Tôhoku Math. Journ.
29 (1977), 375-384.

SMOOTH $Sp(n)$-ACTIONS ON COHOMOLOGY QUATERNION PROJECTIVE SPACES

TÔRU WADA

(Received February 2, 1976)

0. Introduction. In his paper [6], F. Uchida considered smooth $SU(n)$-actions on rational cohomology complex projective spaces. The purpose of this paper is to study smooth $Sp(n)$-actions on rational cohomology quaternion projective spaces and to show similar results as in [6].

In fact, for $8 \leqq n \leqq k \lneqq 2n-4$, each smooth $Sp(n)$-action on rational cohomology $P_k(\boldsymbol{H})$ is characterized by a $4(k-n+1)$-dimensional rationally acyclic compact manifold with smooth $Sp(1)$-action.

Professor Uchida, when he visited our Department in 1974, kindly allowed us to look over the preprint of the paper [6] and suggested to investigate this problem. Without his sincere instruction and advice, this paper could not have been prepared.

It is scarcely enough to pay him the usual thanks for fruitful suggestions.

1. Decomposition of certain $Sp(n)$-manifolds. Throughout this section, we assume $n \geqq 3$. Let M be a closed connected manifold with smooth $Sp(n)$-action. Denote by $F(Sp(n), M)$ the stationary point set. We consider the following condition for each $x \in M - F(Sp(n), M)$, there is $g \in Sp(n)$ such that

$$(*) \qquad Sp(n-1) \subset Sp(n)_{gx} \subset N(Sp(n-1)) = Sp(n-1) \times Sp(1) .$$

Here $N(Sp(n-1))$ is the normalizer of $Sp(n-1)$ in $Sp(n)$, and $Sp(n)_x$ is the isotropy group at x.

THEOREM 1.1. *Let M be a closed connected $Sp(n)$-manifold satisfying the condition $(*)$ and $M \neq F(Sp(n), M)$. Then there is an equivariant deffeomorphism*

$$M \cong \partial(D(\boldsymbol{H}^n) \times X)/Sp(1) .$$

Here X is a compact connected manifold with smooth $Sp(1)$-action.

If $\partial X \neq \emptyset$, the $Sp(1)$-action on ∂X is free. Furthermore,

$$F(Sp(n), M) \cong \partial X/Sp(1) .$$

Here, $\boldsymbol{H}^n$ is an n-dimensional quaternion vector space on which $Sp(n)$ acts naturally and $D(\boldsymbol{H}^n)$ is a unit disk of $\boldsymbol{H}^n$.

To prove Theorem 1.1, we prepare some lemmas.

LEMMA 1.2. *Let* $g \in Sp(n)$. *If* $g\,Sp(n-1)g^{-1} \subset N(Sp(n-1))$ *then* $g \in N(Sp(n-1))$.

This is proved easily by linear algebra, so we omit the proof.

LEMMA 1.3. *Let E be an $Sp(n)$-manifold satisfying the condition* $(*)$. *If $F(Sp(n), E)$ is empty, then*

$$E \cong Sp(n) \underset{NSp(n-1)}{\times} X \cong \frac{Sp(n)}{Sp(n-1)} \underset{Sp(1)}{\times} X$$

where $X = F(Sp(n-1), E)$ on which

$$Sp(1) = \frac{N(Sp(n-1))}{Sp(n-1)} \text{ acts naturally} .$$

PROOF. $[g\,Sp(n-1), x] \to gx$ is bijective by Lemma 1.2, and hence it is an equivariant diffeomorphism.

LEMMA 1.4. *Assume* $n \geqq 3$. *If $Sp(n)$ acts on $\boldsymbol{R}^k$ via a representation $\rho: Sp(n) \to O(k)$ which satisfies the condition* $(*)$ *and* $F(Sp(n), \boldsymbol{R}^k) = \{o\}$, *then $k = 4n$ and ρ is equivalent to the canonical inclusion* $Sp(n) \subset O(4n)$.

