same way, respectively, as (2.26), (2.22), and (2.20). (2.32) follows directly from (2.27) and (2.28). Since $\langle X, \phi(\nabla_W A)Z\rangle = -\langle (\nabla_W A)\phi X, Z\rangle$, we get (2.33) by virtue of (2.27). Let $\{E_i\}$ be locally defined parallel orthonormal fields. Then

$$\sum_i \langle E_i, \phi(\nabla_Z A)E_i\rangle = \sum_i (\nabla_Z \langle E_i, \phi A E_i\rangle - \langle E_i, (\nabla_Z \phi)A E_i\rangle) = 0\ ,$$

which proves (2.34).

PROOF OF PROPOSITION 6. By the definition of L, we see

$$\begin{aligned}L(X, Y)W = &\langle W, Y\rangle AX - \langle W, AX\rangle Y - \eta(W)\eta(Y)AX + \langle W, AX\rangle \eta(Y)\xi \\ &+ \langle W, AY\rangle X - \langle W, X\rangle AY + \eta(W)\eta(X)AY \\ &- \xi\langle W, AY\rangle \eta(X) + \langle W, \phi Y\rangle \phi AX - \langle W, \phi AX\rangle \phi Y\end{aligned}$$

$$+ \langle W, \phi AY\rangle\phi X - \langle W, \phi X\rangle\phi AY$$
$$+ 2\langle AX, \phi Y\rangle\phi W + 2\langle X, \phi Y\rangle\phi AW .$$

From this follows

$$\begin{aligned}(\nabla_Z L)(X, Y)W = {} & \langle W, Y\rangle(\nabla_Z A)X - \langle W, (\nabla_Z A)X\rangle Y \\ & - (\nabla_Z\eta)(W)\eta(Y)AX - \eta(W)((\nabla_Z\eta)(Y)AX + \eta(Y)(\nabla_Z A)X) \\ & + \langle W, AX\rangle\eta(Y)\nabla_Z\xi + (\langle W, (\nabla_Z A)X\rangle\eta(Y) + \langle W, AX\rangle(\nabla_Z\eta)(Y))\xi \\ & + \langle W, (\nabla_Z A)Y\rangle X - \langle W, X\rangle(\nabla_Z A)Y \\ & + (\nabla_Z\eta)(W)\eta(X)AY + \eta(W)((\nabla_Z\eta)AY + \eta(X)(\nabla_Z A)Y) \\ & - \langle W, AY\rangle\eta(X)\nabla_Z\xi - (\langle W, (\nabla_Z A)Y\rangle\eta(X) + \langle W, AY\rangle(\nabla_Z\eta)(X))\xi \\ & + \langle W, (\nabla_Z\phi)Y\rangle\phi AX + \langle W, \phi Y\rangle(\nabla_Z\phi)AX + \langle W, \phi Y\rangle\phi(\nabla_Z A)X \\ & - \langle W, (\nabla_Z\phi)AX\rangle\phi Y - \langle W, \phi(\nabla_Z A)X\rangle\phi Y - \langle W, \phi AX\rangle(\nabla_Z\phi)Y \\ & + \langle W, (\nabla_Z\phi)AY\rangle\phi X + \langle W, \phi(\nabla_Z A)Y\rangle\phi X + \langle W, \phi AY\rangle(\nabla_Z\phi)X \\ & - \langle W, (\nabla_Z\phi)X\rangle\phi AY - \langle W, \phi X\rangle(\nabla_Z\phi)AY - \langle W, \phi X\rangle\phi(\nabla_Z A)Y \\ & + 2\langle(\nabla_Z A)X, \phi Y\rangle\phi W + 2\langle AX, (\nabla_Z\phi)Y\rangle\phi W + 2\langle AX, \phi Y\rangle(\nabla_Z\phi)W \\ & + 2\langle X, (\nabla_Z\phi)Y\rangle\phi AW + 2\langle X, \phi Y\rangle(\nabla_Z\phi)AW + 2\langle X, \phi Y\rangle\phi(\nabla_Z A)W .\end{aligned}$$

