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ON A GENERALIZATION OF THE HOPF FIBRATION, I*

(Contact structures on the generalized Brieskorn manifolds)

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1. Introduction. In the study of differential geometry, the Hopf fibration is, perhaps, one of the most inspiring and informative objects. It is not only simple and lucid in its definition, but also intimately related in its depth to other areas of mathematics in equally fruitful ways. This paper (and Part II [3]) is devoted to an attempt to extend some of the characteristics of the Hopf fibration to a certain class of smooth manifolds.

Let $(S^{2n-1}, \pi, CP^{n-1})$ be the triple of the Hopf fibration. As is well known, this fibration is a principal S^1-bundle over CP^{n-1}. In the language of group actions, S^1 acts freely on S^{2n-1}, and its orbit space is CP^{n-1}. The latter view can be readily refined to get more general fibrations (not necessarily fiber bundle). Let V be an irreducible complex analytic subvariety of $\mathbf{C}^{n+1}$ and let S be the ellipsoid given by the equation $\sum_{i=0}^{n} b_i |Z_i|^2 = \varepsilon^2$ for positive numbers b_i $(i = 0, \cdots, n)$ and ε. Furthermore, assume that V is invariant under a $\mathbf{C}$-action on $\mathbf{C}^{n+1}$ of the form

$$t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t} Z_0, \cdots, e^{2\pi q_n t} Z_n) , \quad t \in \mathbf{C} .$$

Here $q_0, \cdots, q_n$ are positive numbers. Then it is shown (Lemma 1) that $\Sigma = S \cap V$ is a smooth manifold with the induced S^1-action provided that the origin is either a regular or an isolated singular point of V and that $q_0, \cdots, q_n$ are rational numbers. This Σ is called the generalized Brieskorn manifold. Clearly Σ represents all the original Brieskorn manifolds and other similar manifolds. In particular, if $V = C^n$ and $b_0 = \cdots = b_n = \varepsilon = q_0 = \cdots = q_n = 1$, Σ is the total space of the Hopf fibration.

Going back to the Hopf fibration, let us consider the following two basic properties of the fibration. First, the connection 1-form ω of the fibration satisfies $\omega \wedge (d\omega)^{n-1} \neq 0$ everywhere. In other words, ω is a contact structure on S^{2n-1}. Next, $S^{2p-1} \times S^{2q-1}$ admits a complex structure, and furthermore the triple $(S^{2p-1} \times S^{2q-1}, \pi, CP^{p-1} \times CP^{q-1})$ is a holomorphic principal torus bundle over $CP^{p-1} \times CP^{q-1}$. This complex structure is otherwise known as a Calabi-Eckmann structure [7].

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In this paper, we focus our attention to the contact structure on S^{2n-1}, and generalize it on Σ. A generalization of the Calabi-Eckmann structure is treated in Part II. Throughout Parts I and II, a special emphasis is placed on the discussion of the inter-relations between the above two structures from the differential geometric point of view; therefore, this part should be considered as the preliminary to Part II. Also emphasized are examples. Some of the proofs are given via typical examples.

In Chapter 2, we give the fundamental definitions and properties of generalized Brieskorn manifolds, and some typical examples as well. These properties and examples are basically well known in such cases as the original Brieskorn manifolds and the weighted homogeneous manifolds [5] [17].

In Chapter 3, we first show that Σ admits 1-parameter families of almost contact structures and a 1-parameter family of contact structures. An observation concerning the behavior of the leaves of the associated foliations to these structures is made. It is shown that these structures are, in general, non-regular. As a more refined case, we show that Σ admits a normal contact structure. In doing so, we observe that there are two natural ways to generalize ω on S^{2n-1} to Σ. One is the contact structure constructed in the previous paper of Erbacher and the author [2], and the other is the one given in this paper. Although the class of manifolds that admit the former structure seems larger than that of the latter [2], we choose the latter as the generalization of ω for the following reasons. First, the structure in this paper is normal, and secondly, the leaves of the associated foliation are closed curves. In fact, these two structures on Σ are not much different from each other in the sense that there is a 1-parameter family of contact structures connecting them. After establishing a certain criterion to classify the normal contact structures, we show that there exist infinitely many distinct normal contact structures on the Brieskorn spheres (exotic or standard), the generalized lens spaces and $S^n \times S^{n+1}$ (n: even). Some observations are also made to establish a sort of Boothby-Wang fibration theorem on an open dense subset of Σ. This includes a construction of certain Kählerian metric in the base space.

