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AN INVERSION OF THE HANKEL POTENTIAL TRANSFORM OF GENERALIZED FUNCTIONS

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1. Introduction. After Schwartz's [8] extension of the Fourier transform to generalized functions, the extension of classical integral transformations to generalized functions have comprised an active and interesting area of research (see, for example, Cooper [2], Dube and Pandey [4], Koh and Zemanian [5], Pandey [6], Pandey and Zemanian [7], and Zemanian [11], [12]). Our main objective in this paper is to extend the classical Hankel potential transform [1] to generalized functions, and to prove an inversion formula for the generalized Hankel potential transform. It should be noted that the limit operation in the inversion formula is interpreted in the weak topology of the testing function space $T_{\alpha,\beta}$. This yields a more general result than a corresponding formula with the limit operation interpreted in the weak topology of $D(I)$.

The classical Hankel potential transform of a function f has been defined by the following convergent integral:

$$(1.1)\qquad F(x) = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} f(t)\,dm(t)\,,$$
$$dm(t) = \left[2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)\right]^{-1} t^{2\nu}dt\,,\quad \nu>0 \quad\text{and}\quad x>0\,.$$

A function $f(x)$ is said to belong to L if

$$\int_0^R |f(x)|\,dm(x) < \infty \quad\text{for}\quad x\in[0,R]\,,\quad 0<R\leqq\infty\,.$$

Cholewinski and Haimo [1] proved the following inversion formula for the transform (1.1).

THEOREM. *Let $f(t)$ belong to L for $t\in[0,R]$ for every positive R, and let the integral*

$$F(x) = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} f(t)\,dm(t)$$

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converge for some $x \neq 0$. Then

$$\lim_{n\to\infty} L_{n,x}F(x) = f(x) \tag{1.2}$$

at all points x of the Lebesgue set for the function $f(x)$, where the operator $L_{n,x}$ is defined as follows:

$$L_{1,x}[F] = \frac{\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{5/2+\nu}\left[\Gamma\left(1+\frac{\nu}{2}+1\right)\right]^2 x^{1+2\nu}} Dx^{5+2\nu}D\frac{1}{x}[DF]$$

and

$$L_{n,x}[F] = \frac{(-1)^{n+1}\sqrt{\pi}\,\Gamma(2\nu+1)}{2^{2n+\nu+1/2}\left[\Gamma\left(n+\frac{\nu}{2}+1\right)\right]^2 x^{2n+2\nu-1}} \tag{1.3}$$

$$\times\, Dx^{4n+2\nu+1}\left[\prod_{k=1}^{n-1} D\frac{1}{x^{4k+2\nu+1}}Dx^{4k+2\nu+1}\right]\cdot D\frac{1}{x}D(F)\,,$$

$$n = 2, 3, \cdots, D = \frac{d}{dx}\,.$$

An alternate form of the same operator is given as

$$L_{n,x} = -\frac{2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(n+\nu+1)n!}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2} \tag{1.3$'$}$$

$$\times\, x\frac{d}{dx}\prod_{k=1}^{n}\left(1-\frac{x\frac{d}{dx}}{2k}\right)\left(1+\frac{x\frac{d}{dx}}{2k+2}\right)\,,$$

$$n = 1, 2, 3\cdots\,.$$

Our main object in this paper is to extend the above inversion formula to certain space of generalized functions.

The notation and terminology of this work will follow that of [3], [5], and [10]. I denotes the open interval $(0, \infty)$ and all testing functions herein are defined on I. Throughout this work, x and t are variables over I. If f is a generalized function on I, the notation $f(t)$ simply indicates that the testing functions on which f is defined have t as their independent variable. $\langle f(t), \phi(t)\rangle$ denotes the number assigned to some element $\phi(t)$ in a testing function space by a member f of the dual space. Finally $D(I)$ is the space of infinitely differentiable functions defined on I having compact support. The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of the Schwartz distribution.