Applying (1.14) and (1.15) to this, we obtain

$$\begin{aligned}(2.38)\quad (\nabla_Z L)(X, Y)W = {} & \langle W, Y\rangle(\nabla_Z A)X - \langle W, (\nabla_Z A)X\rangle Y \\ & - \langle W, \phi Z\rangle\eta(Y)AX - \eta(W)(\langle Y, \phi Z\rangle AX + \eta(Y)(\nabla_Z A)X) \\ & + \langle W, AX\rangle\eta(Y)\phi Z + (\langle W, (\nabla_Z A)X\rangle\eta(Y) + \langle W, AX\rangle\langle Y, \phi Z\rangle)\xi \\ & + \langle W, (\nabla_Z A)Y\rangle X - \langle W, X\rangle(\nabla_Z A)Y \\ & + \langle W, \phi Z\rangle\eta(X)AY + \eta(W)(\langle X, \phi Z\rangle AY + \eta(X)(\nabla_Z A)Y) \\ & - \langle W, AY\rangle\eta(X)\phi Z - (\langle W, (\nabla_Z A)Y\rangle\eta(X) + \langle W, AY\rangle\langle X, \phi Z\rangle)\xi \\ & + \langle W, \eta(Y)Z - \langle Z, Y\rangle\xi\rangle\phi AX - \langle W, \phi Y\rangle\langle Z, AX\rangle\xi + \langle W, \phi Y\rangle\phi(\nabla_Z A)X \\ & + \langle W, \langle Z, AX\rangle\xi\rangle\phi Y - \langle W, \phi(\nabla_Z A)X\rangle\phi Y - \langle W, \phi AX\rangle(\eta(Y)Z - \langle Z, Y\rangle\xi) \\ & - \langle W, \langle Z, AY\rangle\xi\rangle\phi X + \langle W, \phi(\nabla_Z A)Y\rangle\phi X + \langle W, \phi AY\rangle(\eta(X)Z - \langle Z, X\rangle\xi) \\ & - \langle W, \eta(X)Z - \langle Z, X\rangle\xi\rangle\phi AY + \langle W, \phi X\rangle\langle Z, AY\rangle\xi - \langle W, \phi X\rangle\phi(\nabla_Z A)Y \\ & + 2\langle(\nabla_Z A)X, \phi Y\rangle\phi W + 2\langle AX, \eta(Y)Z\rangle\phi W + 2\langle AX, \phi Y\rangle(\eta(W)Z - \langle Z, W\rangle\xi) \\ & + 2\langle X, \eta(Y)Z - \langle Z, Y\rangle\xi\rangle\phi AW - 2\langle X, \phi Y\rangle\langle Z, AW\rangle\xi + 2\langle X, \phi Y\rangle\phi(\nabla_Z A)W .\end{aligned}$$

Making use of (2.27), (2.31), (2.32), (2.33), and (2.38), we get

$$\begin{aligned}(2.29)\qquad \sum_i (\nabla_{e_i} L)(X, Y)e_i = {} & (\nabla_Y A)X - (\nabla_X A)Y + 2(2n + 3)\langle Y, \phi AX\rangle\xi \\ & + 2(n + 2)\eta(Y)\phi AX - 2(n + 1)\eta(X)\phi AY \\ & + \phi((\nabla_{\phi Y}A)X) - \phi((\nabla_{\phi X}A)Y)\end{aligned}$$

$$+ \operatorname{tr} A(\eta(Y)\phi X - \eta(X)\phi Y) - 2\langle X, \phi Y\rangle(\operatorname{tr} A)\xi$$
$$+ 2\sum_i \langle(\nabla_{e_i}A)X, \phi Y\rangle\phi e_i .$$

Making use of (2.38) and (2.34), we get

$$\begin{aligned}\sum_i (\nabla_Z L)(X, e_i)e_i = {} & (2n+1)(\nabla_Z A)X - 3(\nabla_Z A)X + \langle\xi, (\nabla_Z A)X\rangle\xi \\ & - 5\langle\phi Z, AX\rangle\xi - 2\eta(X)\phi AZ + \eta(X)(\nabla_Z A)\xi \\ & - (\operatorname{tr} A)\eta(X)\phi Z - \operatorname{tr} A\langle X, \phi Z\rangle\xi - 3\phi^2(\nabla_Z A)X \\ & - 3\phi((\nabla_Z A)\phi X) .\end{aligned}$$

Applying (2.29) to this, we get

(2.40) $$\begin{aligned}\sum_i (\nabla_Z L)(X, e_i)e_i = {} & (2n+1)(\nabla_Z A)X - 3\langle AX, \phi Z\rangle\xi - 3\eta(X)\phi AZ \\ & - (\operatorname{tr} A)\eta(X)\phi Z - \operatorname{tr} A\langle X, \phi Z\rangle\xi - 3\phi((\nabla_Z A)\phi X) .\end{aligned}$$