In concluding the introduction, we would like to point out that the above classification of contact structures is still quite crude, and we hope that more precise classification will be made in the near future. It also seems reasonable that some sort of classification can be made in terms of deformation; for example, the deformation in the sense of

Gray [8].

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2. Generalized Brieskorn manifolds. Let $\mathbb{C}^{n+1}$ denote complex Euclidean space of complex dimension $n+1$. For any $(n+1)$-tuple $(q_0, \cdots, q_n)$ of positive numbers, there exists a natural $\mathbb{C}$-action on $\mathbb{C}^{n+1}$ given as follows:

$$t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t}Z_0, \cdots, e^{2\pi q_n t}Z_n)\ , \quad \text{for all } t \in \mathbb{C}\ .$$

In what follows, call this type of $\mathbb{C}$-action on various spaces the natural $\mathbb{C}$-action.

Let V be an irreducible complex analytic subvariety of $\mathbb{C}^{n+1}$, and let us assume that V is invariant under a natural $\mathbb{C}$-action on $\mathbb{C}^{n+1}$; hence, V has a natural $\mathbb{C}$-action induced from that of $\mathbb{C}^{n+1}$. Next let us denote by $S(\varepsilon)$ an ellipsoid in $\mathbb{C}^{n+1}$ given as follows:

$$S(\varepsilon) = \{(Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}: b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon^2\}\ .$$

Here $b_0, \cdots, b_n$ and ε are positive numbers. Notice here that if $b_0 = \cdots = b_n = 1$, $S(\varepsilon)$ turns out to be the sphere of radius ε in $\mathbb{C}^{n+1}$ which has the origin 0 of $\mathbb{C}^{n+1}$ as its center. Denote by $\Sigma(\varepsilon)$ the intersection of $S(\varepsilon)$ and V. From now on, we denote S or Σ for $S(\varepsilon)$ or $\Sigma(\varepsilon)$ unless any possibility of confusion occurs. Now we have,

LEMMA 1. a) *Let* $V = M^k \cup M^{k-1} \cup \cdots \cup M^1 \cup M^0$ *be the partition of* V *by dimension, where* k *is the dimension of* V *and* M^i $(0 \leqq i \leqq k)$ *is a complex submanifold of* $\mathbb{C}^{n+1}$ *of complex dimension* i. *For the details, see Whitney* [26]. *Then* M^i $(0 \leqq i \leqq k)$ *is invariant under the* $\mathbb{C}$-*action. In particular,* $M^0 = \{0\}$ *or empty.*

b) 0 *belongs to the closure of each* M^i *in* V $(1 \leqq i \leqq k)$. *Thus if* 0 *is a regular point of* V, i.e., *if* $0 \in M^k$, $M^i = \varnothing$ $(0 \leqq i \leqq k-1)$. *Therefore* V *is a complex submanifold of* $\mathbb{C}^{n+1}$ *of complex dimension* k. *If* 0 *is an isolated singular point of* V, *then* $M^{k-1} = M^{k-2} = \cdots = M^1 = \varnothing$ *and* $M^0 = \{0\}$. *This implies that* $V_0 = V - \{0\} = M^k$, *so it is a complex* k-*dimensional submanifold of* $\mathbb{C}^{n+1}$.

c) V intersects $S(\varepsilon)$ *transversally and* $V \cap S(\varepsilon) = \Sigma(\varepsilon)$. *Furthermore if* 0 *is a regular* (*or isolated singular*) *point of* V, $\Sigma(\varepsilon)$ *is a* *compact smooth* $(2k-1)$*-dimensional manifold with the naturally induced smooth structure from that of* $S(\varepsilon)$.

d) V *is, in general, homeomorphic to the cone built on* $\Sigma(\varepsilon)$ *whose generator is the real line* $\boldsymbol{R}$. *If* 0 *is a regular* (*or isolated singular*) *point of* V, V_0 *is diffeomorphic to* $\boldsymbol{R}\times\Sigma(\varepsilon)$, *where* $\boldsymbol{R}\times\Sigma(\varepsilon)$ *has the product differentiable structure.*

We call this Σ a generalized Brieskorn manifold (associated to V).