2. The testing function space. *$T_{\alpha,\beta}(I)$ and its dual $T'_{\alpha,\beta}(I)$.* Let $\nu > 0$ be a fixed real number and let α, β be a pair of real numbers such that $0 < \alpha < 2\nu + 1$ and $0 \leqq \beta < 1$. Let $\xi(t)$ be the positive continuous function defined on I as

$$\xi(t) = \begin{cases} t^{-\beta} & \text{for} \quad 0 < t < 1 \\ t^{\alpha} & \text{for} \quad t \geqq 1 \, . \end{cases}$$

We define $T_{\alpha,\beta}(I)$ as the collection of infinitely differentiable complex-valued functions $\phi(t)$ defined on I such that

$$\gamma_k(\phi) = \sup_{0<t<\infty} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \left(\frac{\phi(t)}{m'(t)} \right) \right| < \infty \, , \tag{2.1}$$

for each $k = 0, 1, 2, 3, \cdots$, where $m'(t) = [2^{\nu-1/2}\Gamma(\nu + 1/2)]^{-1} t^{2\nu}$. We assign to $T_{\alpha,\beta}(I)$ the topology generated by the semi-norms $\{\gamma_k\}_{k=0}^{\infty}$, thereby making it a countably multi-normed space. A sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ where each $\phi_n(x) \in T_{\alpha,\beta}(I)$, converges in $T_{\alpha,\beta}(I)$ to $\phi(x)$ if $\gamma_k(\phi_n - \phi)$ tends to zero as $n \rightarrow \infty$ for each $k = 0, 1, 2, \cdots$. A sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ in $T_{\alpha,\beta}(I)$ is said to be a Cauchy sequence if $\gamma_k(\phi_m - \phi_n)$ tends to zero as m and n both tend to infinity independently of each other, for each $k = 0, 1, 2, \cdots$. It can be seen that $T_{\alpha,\beta}(I)$ is a Frechet space, i.e., a complete countably multi-normed-space [10, p. 12]. The dual space $T'_{\alpha,\beta}(I)$ consists of all continuous linear functionals on $T_{\alpha,\beta}(I)$. $T'_{\alpha,\beta}(I)$ is also a linear space to which we assign the weak topology generated by the multi-norm $\{\eta_\phi\}$, where $\eta_\phi(f) = |\langle f, \phi \rangle|$ and ϕ ranges through $T_{\alpha,\beta}(I)$. From now on the space $T_{\alpha,\beta}(I)$ and its dual will be denoted by the symbol $T_{\alpha,\beta}$ and $T'_{\alpha,\beta}$ respectively.

It is obvious that the space $D(I)$ is contained in $T_{\alpha,\beta}$, and the topology of $D(I)$ is stronger than that induced on it by $T_{\alpha,\beta}$. Hence the restriction of any $f \in T'_{\alpha,\beta}$ to $D(I)$ is in $D'(I)$.

One can easily check that if $f(x)$ is a function on I such that

$$\int_0^\infty \frac{|f(x)|}{\xi(x)} dm(x) < \infty \, ,$$

then $f(x)$ generates a regular generalized function on $T_{\alpha,\beta}$ defined by

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)dx \, , \quad \phi \in T_{\alpha,\beta} \, . \tag{2.2}$$

Using this fact one can show that if $f(x) \in L$ then $f(x)$ generates a regular generalized function on $T_{\alpha,\beta}$.

3. The Hankel potential transform of generalized functions. Motivation. For $f(x) \in L$, we rewrite the Hankel potential transform (1.1) of f as

$$T(f) \equiv (Tf)(x) = \int_0^\infty k_\nu(x, t) f(t) dt$$

where

$$k_\nu(x, t) = \left[2^{\nu-1/2}\Gamma\left(\nu + \frac{1}{2}\right)\right]^{-1} t^{2\nu+1}(x^2 + t^2)^{-\nu-1} .$$

Since $x^{2\nu+1}k_\nu(x, t) = t^{2\nu+1}k_\nu(t, x)$, it follows from Fubini's theorem that for any $\phi \in T_{\alpha,\beta}$

$$(3.1) \qquad \int_0^\infty \phi(x) \int_0^\infty k_\nu(x, t) f(t) dt dx = \int_0^\infty f(t) t^{2\nu+1} \int_0^\infty k_\nu(t, x) \frac{1}{x^{2\nu+1}} \phi(x) dx dt ,$$

i.e.,

$$(3.2) \qquad \int_0^\infty T(f(t))(x)\phi(x) dx = \int_0^\infty f(t) t^{2\nu+1} T(x^{-(2\nu+1)}\phi(x))(t) dt ,$$

which is a Parseval type relation for the Hankel potential transformation.

We can show that $t^{2\nu+1}T(x^{-(2\nu+1)}\phi(x))(t) \in T_{\alpha,\beta}$ wherever $\phi \in T_{\alpha,\beta}$. This fact enables us to define the Hankel potential transform of generalized functions in $T'_{\alpha,\beta}$ by extending the relation (3.2) to generalized functions.