Since L is proper, we see

(2.41) $$\sum_i (\nabla_{e_i}L)(X, Y)e_i = -\sum_i (\nabla_X L)(Y, e_i)e_i + \sum_i (\nabla_Y L)(X, e_i)e_i .$$

On the basis of (2.39), (2.40), and (2.41), we obtain

(2.42) $$\begin{aligned}2n\{(\nabla_Y A)X - (\nabla_X A)Y\} = {} & \phi((\nabla_{\phi Y}A)X) - \phi((\nabla_{\phi X}A)Y) \\ & + 3\phi((\nabla_Y A)\phi X - (\nabla_X A)\phi Y) + 4n\langle Y, \phi AX\rangle\xi + (2n+1)\eta(Y)\phi AX \\ & - (2n-1)\eta(X)\phi AY + 2\sum_i \langle(\nabla_{e_i}A)X, \phi Y\rangle\phi e_i .\end{aligned}$$

By virtue of (2.28) and (2.29), we get

$$\begin{aligned}\phi((\nabla_{\phi Y}A)X) - \phi((\nabla_{\phi X}A)Y) = {} & (\nabla_{\phi Y}A)\phi X - (\nabla_{\phi X}A)\phi Y \\ & + \eta(X)A\phi Y - \eta(Y)A\phi X + 2\langle\phi Y, AX\rangle\xi ;\end{aligned}$$

$$\begin{aligned}\phi((\nabla_Y A)\phi X) - \phi((\nabla_X A)\phi Y) = {} & \eta(Y)\phi AX - \eta(X)\phi AY \\ & + (\nabla_X A)Y - (\nabla_Y A)X + 2\langle AY, \phi X\rangle\xi .\end{aligned}$$

Putting these two formulas into (2.42), we get

$$\begin{aligned}(2n+3)\{(\nabla_Y A)X - (\nabla_X A)Y\} = {} & (\nabla_{\phi Y}A)\phi X - (\nabla_{\phi X}A)\phi Y \\ & + (2n+3)\eta(Y)\phi AX - (2n+1)\eta(X)\phi AY \\ & + 4(n+1)\langle Y, \phi AX\rangle\xi + 2\sum_i \langle(\nabla_{e_i}A)X, \phi Y\rangle\phi e_i .\end{aligned}$$

Putting (2.30) into this, we obtain

$$\begin{aligned}(n+1)\{(\nabla_Y A)X - (\nabla_X A)Y\} = {} & -n\eta(X)\phi AY + (n+1)\eta(Y)\phi AX \\ & + 2(n+1)\langle Y, \phi AX\rangle\xi + \sum_i \langle(\nabla_{e_i}A)X, \phi Y\rangle\phi e_i ,\end{aligned}$$

that is,

(2.43) $(n+1)\langle Z, (\nabla_Y A)X - (\nabla_X A)Y\rangle = -n\eta(X)\langle \phi AY, Z\rangle$
$+ (n+1)\eta(Y)\langle \phi AX, Z\rangle + 2(n+1)\eta(Z)\langle Y, \phi AX\rangle - \langle(\nabla_{\phi Z}A)X, \phi Y\rangle$.

We see easily

$$\sigma\langle Z, (\nabla_Y A)X - (\nabla_X A)Y\rangle = 0 ;$$
$$\sigma(-n\eta(X)\langle \phi AY, Z\rangle + (n+1)\eta(Y)\langle \phi AX, Z\rangle + 2(n+1)\eta(Z)\langle Y, \phi AX\rangle) = \sigma(\eta(X)\langle \phi AY, Z\rangle) .$$

By virtue of these two formulas and (2.43), we obtain

$$0 = -\sigma(\eta(Y)\langle A\phi X, Z\rangle) - \sigma(\langle(\nabla_{\phi Z}A)X, \phi Y\rangle) ,$$

that is,

(2.44) $\langle(\nabla_{\phi Y}A)\phi X, Z\rangle - \langle \phi((\nabla_{\phi X}A)Y), Z\rangle = -\langle(\nabla_{\phi Z}A)X, \phi Y\rangle$
$- (\eta(Y)\langle A\phi X, Z\rangle + \eta(Z)\langle A\phi Y, X\rangle + \eta(X)\langle A\phi Z, Y\rangle)$.