PROOF. Let $\xi=(\xi_0,\cdots,\xi_n)$ be a point in M^i $(0\leqq i\leqq k)$. Note here that M^i is a disjoint union of i-dimensional complex submanifolds of $\boldsymbol{C}^{n+1}$, see p. 93 in [26]. Denote by M^i_ξ the connected component containing ξ. Then there exists an open neighborhood U of ξ in $\boldsymbol{C}^{n+1}$ such that $U_\xi=U\cap M^i_\xi$ is a connected complex submanifold of U of dimension i, and such that U_ξ is open in M^i_ξ. Now let $t\in\boldsymbol{C}$ be any complex number. Then $(Z_0,\cdots,Z_n)\mapsto t(Z_0,\cdots,Z_n)$ is a biholomorphism of $\boldsymbol{C}^{n+1}$. Since V is invariant under the $\boldsymbol{C}$-action, $t(U_\xi)$ is contained in V. By the definition of M^j $(0\leqq j\leqq k)$, $t(U_\xi)$, then, is contained in M^i. Now let st $(0\leqq s\leqq 1)$ be the line segment in $\boldsymbol{C}$ between 0 and t. It is easy to see that $(st)(\xi)$ $(0\leqq s\leqq 1)$ is a curve connecting ξ and $t(\xi)$. By the above observation, we know that $(st)(\xi)$ $(0\leqq s\leqq 1)$ belongs to M^i. Thus $(st)(\xi)$ must belong to M^i_ξ. As t is any complex number, the action of $\boldsymbol{C}$ leaves M^i_ξ invariant. If $i=0$, M^0 consists of isolated points. It is clear that the $\boldsymbol{C}$-action is nowhere trivial, i.e., the $\boldsymbol{C}$-orbit of any point in $\boldsymbol{C}^{n+1}$ is not a point except for 0. Combining this fact with the above observation, we see that 0 is the only possible point in M^0. In any case, the $\boldsymbol{C}$-action leaves M^i $(i=0,\cdots,k)$ invariant. This proves a).

Let ξ be an element of M^i $(i=0,\cdots,k)$. We restrict the $\boldsymbol{C}$-action on V to the subgroup of $\boldsymbol{C}$ consisting of the real numbers. Then we have an induced $\boldsymbol{R}$-action on V defined by $t(Z_0,\cdots,Z_n)=(e^{2\pi q_0 t}Z_0,\cdots,e^{2\pi q_n t}Z_n)$, $t\in\boldsymbol{R}$. The orbit of ξ under this $\boldsymbol{R}$-action is a curve in V. By the similar argument to the one used in a), $t(\xi)$ belongs to M^i_ξ for all $t\in\boldsymbol{R}$. On the other hand, $t(\xi)=(e^{2\pi q_0 t}\xi_0,\cdots,e^{2\pi q_n t}\xi_n)$ approaches the origin as t approaches $-\infty$. Thus the origin 0 of $\boldsymbol{C}^{n+1}$ is in the closure of M^i_ξ; therefore, in the closure of M^i. Next let 0 be a regular point. Then 0 belongs to M^k which is open and dense in V, see Gunning-Rossi [9]. Since 0 is in the closure of M^i, $M^i=\varnothing$ $(0\leqq i\leqq k-1)$.

If 0 is an isolated singular point of V, it can be a limit point of M^k alone again by the first half of b). Therefore, $M^{k-1}=\cdots=M^1=\varnothing$ and $M^0=\{0\}$. This completes the proof of b).

In order to prove c), let $(Z_0,\cdots,Z_n)$ be a point in $\Sigma(\varepsilon)=V\cap S(\varepsilon)$.

As is given in the proof of b), $t(Z_0, \cdots, Z_n)$, $t \in \boldsymbol{R}$, is a curve in V passing through $(Z_0, \cdots, Z_n)$ at $t = 0$. The velocity vector of this orbit at $(Z_0, \cdots, Z_n)$ is given by $(2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n)$. Let $r(Z_0, \cdots, Z_n) = \sum_{i=0}^{n} b_i |Z_i|^2 - \varepsilon^2$ be a function defined in $\mathbb{C}^{n+1}$. Then the set $\{(Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}: r(Z_0, \cdots, Z_n) = 0\}$ is exactly the ellipsoid $S(\varepsilon)$. It is easy to show that the gradient of r, say $\operatorname{grad} r$, is given by $(2b_0 Z_0, \cdots, 2b_n Z_n)$ at $(Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}$. Now denote by $\langle\ \rangle$ the standard hermitian metric of $\mathbb{C}^{n+1}$. Then

$$\langle (2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n), (2b_0 Z_0, \cdots, 2b_n Z_n) \rangle = 4\pi(b_0 q_0 |Z_0|^2 + \cdots + b_n q_n |Z_n|^2) .$$