DEFINITION 3.1. The Hankel potential transform $H_\nu f$ of $f \in T'_{\alpha,\beta}$ is that element of $T'_{\alpha,\beta}$ which assigns the same number to $\phi(x) \in T_{\alpha,\beta}$ as f assigns to $t^{2\nu+1}T(x^{-(2\nu+1)}\phi(x))(t)$. More precisely, $H_\nu f$ is given by

$$(3.3) \qquad \langle H_\nu f, \phi \rangle = \langle f(t), t^{2\nu+1}T(x^{-(2\nu+1)}\phi(x))(t) \rangle ,$$

for all $\phi \in T_{\alpha,\beta}$.
The above definition is meaningful, as function $t^{2\nu+1}T(x^{-(2\nu+1)}\phi(x))(t) \in T_{\alpha,\beta}$ whenever $\phi(x) \in T_{\alpha,\beta}$.

4. **The inversion theorem.** First we notice that for a fixed $\nu > 0$, $k_\nu(x, t)$ belongs to the space $T_{\alpha,\beta}$ but $\partial k_\nu(x, t)/\partial t$ does not belong to the same space. This shows that the testing function space $T_{\alpha,\beta}$ is not closed with respect to differentiation. Therefore, it will not be possible for us to define the generalized derivative of $f \in T'_{\alpha,\beta}$ by

$$(4.1) \qquad \langle Df, \phi \rangle = \langle f, -D\phi \rangle , \quad \forall \phi \in T_{\alpha,\beta} .$$

Now, noting that $T_{\alpha,\beta}$ is closed with respect to the operator $t(d/dt)$, we define an operator θ as a mapping from $T'_{\alpha,\beta}$ to itself by the following relation

$$(4.2) \qquad \langle \theta f, \phi \rangle = \left\langle f, -\left(t\frac{d}{dt} + 1\right)\phi \right\rangle , \quad \forall \phi \in T_{\alpha,\beta} .$$

For a regular distribution f generated by $f \in D'(I)$, we have

$$\left\langle t\frac{d}{dt}f,\phi\right\rangle = \left\langle f,\ -\left(t\frac{d}{dt}+1\right)\phi\right\rangle, \quad \forall\phi\in D(I).$$

Therefore it follows that the operator θ defined by (4.2) is analogue of the operator $t(d/dt)$ on $D'(I)$.

Define the operators $\tilde{L}_{n,x}$ and $Q_{n,x}$ as follows

$$\tilde{L}_{n,x} = \frac{2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(n+\nu+1)n!}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2}\theta\prod_{k=1}^{n}\left(1-\frac{\theta}{2k}\right)\left(1+\frac{\theta}{2k+2\nu}\right) \tag{4.3}$$

$$n = 1, 2, 3\cdots,$$

$$\begin{aligned} Q_{n,x} = {} & \frac{2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(n+\nu+1)n!}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2} \\ & \times\left\{-x\frac{d}{dx}-(2\nu+1)\right\}\prod_{k=1}^{n}\left(\left\{1+\frac{x\frac{d}{dx}+2\nu+1}{2k}\right\}\right. \\ & \times\left.\left\{1-\frac{x\frac{d}{dx}+2\nu+1}{2k+2\nu}\right\}\right) \qquad n = 1, 2, 3, \cdots. \end{aligned} \tag{4.4}$$

In (4.3), the suffix x represents the argument of the testing function of the generalized function upon which the operator $\tilde{L}_{n,x}$ works. A simple computation shows that for $f\in T'_{\alpha,\beta}$ and all $\phi\in T_{\alpha,\beta}$ we have

$$\langle\theta f,\phi\rangle = \left\langle f,\ -m'(x)\left\{x\frac{d}{dx}+(2\nu+1)\right\}\left(\frac{\phi(x)}{m'(x)}\right)\right\rangle,$$

where $m'(x) = [2^{\nu-1/2}\Gamma(\nu+(1/2))]^{-1}x^{2\nu}$.
Therefore it can be shown that

$$\langle\tilde{L}_{n,x}f,\phi\rangle = \left\langle f,\ m'(x)Q_{n,x}\left(\frac{\phi(x)}{m'(x)}\right)\right\rangle \tag{4.5}$$

$$\forall\phi\in T_{\alpha,\beta} \text{ and } f\in T'_{\alpha,\beta}.$$

We now wish to show that if $H_\nu f$ is the Hankel potential transform of $f\in T'_{\alpha,\beta}$, then

$$\lim_{n\to\infty}\tilde{L}_{n,x}(H_\nu f) = f \text{ in the topology of } T'_{\alpha,\beta},$$

i.e., for $f\in T'_{\alpha,\beta}$,

$$\lim_{n\to\infty}\langle\tilde{L}_{n,x}H_\nu f,\phi\rangle = \langle f,\phi\rangle, \quad \forall\phi\in T_{\alpha,\beta}. \tag{4.6}$$

To prove the inversion formula (4.6), we need the following lemmas.