PROOF. The unit sphere S^{k-1} is $Sp(n)$-invariant. By Lemma 1.3, there is an equivariant diffeomorphism

$$S^{k-1} \cong \frac{Sp(n)}{Sp(n-1)} \underset{Sp(1)}{\times} F \tag{1.4.1}$$

where $F = F(Sp(n-1), S^{k-1})$ is a unit sphere of $F(Sp(n-1), \boldsymbol{R}^k)$.

Since $Sp(n)$ is a simple Lie group, we have

$$2n^2 + n = \dim Sp(n) \leqq \dim O(k) = \frac{k(k-1)}{2}$$

and hence $k > 2n$.

By (1.4.1), there is an exact sequence

$$\pi_4(S^{k-1}) \to \pi_4(P_{n-1}(\boldsymbol{H})) \to \pi_3(F) \to \pi_3(S^{k-1}) .$$

Here $\pi_4(P_{n-1}(\boldsymbol{H})) = \boldsymbol{Z}$ and $\pi_4(S^{k-1}) = \pi_3(S^{k-1}) = 0$ by $k > 2n$ and $n \geqq 3$.

Therefore $\pi_3(F) = \boldsymbol{Z}$ and hence $F = S^3$. Then $k = 4n$ by (1.4.1). Next we can prove that $Sp(1)$ acts freely and transitively on $F = S^3$. Therefore $Sp(n)$ acts transitively on S^{k-1} by (1.4.1). Hence ρ is equivalent to the canonical inclusion $Sp(n) \to O(4n)$ by Lemma 5.6 of [4].

LEMMA 1.5. *Let $V \to X$ be a real $Sp(n)$-vector bundle. Assume that*

$Sp(n)$ acts trivially on X and each fiber of V is isomorphic to $\boldsymbol{H}^n$ as *real $Sp(n)$-vector space. Then V is naturally equivalent to $\boldsymbol{H}^n \otimes_H \mathrm{Hom}^{Sp(n)}(\boldsymbol{H}^n, V)$ as real $Sp(n)$-vector bundles. Here* $\mathrm{Hom}^{Sp(n)}(\boldsymbol{H}^n, V)$ *is a quaternion line bundle over X via the right scalar multiplication on $\boldsymbol{H}^n$.*

PROOF. The natural map $\boldsymbol{H}^n \otimes_H \mathrm{Hom}^{Sp(n)}(\boldsymbol{H}^n, V) \to V$ defined by

$$u \otimes h \to h(u)$$

is an isomorphism as real $Sp(n)$-vector bundles.

REMARK 1.6. We can prove $F(Sp(n-1), V) = \mathrm{Hom}^{Sp(n)}(\boldsymbol{H}^n, V)$ as $Sp(1)$-vector bundles. Denote by $D(V), S(V)$ the unit disk bundle and the unit sphere bundle of V, resp.

Then we can prove by Lemma 1.5

$$D(V) = D(\boldsymbol{H}^n) \underset{Sp)1)}{\times} F(Sp(n-1), S(V))$$

as $Sp(n)$-manifolds.

PROOF OF THEOREM 1.1. If $F(Sp(n), M)$ is empty, then the result is proved in Lemma 1.3. So we assume $F = F(Sp(n), M)$ is non-empty. Let $N \to F$ be an equivariant normal bundle of F in M. Then each fiber of N is isomorphic to $\boldsymbol{H}^n$ as real $Sp(n)$-vector spaces by Lemma 1.4. Thus we can apply Lemma 1.5 and Remark 1.6 on N.