Replacing X and Z in (2.28), respectively, with Y and ϕX, we get

$$(\nabla_{\phi X}A)\phi Y = -\eta(Y)A\phi X - \langle \phi X, AY\rangle\xi + \phi((\nabla_{\phi X}A)Y) .$$

Putting this into (2.30), we get

$$(\nabla_{\phi Y}A)\phi X - \phi((\nabla_{\phi X}A)Y) = -2\eta(Y)A\phi X - \langle \phi X, AY\rangle\xi + \eta(X)A\phi Y + (\nabla_Y A)X - (\nabla_X A)Y .$$

Putting this into (2.44), we get

$$-\langle(\nabla_{\phi Z}A)X, \phi Y\rangle = \langle(\nabla_Y A)X - (\nabla_X A)Y, Z\rangle - 2\eta(Z)\langle \phi X, AY\rangle - \eta(Y)\langle A\phi X, Z\rangle .$$

Putting this into (2.43), we get

$$\langle(\nabla_Y A)X - (\nabla_X A)Y, Z\rangle = -\eta(X)\langle \phi AY, Z\rangle + \eta(Y)\langle \phi AX, Z\rangle + 2\eta(Z)\langle Y, \phi AX\rangle ,$$

which proves our assertion.

PROOF OF PROPOSITION 7. By virtue of (1.19), we can easily prove

$$-\eta(W)\eta(Y)(\nabla_Z A)X + \xi\langle W, (\nabla_Z A)X\rangle\eta(Y) + \eta(W)\eta(X)(\nabla_Z A)Y - \langle W, (\nabla_Z A)Y\rangle\eta(X)\xi = 0 .$$

The following formulas can be proved easily:

$$\sigma(\langle W, AX\rangle\langle Y, \phi Z\rangle - \langle W, AY\rangle\langle X, \phi Z\rangle - 2\langle X, \phi Y\rangle\langle AW, Z\rangle) = 0 ;$$
$$\sigma(-\langle W, \langle Z, Y\rangle\xi\rangle\phi AX + \langle W, \langle X, Z\rangle\xi\rangle\phi AY) = 0 ;$$
$$\sigma(-\langle W, \phi Y\rangle\langle AX, Z\rangle + \langle W, \phi X\rangle\langle AY, Z\rangle) = 0 ;$$
$$\sigma(\langle W, \langle AX, Z\rangle\xi\rangle\phi Y - \langle W, \langle AY, Z\rangle\xi\rangle\phi X) = 0 ;$$

$$\sigma(\langle W, \phi AX\rangle\langle Y, Z\rangle - \langle W, \phi AY\rangle\langle X, Z\rangle) = 0 ;$$
$$\sigma(\langle X, \eta(Y)Z - \langle Y, Z\rangle\xi\rangle) = 0 ,$$

where σ denotes the cyclic sum over X, Y, and Z. Applying these formulas to the cyclic sum of (2.38) over X, Y, and Z, we obtain

$$\begin{aligned}(2.45)\quad \sigma((\nabla_Z L)(X, Y)W)
&= \sigma(\langle W, Y\rangle(\nabla_Z A)X - \langle W, (\nabla_Z A)X\rangle Y - \langle W, \phi Z\rangle\eta(Y)AX \\
&\quad - \eta(W)\langle Y, \phi Z\rangle AX + \langle W, AX\rangle\eta(Y)\phi Z \\
&\qquad + \langle W, (\nabla_Z A)Y\rangle X - \langle W, X\rangle(\nabla_Z A)Y \\
&\quad + \langle W, \phi Z\rangle\eta(X)AY + \eta(W)\langle X, \phi Z\rangle AY \\
&\qquad - \langle W, AY\rangle\eta(X)\phi Z + \langle W, \eta(Y)Z\rangle\phi AX \\
&\quad + \langle W, \phi Y\rangle\phi((\nabla_Z A)X) - \langle W, \phi(\nabla_Z A)X\rangle\phi Y - \langle W, \phi AX\rangle\eta(Y)Z \\
&\quad + \langle W, \phi(\nabla_Z A)Y\rangle\phi X + \langle W, \phi AY\rangle\eta(X)Z \\
&\qquad - \langle W, \eta(X)Z\rangle\phi AY - \langle W, \phi X\rangle\phi((\nabla_Z A)Y) \\
&\quad + 2\langle(\nabla_Z A)X, \phi Y\rangle\phi W + 2\langle AX, \eta(Y)Z\rangle\phi W \\
&\qquad + 2\langle AX, \phi Y\rangle(\eta(W)Z - \langle Z, W\rangle\xi) \\
&\quad + 2\langle X, \phi Y\rangle\phi((\nabla_Z A)W)) .\end{aligned}$$