LEMMA 4.1. *Let $\xi(t)$ be the function as defined in Section 2. Then, for an arbitrary $\varepsilon > 0$, there exists a positive $\eta < 1$ such that*

$$\left|\frac{\xi(t)}{\xi(tx)} - 1\right| < \varepsilon \qquad \forall t > 0 \quad and \quad 1 - \eta < x < 1 + \eta \,. \tag{4.7}$$

PROOF. The lemma follows quite easily by considering the behaviour, of $\xi(t)$ as $t \to 0+$ and $t \to \infty$, and observing that $\xi(t)$ is uniformly continuous function of t over any compact subset of I.

LEMMA 4.2. *If $\phi(t)$ is continuous in $0 < t < \infty$ and the limits as t approaches to $0+$ and ∞ exist then to an arbitrary positive ε there corresponds a number η such that*

$$|\phi(tx) - \phi(t)| < \varepsilon$$

for $0 < t < \infty$, $0 < 1 - \eta < x < 1 + \eta$.

PROOF. The proof follows from [9, Lemma 5, p. 287].

LEMMA 4.3. *Let $\phi(t) \in T_{\alpha,\beta}$ with $0 < \beta \leqq 1$ and $0 < \alpha < 2\nu + 1$ then for $\varepsilon > 0$ there exists a positive $\eta < 1$ such that*

$$\left|\xi(tx)\left\{\frac{\phi(tx)}{m'(tx)}\right\} - \xi(t)\left\{\frac{\phi(t)}{m'(t)}\right\}\right| < \varepsilon\,, \qquad \forall t > 0 \quad and \quad 1 - \eta < x < 1 + \eta\,.$$

PROOF. This is a consequence of Lemma 4.2.

LEMMA 4.4. *Let $\phi(t) \in T_{\alpha,\beta}$ with $0 < \alpha < 2\nu + 1$, $0 \leqq \beta < 1$. Then for $\varepsilon > 0$ there exists a positive $\eta < 1$ such that*

$$\xi(t)\left|\frac{\phi(tx)}{m'(tx)} - \frac{\phi(t)}{m'(t)}\right| < \varepsilon\,, \qquad \forall t \text{ and } x \text{ satisfying}$$

$$1 - \eta < x < 1 + \eta \quad and \quad t > 0\,.$$

PROOF. In view of Lemmas 4.1 and 4.3, for $\varepsilon > 0$ there exists positive $\eta < 1$ such that for all $t > 0$ and $1 - \eta < x < 1 + \eta$

$$(1 - \varepsilon)\xi(tx) < \xi(t) < \xi(tx)(1 + \varepsilon) \tag{4.8}$$

and

$$-\varepsilon < \xi(tx)\frac{\phi(tx)}{m'(tx)} - \xi(t)\frac{\phi(t)}{m'(t)} < \varepsilon\,. \tag{4.9}$$

Now, we write $q(t, x) = \xi(t) - \xi(tx)$.
Clearly,

$$|q(t, x)| < \varepsilon\xi(tx) \qquad \forall t > 0 \quad \text{and} \quad 1 - \eta < x < 1 + \eta\,.$$

Now,

$$\xi(t)\left[\frac{\phi(tx)}{m'(tx)} - \frac{\phi(t)}{m'(t)}\right] = [q(t, x) + \xi(tx)]\frac{\phi(tx)}{m'(tx)} - \xi(t)\left(\frac{\phi(t)}{m'(t)}\right)$$

$$= \xi(tx)\left(\frac{\phi(tx)}{m'(tx)}\right) - \xi(t)\left(\frac{\phi(t)}{m'(t)}\right) + q(t, x)\left(\frac{\phi(tx)}{m'(tx)}\right) .$$

Therefore, using Lemmas 4.1 and 4.3 we get

$$\xi(t)\left|\left[\frac{\phi(tx)}{m'(tx)} - \frac{\phi(t)}{m'(t)}\right]\right| < \varepsilon + \varepsilon\xi(tx)\left|\frac{\phi(tx)}{m'(tx)}\right| \leqq \varepsilon(1 + M)$$

where $\quad M = \sup_{0<t<\infty}\left|\frac{\phi(t)}{m'(t)}\right|\xi(t) .$

Since ε is arbitrary the proof of the lemma is complete.