Let U be a closed invariant tubular neighborhood of F in M. Then U is identified with $D(N)$ as $Sp(n)$-manifold. Hence there is an equivariant diffeomorphism

$$U \cong D(\boldsymbol{H}^n) \underset{Sp(1)}{\times} F(Sp(n-1), \partial U)$$

as $Sp(n)$-manifolds from Remark 1.6. Applying Lemma 1.3 on $E = M - \mathrm{int}\, U$, we have an equivariant diffeomorphism

$$E \cong \frac{Sp(n)}{Sp(n-1)} \underset{Sp(1)}{\times} X$$

as $Sp(n)$-manifolds.
Here $X = F(Sp(n-1), E)$, $\partial X = F(Sp(n-1), \partial U)$ and

$$S(\boldsymbol{H}^n) = \partial D(\boldsymbol{H}^n) = \frac{Sp(n)}{Sp(n-1)} .$$

Therefore there is an equivariant decomposition

$$M = D(\boldsymbol{H}^n) \underset{Sp(1)}{\times} \partial X \cup_f S(\boldsymbol{H}^n) \underset{Sp(1)}{\times} X .$$

Here $Sp(1)$ acts freely on ∂X and f is an equivariant diffeomorphism on

$S(\boldsymbol{H}^n) \times_{Sp(1)} \partial X$. Because

$$F(Sp(n-1), S(\boldsymbol{H}^n) \underset{Sp(1)}{\times} \partial X)$$

is canonically identified with ∂X, f induces an equivariant diffeomorphism f' on ∂X as $Sp(1)$-manifold and $f = 1 \times_{Sp(1)} f'$. Therefore f is extendable to an equivariant diffeomorphism on $D(\boldsymbol{H}^n) \times_{Sp(1)} \partial X$, and hence there is an equivariant diffeomorphism

$$M \cong \partial(D(\boldsymbol{H}^n) \times X)/Sp(1)$$

as $Sp(n)$-manifolds.

THEOREM 1.7. *Let M be a closed connected manifold with smooth $Sp(n)$-action. Assume $n \geqq 8$, $\dim M < 8n - 16$ and $M \neq F(Sp(n), M)$. Then there is a compact connected manifold X with smooth $Sp(1)$-action, and there is an equivariant diffeomorphism*

$$M = \partial(D(\boldsymbol{H}^n) \times X)/Sp(1)\ .$$

Here $Sp(n)$-action is canonical on $D(\boldsymbol{H}^n)$ and trivial on X, $Sp(1)$-action on ∂X is free if ∂X is non-empty, and

$$F(Sp(n), M) \cong \partial X/Sp(1)\ .$$

PROOF. It is sufficient to prove that M satisfies the condition $(*)$. By the assumption,

$$\dim \frac{Sp(n)}{Sp(n)_x} < 8n - 16 \leqq (n-1)^2 \tag{1.7.1}$$

for each point $x \in M - F(Sp(n), M)$. Hence the identity component of $Sp(n)_x$ is conjugate to a subgroup

$$Sp(k) \times K \subset Sp(k) \times Sp(n-k) \subset Sp(n)$$

where $k > n/2$ and K is a closed subgroup of $Sp(n-k)$, by Theorem 1.20 of [2]. Since

$$\dim Sp(n)/Sp(n-2) \times Sp(2) = 8n - 16\ ,$$

the identity component of $Sp(n)_x$ is conjugate to subgroup $Sp(n-1) \times K$, for each $x \in M - F(Sp(n), M)$. Here K is a closed subgroup of $Sp(1)$. Hence the condition $(*)$ is satisfied.

2. Decomposition of cohomology $P_k(\boldsymbol{H})$ with smooth $Sp(n)$-action. Let A be a commutative ring. A closed connected $4k$-dimensional manifold M is called to be a cohomology $P_k(\boldsymbol{H})$ over A if the cohomology ring of M over A is isomorphic to one of $P_k(\boldsymbol{H})$, a topological space X is called to be acyclic over A if the reduced cohomology ring of X over A is null.