By virtue of (1.19), we can prove the following:

$$\sigma(\langle W, \phi(\nabla_Z A)Y\rangle\phi X - \langle W, AY\rangle\eta(X)\phi Z) = 0 ;$$
$$\sigma(\langle W, Y\rangle(\nabla_Z A)X + \langle W, Z\rangle\langle AY, \phi X\rangle\xi + \eta(Y)\langle W, Z\rangle\phi AX) = 0 ;$$
$$\sigma(\eta(X)\langle W, \phi Z\rangle AY - \langle W, \phi X\rangle\phi((\nabla_Z A)Y) = 0 ;$$
$$\sigma(\langle W, (\nabla_Z A)X\rangle Y + \eta(W)\langle AY, \phi X\rangle Z + \langle W, \phi AX\rangle\eta(Y)Z) = 0 ;$$
$$\sigma(-\eta(W)\langle Y, \phi Z\rangle AX + \eta(W)\langle X, \phi Z\rangle AY + 2\langle X, \phi Y\rangle\phi((\nabla_Z A)W) = 0 ,$$

and we get the counterparts, respectively, of these formulas by interchanging X and Y. We get also

$$\sigma(\langle(\nabla_Z A)X, \phi Y\rangle + \langle AX, \eta(Y)Z\rangle) = 0 .$$

Applying this and the above ten formulas to (2.45), we obtain

$$\sigma((\nabla_Z L)(X, Y)) = 0 ,$$

which proves our assertion.

3. Proof of theorems and corollaries.

PROOF OF COROLLARY 1. If the contact Bochner tensor associated to $L \in \mathscr{L}(V)$ is 0, then we see by Proposition 3

$$L = \frac{1}{2(n+2)} L_{K,I} - \frac{\operatorname{tr} K}{8(n+1)(n+2)} L_{I,I} ,$$

where K is the Ricci tensor of L. By setting,

$$A = \frac{K}{2(n+2)} - \frac{\operatorname{tr} K}{8(n+1)(n+2)} I ,$$

we may write as (1.13). By Example 2 and Proposition 3, the converse is easy to see.

PROOF OF COROLLARY 2. Let A be a symmetric endomorphism of V which commutes with ϕ, and let L be an S-curvature-like tensor defined by (1.13). Then

$$k(P) = 8\langle x, x\rangle\langle Ax, x\rangle \tag{3.1}$$

for $x \in V$ such that $\eta(x) = 0$, where P is a 2-plane spanned by x and ϕx. Conversely if L is an S-curvature-like tensor whose ϕ-sectional curvature for P is given by (3.1), then L satisfies the equality (1.13) (cf. Chapter IX, Proposition 7.1 in [2]). Putting $Q = 4A$, the following follows from (3.1) and vice versa:

$$k(P) = \langle x, x\rangle(\langle Qx, x\rangle + \langle Q\phi x, \phi x\rangle)$$

for $x \in V$ such that $\eta(x) = 0$. This proves our assertion, since $L_B = 0$ if and only if L is given by (1.13).

PROOF OF THEOREM 1. First assume that K satisfies (1.16). By Proposition 5, $\operatorname{tr} K$ is constant on M. Then L_1 defined by

$$L_1 = \frac{\operatorname{tr} K}{8n(n+1)} L_{I,I}$$

is proper as in Example 3. Also L_2 defined by

$$L_2 = \frac{1}{2(n+2)} L_{K,I} - \frac{\operatorname{tr} K}{4n(n+2)} L_{I,I}$$

is proper, L' defined by

$$L' = \frac{1}{2(n+2)} L_{K,I}$$

is proper by Proposition 7. It follows that L_B is proper.

Conversely, assume that L_1, L_B, and L_2 are proper and that $\dim M \geqq 5$. From the assumption on L_1 we see that $\operatorname{tr} k$ is constant on M (see Example 3). Since L_2 is proper, we see that L' defined above is also proper. By Propositions 4 and 6 we conclude that K satisfies the equation (1.16).

This completes the proof of Theorem 1.

We see by Example 3 that Corollary 2 is an immediate consequence of Theorem 1.

The linear mapping defined by (1.20) is one-to-one, because the Ricci tensor field of L_A is precisely A. Noting this, Theorem 2 is now easy to prove.

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CENTRAL RESEARCH LABORATORY
MITSUBISHI ELECTRIC CORP.
AMAGASAKI-SHI, JAPAN