LEMMA 4.5. *Let $\phi \in T_{\alpha,\beta}$ and $\xi(t)$ be the function as defined in Section 2. Then*

$$(4.10) \qquad \frac{\xi(t)2\Gamma(2n + \nu + 2)}{\left\{\Gamma\left(n + \frac{\nu}{2} + 1\right)\right\}^2}\int_0^\infty\left[\frac{\phi(x)}{m'(x)} - \frac{\phi(t)}{m'(t)}\right]\frac{x^{2n+\nu+2}t^{2n+1}}{(x^2 + t^2)^{2n+\nu+2}}dx \longrightarrow 0$$

uniformly for all $t > 0$ as $n \longrightarrow \infty$.

PROOF. Denoting the integral in (4.10) by I and using the substitution $x = ty$, we get

$$(4.11) \qquad I = \int_0^\infty\left[\frac{\phi(tx)}{m'(tx)} - \frac{\phi(t)}{m'(t)}\right]\frac{x^{2n+2\nu+2}}{(1 + x^2)^{2n+\nu+2}}dx .$$

Now we divide the range of integration in (4.11) into $0 < x < 1 - \eta$, $1 - \eta < x < 1 + \eta$ and $1 + \eta < x < \infty$, $(0 < \eta < 1)$, and denote the corresponding integrals by I_1, I_2, and I_3, respectively.

Using Lemma 4.4 and Stirling's approximation formula we get

$$(4.12) \qquad \frac{2\Gamma(2n + \nu + 2)}{\left[\Gamma\left(n + \frac{\nu}{2} + 1\right)\right]^2}\xi(t)|I_2| \leqq M_\nu\varepsilon \quad \text{as} \quad n \longrightarrow \infty ,$$

where M_ν is a constant depending upon ν.

Next, for $x \in (0, 1 - \eta)$, there exist positive constants c and c' satisfying

$$(4.13) \qquad \sup_{0<t<\infty}\xi(t)\left|\frac{\phi(tx)}{m'(tx)} - \frac{\phi(t)}{m'(t)}\right| \leqq \frac{c}{x^\alpha} + c' .$$

A careful computation shows that

$$\xi(t)|I_1| \leqq (c+c')\int_0^{1-\eta}\left(\frac{x}{1+x^2}\right)^{2n+1}dx\,.$$

Using Stirling's approximation formula again, we have for $n \to \infty$

$$\frac{2\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2}\xi(t)|I_1| \leqq \frac{(c+c')2^{\nu+1}}{e^{\sqrt{\pi}}}\sqrt{n}\left(1-\frac{\eta^2}{1+(1-\eta)^2}\right)^{2n+1} \to 0$$

as $n \to \infty$.

Hence

$$\frac{2\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2}\xi(t)I_1 \to 0 \quad \text{as} \quad n \to \infty \tag{4.14}$$

uniformly for all $t > 0$.

Again, for $x > 1+\eta$ we can find positive constants c_1, c_2 and c_3 such that

$$\sup_{0<t<\infty}\xi(t)\left|\frac{\phi(tx)}{m'(tx)}-\frac{\phi(t)}{m'(t)}\right| \leqq c_1x^\beta+\frac{c_2}{x^\alpha}+c_3 \leqq c_1x+c_2+c_3\,. \tag{4.15}$$

Using the bound (4.15) and Stirling's approximation formula, we can show that

$$\frac{\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2}\xi(t)I_3 \to 0 \quad \text{as} \quad n \to \infty \tag{4.16}$$

uniformly for all $t > 0$.

Combining (4.12), (4.14) and (4.16), Lemma 4.5 is proven.

We are now ready to prove our main result.