THEOREM 2.1. *Let K be a commutative field. Let M be a cohomology $P_k(\boldsymbol{H})$ over K with smooth $Sp(n)$-action. Assume $8 \leqq n \leqq k < 2n - 4$ and $M \neq F(Sp(n), M)$. Then there is a compact connected manifold X with smooth $Sp(1)$-action, and there is an equivariant diffeomorphism*

$$M = \partial(D(\boldsymbol{H}^n) \times X)/Sp(1)$$

as $Sp(n)$-manifolds.

Here X is a compact connected orientable $(4k - 4n + 4)$-dimensional manifold which is acyclic over K, the $Sp(1)$-action is free on ∂X, the $Sp(n)$-action is standard on $D(\boldsymbol{H}^n)$ and trivial on X, and $\pi_1(X) = \pi_1(M)$.

Furthermore, if $H^*(M; \boldsymbol{Z}) = H^*(P_{n+k}(\boldsymbol{H}); \boldsymbol{Z})$ then X is acyclic over integers, and $H^*(F; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{H}); \boldsymbol{Z})$.

PROOF. By Theorem 1.7, there is an equivariant diffeomorphism

$$M = \partial(D(\boldsymbol{H}^n) \times X)/Sp(1) \tag{2.1.1}$$

where X is a compact connected orientable $(4k - 4n + 4)$-dimensional manifold with smooth $Sp(1)$-action.

i) First we show that ∂X is non-empty. Assume that ∂X is empty. Then by (2.1.1) there is a smooth fiber bundle

$$X \longrightarrow M \longrightarrow P_{n-1}(\boldsymbol{H}) .$$

Thus

$$k + 1 = \chi(M) = \chi(X)\cdot\chi(P_{n-1}(\boldsymbol{H})) = 0 \pmod n .$$

This is impossible by the assumption $n \leqq k < 2n - 4$. Consequently, we have that ∂X is non-empty.

Then by (2.1.1) there is an equivariant diffeomorphism

$$M = \partial(D(\boldsymbol{H}^n) \times X)/Sp(1) = D(\boldsymbol{H}^n) \underset{Sp)1)}{\times} \partial X \cup_f S(\boldsymbol{H}^n) \underset{Sp(1)}{\times} X$$

as $Sp(n)$-manifolds. Here the $Sp(1)$-action is free on ∂X.

ii) Next we show that X is acyclic over K. Since

$$D(\boldsymbol{H}^n) \underset{Sp)1)}{\times} \partial X \longrightarrow \partial X/Sp(1)$$

is an orientable $4n$-disk bundle, there is an isomorphism

$$H^i(M, S(\boldsymbol{H}^n) \underset{Sp(1)}{\times} X; K) = H^{i-4n}(\partial X/Sp(1); K) .$$

Then we have

$$H^i(M; K) = H^i(S(\boldsymbol{H}^n) \underset{Sp(1)}{\times} X; K) \quad \text{for } i \leqq 4n - 2 . \tag{2.1.2}$$

Now we show that the Euler class $e(p)$ of the principal $Sp(1)$-bundle

$$p\colon \partial(D(\boldsymbol{H}^n) \times X) \longrightarrow M$$

is non-zero in $H^4(M; K)$. Assume $e(p)=0$. Then the Euler class of the principal $Sp(1)$-bundle

$$S(\boldsymbol{H}^n)\times X \to S(\boldsymbol{H}^n)\underset{Sp(1)}{\times} X$$

is zero in $H^4(S(\boldsymbol{H}^n)\times_{Sp(1)} X; K)$, and hence there is an isomorphism

$$H^*(S(\boldsymbol{H}^n); K)\otimes H^*(X; K)\cong H^*(S^3; K)\otimes H^*(S(\boldsymbol{H}^n)\underset{Sp(1)}{\times} X; K)$$

by a Gysin sequence. Therefore, we have the identity;

$$H^{4i}(X; K)=K \quad \text{for} \quad 0\leqq i<n$$

by (2.1.2) and the assumption $H^*(M; K)=H^*(P_k(\boldsymbol{H}); K)$. On the other hand,

$$\dim X = 4k-4n+4\leqq 4n-16\,,$$

which contradicts to the above equality. Therefore $e(p)\neq 0$ and hence

(2.1.3) $$H^*(\partial(D(\boldsymbol{H}^n)\times X); K)=H^*(S^{4k+3}; K)$$

by the Gysin sequence of the principal $Sp(1)$-bundle

$$p\colon \partial(D(\boldsymbol{H}^n)\times X)\to M\,.$$

There is an isomorphism

$$H^i(D(\boldsymbol{H}^n)\times X; K)=H_{4k+4-i}(D(\boldsymbol{H}^n)\times X, \partial(D(\boldsymbol{H}^n)\times X); K)$$

by the Poincaré-Lefschetz duality, and the homomorphism

$$H_{4k+4-i}(D(\boldsymbol{H}^n)\times X; K)\to H_{4k+4-i}(D(\boldsymbol{H}^n)\times X, \partial(D(\boldsymbol{H}^n)\times X); K)$$

is onto for $0<i<4k+4$ by (2.1.3). Since X is a connected $(4k-4n+4)$-dimensional manifold with a non-empty boundary,

$$H_{4k+4-i}(D(\boldsymbol{H}^n)\times X; K)=0 \quad \text{for} \quad 0<i\leqq 4n$$

and hence

$$H^i(X; K)=0 \quad \text{for} \quad 0<i\leqq 4n\,.$$

Therefore X is acyclic over K.

iii) Next we show $\pi_1(X)=\pi_1(M)$.

$$\text{Let } U=D(\boldsymbol{H}^n)\underset{Sp(1)}{\times}\partial X,\ E=S(\boldsymbol{H}^n)\underset{Sp(1)}{\times} X\,.$$

Then $M=U\bigcup_f E$. There is a smooth fiber bundle $X\to E\to S(\boldsymbol{H}^n)/Sp(1)$. Thus we have an isomorphism $\pi_1(X)\cong\pi_1(E)$. Applying the van Kampen theorem (see [1]) to the following diagram

$$\begin{array}{ccccc}
\pi_1(S(\boldsymbol{H}^n)\times_{Sp(1)}\partial X) & = & \pi_1(E\cap U) & \xrightarrow{j'_*} & \pi_1(E)\cong\pi_1(X) \\
\cong\big\downarrow i_* & & \big\downarrow i_* & & \big\downarrow i_* \\
\pi_1(D(\boldsymbol{H}^n)\times_{Sp(1)}\partial X) & = & \pi_1(U) & \xrightarrow{\quad j_*\quad} & \pi_1(M)\,,
\end{array}$$

we have $\pi_1(X) = \pi_1(M)$.
iv) Finally we show that the assumption $H^*(M; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{H}); \boldsymbol{Z})$ implies

$$\tilde{H}(X; \boldsymbol{Z}) = 0 .$$

Since $H^*(M; \boldsymbol{Z}) = \boldsymbol{Z}[x]/(x^{k+1})$, we have $H^*(M; \boldsymbol{Z}_p) = \boldsymbol{Z}_p[x]/(x^{k+1})$ for each prime p. Applying ii), we have

$$\tilde{H}^*(X; \boldsymbol{Z}_p) = 0 \quad \text{for each prime } p .$$

Then

$$\tilde{H}^*(X; \boldsymbol{Z}) = 0$$

by the universal coefficient theorem.

3. Construction of cohomology $P_k(\boldsymbol{H})$ with smooth $Sp(n)$-action. In this section we construct $Sp(n)$-actions on cohomology quaternion projective spaces, and we have the following results.

THEOREM 3.1. *Let $n \geqq 1$, $k \geqq 1$ and $p \geqq 1$. Then there is a compact orientable $4(n+k)$-dimensional manifold M such that*

$$\pi_1(M) = \boldsymbol{Z}/p\boldsymbol{Z} \text{ and } H^*(M; \boldsymbol{Q}) = H^*(P_{n+k}(\boldsymbol{H}); \boldsymbol{Q})$$

and M admits a smooth $Sp(n)$-action with

$$F(Sp(n), M) = P_k(\boldsymbol{H}) .$$

Here $\boldsymbol{Q}$ is the field of rational numbers.