THEOREM 4.1. *Let f be an arbitrary generalized function in $T'_{\alpha,\beta}$ $(0<\alpha<2\nu+1,\ 0\leqq\beta<1;\ \nu>0)$ and let $H_\nu f$ be the Hankel potential transform of f as defined by* (3.3). *Then*

$$\lim_{n\to\infty}\langle\widetilde{L}_{n,x}H_\nu f,\phi\rangle = \langle f,\phi\rangle\,, \quad \forall\phi\in T_{\alpha,\beta}\,. \tag{4.17}$$

PROOF. For any ϕ in $T_{\alpha,\beta}$, we have

$$\langle\widetilde{L}_{n,x}H_\nu f,\phi\rangle = \left\langle H_\nu f,\ Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\}m'(x)\right\rangle, \qquad \text{(by (4.5))}$$

$$= \left\langle f(t),\ t^{2\nu+1}T\left(x^{-(2\nu+1)}Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\}m'(x)\right)(t)\right\rangle$$

$$= \left\langle f(t), \frac{t^{2\nu+1}}{2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty \frac{Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\}}{(t^2+x^2)^{\nu+1}} dm(x) \right\rangle$$

$$= \left\langle f(t), m'(t) \int_0^\infty \frac{t}{(t^2+x^2)^{\nu+1}} Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\} dm(x) \right\rangle$$

$$= \langle f(t), \theta_{n,\nu}(t)\rangle , \quad \text{say.}$$

We wish to show that,

$$\theta_{n,\nu}(t) \to \phi(t) \quad \text{in} \quad T_{\alpha,\beta} \quad \text{as} \quad n \to \infty .$$

Now,

$$\left(t\frac{d}{dt}\right)^k \frac{1}{m'(t)}[\theta_{n,\nu}(t)] = \int_0^\infty \left(t\frac{d}{dt}\right)^k \left(\frac{t}{(t^2+x^2)^{\nu+1}}\right) Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\} dm(x)$$

$$= \int_0^\infty \left\{-x\frac{d}{dx} - (2\nu+1)\right\}^k \left(\frac{t}{(t^2+x^2)^{\nu+1}}\right) Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\} dm(x)$$

$$= \int_0^\infty \frac{t}{(t^2+x^2)^{\nu+1}} \left(x\frac{d}{dx}\right)^k Q_{n,x}\left\{\frac{\phi(x)}{m'(x)}\right\} dm(x)$$

(by integration by parts)

$$= \int_0^\infty \frac{t}{(t^2+x^2)^{\nu+1}} Q_{n,x} \left(x\frac{d}{dx}\right)^k \left(\frac{\phi(x)}{m'(x)}\right) dm(x)$$

$$(4.18) \qquad = \int_0^\infty L_{n,x}\left(\frac{t}{(t^2+x^2)^{\nu+1}}\right) \left(x\frac{d}{dx}\right)^k \left(\frac{\phi(x)}{m'(x)}\right) dm(x)$$

(by integration by parts)

where $L_{n,x}$ is the operator defined by (1.3).

(4.18) can be written in view of [1, (2.4), p. 320] as

$$\frac{2\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2} \int_0^\infty \frac{x^{2n+2\nu+2}}{(x^2+t^2)^{2n+\nu+2}} t^{2n+1} \left(x\frac{d}{dx}\right)^k \left(\frac{\phi(x)}{m'(x)}\right) dx .$$

Moreover, it can be seen that

$$\frac{2\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2} \int_0^\infty \frac{x^{2n+2\nu+2}}{(x^2+t^2)^{2n+\nu+2}} t^{2n+1} dx \to 1 ,$$

as $n \to \infty$. In fact, this follows by evaluating the above integral with the help of the substitution $x = t \tan u$, and then using Sterling's formula. Hence, as $n \to \infty$,

$$\xi(t)\left(t\frac{d}{dt}\right)^k\left[\frac{1}{m'(t)}\theta_{n,\nu}(t)-\frac{\phi(t)}{m'(t)}\right]=\xi(t)\frac{2\Gamma(2n+\nu+2)}{\left\{\Gamma\left(n+\frac{\nu}{2}+1\right)\right\}^2}\tag{4.19}$$
$$\times\int_0^\infty\frac{x^{2n+2\nu+2}}{(x^2+t^2)^{2n+\nu+2}}\,t^{2n+1}\left[\left(x\frac{d}{dx}\right)^k\left(\frac{\phi(x)}{m'(x)}\right)-\left(t\frac{d}{dt}\right)^k\left(\frac{\phi(t)}{m'(t)}\right)\right]dx\,.$$

Also,

$$\left(x\frac{d}{dx}\right)\left(\frac{\phi(x)}{m'(x)}\right)=\frac{1}{m'(x)}\left[\left(x\frac{d}{dx}\right)\phi(x)-2\nu\phi(x)\right]$$
$$=\frac{\psi(x)}{m'(x)}\,,\quad\text{say,}$$

where $\psi(x)$ is clearly an element of $T_{\alpha,\beta}$. Hence, in view of Lemma 4.5, (4.19) converges to zero as $n\to\infty$, uniformly for all $t>0$.