THEOREM 3.2. *Let $n \geqq 1$ and $k \geqq 2$. Let G be a finitely presentable group with $H_1(G; \boldsymbol{Z}) = H_2(G; \boldsymbol{Z}) = 0$. Then*
(a) *there is a compact orientable $4(n+k)$-dimensional manifold M such that*

$$\pi_1(M) = G \text{ and } H^*(M; \boldsymbol{Z}) = H^*(P_{n+k}(\boldsymbol{H}); \boldsymbol{Z})$$

and M admits a smooth $Sp(n)$-action with

$$F(Sp(n), M) = P_k(\boldsymbol{H}) ,$$

(b) *there is a smooth $Sp(n)$-action on $P_{n+k}(\boldsymbol{H})$ such that*

$$\pi_1(F) = G \text{ and } H^*(F; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{H}); \boldsymbol{Z}) ,$$

where $F = F(Sp(n), P_{n+k}(\boldsymbol{H}))$.

REMARK 3.3. It is known that if G is a finitely presentable group with $H_1(G; \boldsymbol{Z}) = H_2(G; \boldsymbol{Z}) = 0$, then for each $m \geqq 7$, there is a compact contractible smooth m-dimensional manifold P such that

$$\pi_1(\partial P) = G \quad \text{(see [5])} .$$

It is known that there are infinitely many groups satisfying the above condition.

The above theorems are proved similarly as in the proof of Theorems 3.1 and 3.2 of [6], by making use of the following lemma. So we omit the proof.

LEMMA 3.4. *Let X be a compact orientable $4(k+1)$-dimensional manifold which is acyclic over $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$). Assume that X admits a smooth semi-free $Sp(1)$-action which is free on ∂X. If $n \geqq 1$, then*
(a) *$M = \partial(D(\boldsymbol{H}^n)\times X)/Sp(1)$ is a cohomology $P_{n+k}(\boldsymbol{H})$ over $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$),*
(b) *$\pi_1(M) = \pi_1(X)$.*
Moreover if $n+k \geqq 2$ and X is contractible, then $M = P_{n+k}(\boldsymbol{H})$.

PROOF. Let $Sp(n)$-action and $Sp(1)$-action on $D(\boldsymbol{H}^n)$ be the left standard action and the right scalar multiplication on $D(\boldsymbol{H}^n)$, resp.

The actions of $g \in Sp(n)$ and $u \in Sp(1)$ on $D(\boldsymbol{H}^n)\times X$ defined by

$$g(q, x) = (g\cdot q, x)$$
$$u(q, x) = (q\cdot u^{-1}, u\cdot x)\ ,$$

are compatible.

This $Sp(1)$-action is free on $\partial(D(\boldsymbol{H}^n)\times X)$, and $\partial(D(\boldsymbol{H}^n)\times X)$ is $Sp(n)$-invariant.

Therefore, $Sp(n)$-action on the orbit manifold

$$M = \partial(D(\boldsymbol{H}^n)\times X)/Sp(1)$$

is naturally induced.

Next, M is an orientable manifold, because X is an orientable manifold. The stationary points of $Sp(n)$-action on $D(\boldsymbol{H}^n)$ are an only origin, and $Sp(n)$-action on X is a trivial. Then we have

$$F = F(Sp(n), M) = \frac{\{o\}\times\partial X}{Sp(1)} \cong \partial X/Sp(1)\ .$$

i) Since $\tilde{H}^*(X; A) = 0$ for $A = \boldsymbol{Z}$ (resp. $\boldsymbol{Q}$) and

$$\dim(D(\boldsymbol{H}^n)\times X) = 4n + 4k + 4\ .$$

Applying Poincaré-Lefschetz duality to the pair $(D(\boldsymbol{H}^n)\times X, \partial(D(\boldsymbol{H}^n)\times X))$, we have

$$H_j(D(\boldsymbol{H}^n)\times X, \partial(D(\boldsymbol{H}^n)\times X); A) = 0 \quad \text{for } j \neq 4n+4k+4\ ,$$

and hence

$$H_j(\partial(D(\boldsymbol{H}^n)\times X); A) = 0 \quad \text{for } j \neq 0, 4n+4k+3\ .$$

Namely, $\partial(D(\boldsymbol{H}^n)\times X)$ is a homology $S^{4n+4k+3}$ over A. Here $n \geqq 1$, $k \geqq 0$.

By the Gysin sequence of the $Sp(1)$-bundle

$$\partial(D(\boldsymbol{H}^n) \times X) \to \partial(D(\boldsymbol{H}^n) \times X)/Sp(1) = M ,$$

we have

$$H^i(M; A) = \begin{cases} 0 & (i \not\equiv 0 \mod 4) \\ A & (i \equiv 0 \mod 4) \end{cases} \quad \text{for } 0 \leqq i \leqq 4n + 4k.$$

Namely, M is a cohomology $P_{n+k}(\boldsymbol{H})$ over A.
ii) There is an $Sp(1)$-bundle

$$Sp(1) \to \partial(D(\boldsymbol{H}^n) \times X) \to M .$$

Thus we have an isomorphism

$$\pi_1(\partial(D(\boldsymbol{H}^n) \times X)) \cong \pi_1(M) .$$

Applying the van Kampen theorem to the following diagram

$$\begin{array}{ccc} \pi_1(S(\boldsymbol{H}^n) \times \partial X) & \xrightarrow{\cong} & \pi_1(D(\boldsymbol{H}^n) \times \partial X) \\ \downarrow & & \downarrow \\ \pi_1(X) = \pi_1(S(\boldsymbol{H}^n) \times X) & \longrightarrow & \pi_1(\partial(D(\boldsymbol{H}^n) \times X)) \end{array}$$

we have $\pi_1(X) \cong \pi_1(\partial(D(\boldsymbol{H}^n) \times X))$.
iii) Let U be an $Sp(1)$-invariant disk around the just one fixed point in $D(\boldsymbol{H}^n) \times X$. We can assume $U = D(\boldsymbol{H}^{n+k+1})$ as $Sp(1)$-manifolds. Put

$$W = (D(\boldsymbol{H}^n) \times X - \operatorname{int} U)/Sp(1) .$$

Then, $\partial W = M \cup P_{n+k}(\boldsymbol{H})$. Since

$$\pi_1(M) = \pi_1(W) = 0$$
$$H_*(W, M; Z) = 0$$

and

$$\dim W = 4n + 4k + 1 \geqq 6 ,$$

we have

$$M = P_{n+k}(\boldsymbol{H}) ,$$

by applying the h-cobordism theorem (see [3], Theorem 9.1) to the triad $(W; M, P_{n+k}(\boldsymbol{H}))$.

REFERENCES

[1] R. CROWELL AND R. FOX, Introduction to Knot Theory, Guin and Co., 1963.
[2] W. C. HSIANG AND W. Y. HSIANG, Differentiable action of compact connected classical groups, I, Amer. J. Math., 89 (1967), 705-786.

[3] J. Milnor, Lectures on the h-cobordism theorem, Princeton Math. Notes, 1965.
[4] T. Nagano, Homogeneous sphere bundles and the isotropic Riemann manifolds, Nagoya Math. J., 15 (1959), 29-55.
[5] I. Tamura, Variety of varieties (Japanese), Sugaku, 21 (1969), 275-285.
[6] F. Uchida, Smooth actions of special unitary groups on cohomology complex projective spaces, Osaka J. Math., 12 (1975), 375-400.

Department of Mathematics
Faculty of Science
Yamagata University