This completes the proof of the theorem.

5. **Another approach.** In this section, the Hankel potential transform of generalized functions is defined in a different way. An inversion formula for the Hankel potential transform of generalized functions is also established, interpreting the limit operation in $D'(I)$. The testing function space $T_{\alpha,\beta}$ and its dual $T'_{\alpha,\beta}$ are taken to be the same as before.

DEFINITION 5.1. The Hankel potential transform of a generalized function $f\in T'_{\alpha,\beta}$ is defined as a function $F(x)$ obtained by applying $f(t)$ on the kernel $m'(t)(t/(x^2+t^2)^{\nu+1})$, i.e.,

$$F(x)=\left\langle f(t),\,m'(t)\frac{t}{(x^2+t^2)^{\nu+1}}\right\rangle,\quad x>0\,.\tag{5.1}$$

It can be easily seen that for a fixed $x>0$, $m'(t)(t/(x^2+t^2)^{\nu+1})$ belongs to the space $T_{\alpha,\beta}$, and therefore the right-hand side of (5.1) has a sense.

REMARK 5.1. A similar approach as above has also been used in defining Hankel transform [4], [5], Laplace transform [11], Mellin transform [11], and Stieltjes transform [6], of generalized functions.

Now following the standard technique [3, p. 70], one can prove that $F(x)$ is an infinitely differentiable function and that

$$F^{(k)}(x)=\left\langle f(t),\,\frac{\partial^k}{\partial x^k}\left[m'(t)\frac{t}{(x^2+t^2)^{\nu+1}}\right]\right\rangle,\quad k=1,2,\cdots.$$

The following theorem gives the behaviour of $F(x)$ near zero and infinity.

THEOREM 5.1. *Let* $f\in T'_{\alpha,\beta}$ $(0\leqq\beta<1,\,0<\alpha<2\nu+1,\,\nu>0)$ *and let*

$F(x)$ be the Hankel potential transform of f as defined by (5.1). *Then*

$$|F(x)| = O(x^{-(2\nu+1+\beta)}) \quad as \quad x \longrightarrow 0+$$

and

$$|F(x)| = O(x^{\alpha-(2\nu+1)}) \quad as \quad x \longrightarrow \infty \ .$$

PROOF. In view of the result [10, Th. 8-1, p. 18], there exists a positive constant c and a non-negative integer r such that

$$|F(x)| \leqq c \max_{0\leqq k\leqq r} \sup_{0<t<\infty} \left| \xi(t)\Big(t\frac{d}{dt}\Big)^k \frac{t}{(x^2+t^2)^{\nu+1}} \right| .$$

Clearly

$$\left| \xi(t)\Big(t\frac{d}{dt}\Big)^k \frac{t}{(x^2+t^2)^{\nu+1}} \right| \longrightarrow 0 \text{ when either } t \longrightarrow 0+ \quad \text{or} \quad t \longrightarrow \infty \ . \tag{5.2}$$

Therefore, the expression in (5.2) assumes global maximum for each $k = 0, 1, 2, \cdots$ somewhere in $(0, \infty)$. Carrying on the principle of maxima and minima, we see that the maximum value of the expression in (5.1) lies at points like $a_{k,\nu}x$. Using this fact, our result stated in the theorem follows.

THEOREM 5.2. (Inversion). *Let* $f \in T'_{\alpha,\beta}$ $(0 \leqq \beta < 1, 0 < \alpha < 2\nu + 1)$ *and let $F(x)$ be the Hankel potential transform of f. Then*

$$\lim_{n\to\infty} \langle L_{n,x}F(x), \phi(x)\rangle = \langle f, \phi\rangle ,$$

for each $\phi \in D(I)$, where $L_{n,x}$ is the operator defined by (1.3)$'$.

PROOF. The theorem is proved by justifying the steps in the following manipulations.

$$\langle L_{n,x}F(x), \phi(x)\rangle = \int_0^\infty L_{n,x}(F(x))\phi(x)dx \tag{5.3}$$

$$= \int_0^\infty \Big\langle f(t), \frac{m'(t)t}{(x^2+t^2)^{\nu+1}} \Big\rangle Q_{n,x}\Big(\frac{\phi(x)}{m'(x)}\Big)dm(x) \tag{5.4}$$

$$= \Big\langle f(t), m'(t)\int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} Q_{n,x}\Big(\frac{\phi(x)}{m'(x)}\Big)dm(x) \Big\rangle \tag{5.5}$$

$$\longrightarrow \langle f, \phi\rangle \quad \text{as} \quad n \longrightarrow \infty \ . \tag{5.6}$$

The step (5.3) is obvious in view of Theorem 5.1 and the facts that both $F(x)$ and $\phi(x)$ are smooth functions and $\phi(x)$ has a compact support on $(0, \infty)$. The successive integration by parts in (5.3) leads to (5.4); the operator $Q_{n,x}$ has been defined by (4.4). That (5.4) equals to (5.5) can be proved by the technique of Riemann sums [2, p. 76]. The last step follows by showing that

$$m'(t)\int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} Q_{n,x}\left(\frac{\phi(x)}{m'(x)}\right)dm(x) \to \phi(t) \tag{5.7}$$

in $T_{\alpha,\beta}$ as $n \to \infty$, which has been established while proving Theorem 4.1 for $\phi \in T_{\alpha,\beta}$ and is therefore true for $\phi \in D(I)$.

This completes the proof of the theorem.

REMARK 5.2. We could not succeed in proving the above theorem for $\phi \in T_{\alpha,\beta}$. The difficulty arises in justifying the step (5.3) for $\phi \in T_{\alpha,\beta}$.

REMARK 5.3. The Hankel transformations $F(x)$ and $H_\nu(f)$ of f as defined by (5.1) and (3.3) respectively, agree on the space $D(I)$. In fact for any $\phi \in D(I)$,

$$\begin{aligned}
\langle F(x), \phi(x)\rangle &= \int_0^\infty F(x)\phi(x)dx \\
&= \int_0^\infty \left\langle f(t),\, m'(t)\frac{t}{(x^2+t^2)^{\nu+1}}\right\rangle \phi(x)dx \\
&= \left\langle f(t),\, \int_0^\infty m'(t)\frac{t}{(x^2+t^2)^{\nu+1}}\phi(x)dx \right\rangle \\
&\qquad \text{(by Riemann sum technique)} \\
&= \langle f(t),\, t^{2\nu+1}T(x^{-(2\nu+1)}\phi(x))(t)\rangle \\
&= \langle H_\nu f, \phi\rangle\ .
\end{aligned}$$

REFERENCES

[1] F. M. CHOLEWINSKI AND D. T. HAIMO, Inversion of the Hankel potential transform, Pacific J. Math., 37, No. 2 (1971), 319-330.

[2] J. L. B. COOPER, Laplace transformation of distributions, Canadian J. Math., Vol. 18 (1966), 1325-1332.

[3] L. S. DUBE, Some Hankel transformations of generalized functions, Ph. D. Thesis, Carleton University (1973).

[4] L. S. DUBE AND J. N. PANDEY, On the Hankel transform of distributions, Tôhoku Math. J., 27 (3), (1975), 337-354.

[5] E. L. KOH AND A. H. ZEMANIAN, The complex Hankel and I-transformations of generalized functions, SIAM J. Appl. Math., 16, No. 5 (1968) 945-957.

[6] J. N. PANDEY, On the Stieltjes transform of generalized functions, Proc. Camb. Phil. Soc., 71 (1972), 85-96.

[7] J. N. PANDEY AND A. H. ZEMANIAN, Complex inversion for the generalized convolution transformation, Pacific J. Math., 25 (1968), 147-157.

[8] L. SCHWARTZ, Théorie des Distributions, Vol. II, Hermann, Paris, 1959.

[9] D. V. WIDDER, The Laplace Transform, Princeton Univ. Press, Princeton, N.J., 1941.

[10] A. H. ZEMANIAN, Generalized Integral Transformations, Interscience Publishers, 1968.

[11] A. H. ZEMANIAN, The distributional Laplace and Mellin transformations, SIAM J. Appl. Math., 14 (1966), 41-59.

[12] A. H. ZEMANIAN, A generalized Weierstrass transformation, SIAM J. Appl. Math., 15 (1967), 1088-1105.

[13] A. H. ZEMANIAN, Distribution Theory and Transform Analysis, McGraw-Hill Book Company, 1965.

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