The real part of this inner product is $4\pi \sum_{i=0}^{n} b_i q_i |Z_i|^2$ itself. Since b_i and q_i are positive $(i = 0, \cdots, n)$, the real inner product between the tangent vector to the $\boldsymbol{R}$-orbit and the gradient of r is positive everywhere in Σ. Since $\operatorname{grad} r$ is perpendicular to S at $(Z_0, \cdots, Z_n) \in S$, the tangent space of S at $(Z_0, \cdots, Z_0)$ and $(2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n)$ span the whole $\mathbb{C}^{n+1}$. This implies that V and $S(\varepsilon)$ intersect transversally everywhere. Now let 0 be either regular of isolated singular. First let us point out that $t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t} Z_0, \cdots, e^{2\pi q_n t} Z_n)$, $t \in \boldsymbol{R}$, approaches infinity as $t \to \infty$ since the magnitude $\| t(Z_0, \cdots, Z_n) \|$ of $t(Z_0, \cdots, Z_n)$ equals $(e^{4\pi q_0 t} |Z_0|^2 + \cdots + e^{4\pi q_n t} |Z_n|^2)^{1/2}$, and it goes to ∞ as $t \to \infty$. Therefore for any $\varepsilon > 0$, the intersection of V and $S(\varepsilon)$ is non-empty. By b), $V_0 = V - \{0\}$ is a real $2k$-dimensional smooth submanifold (or complex k-dimensional) of $\mathbb{C}^{n+1}$. By restricting the above function r to V_0, we have a real valued smooth function on V_0. Denote the restriction by the same letter r for the sake of convenience. As before, the gradient of r in $\mathbb{C}^{n+1}$ is given by $2(b_0 Z_0, \cdots, b_n Z_n)$ at $(Z_0, \cdots, Z_n)$, and the gradient of r in V_0 is nothing but the tangential component of $2(b_0 Z_0, \cdots, b_n Z_n)$ to V_0 at each $(Z_0, \cdots, Z_n) \in V_0$. By the previous observation, we know that the gradient has non-vanishing inner product with the velocity vector along the $\boldsymbol{R}$-orbit. This tells us the gradient of r does not vanish on V_0. Thus all the points in V_0 are regular points of r in the sense of Morse theory; i.e., they are not critical points of r. It is well known that any level set of such a function is a smooth $(2k-1)$-dimensional submanifold of V_0 without boundary. For any $\varepsilon > 0$, the level set of $r = \{(Z_0, \cdots, Z_n) \in V_0: r(Z_0, \cdots, Z_n) = \varepsilon^2\} = V \cap \{(Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}: r(Z_0, \cdots, Z_n) = \varepsilon^2\} = V \cap S(\varepsilon) = \Sigma(\varepsilon)$. Thus $\Sigma(\varepsilon)$ is a compact, smooth, $(2k-1)$-dimensional submanifold of V as well as $\mathbb{C}^{n+1}$ and $S(\varepsilon)$ without boundary. This proves c).

Finally, let $[-\infty, \infty) \times \Sigma(\varepsilon)$ be the Cartesian product of $[-\infty, \infty)$ and $\Sigma(\varepsilon)$. By the cone built on $\Sigma(\varepsilon)$ with generator $\boldsymbol{R}$, we mean the topological space obtained from $[-\infty, \infty) \times \Sigma(\varepsilon)$ by identifying $\{-\infty\} \times \Sigma(\varepsilon)$ with a point. The cone is given the natural quotient topology. Now define a mapping $\tilde{F}\colon [-\infty, \infty) \times \Sigma(\varepsilon) \to V$ as follows:

$$\begin{aligned} \tilde{F}(t, (Z_0, \cdots, Z_n)) &= t(Z_0, \cdots, Z_n) \quad \text{if} \quad t \in (-\infty, \infty) \\ &= \text{the origin of } 0 \text{ of } \mathbb{C}^{n+1} \\ &\quad\ \text{for all points in } \{-\infty\} \times \Sigma(\varepsilon)\,. \end{aligned}$$

Clearly $\tilde{F}$ is continuous in $(-\infty, \infty) \times \Sigma(\varepsilon)$. Let $(-\infty, (Z_0, \cdots, Z_n))$ be a point such that $\tilde{F}(-\infty, (Z_0, \cdots, Z_n)) = 0$. Let $B(\delta)$ be the open ball in $\mathbb{C}^{n+1}$ about 0 with radius δ. Since $\|t(Z_0, \cdots, Z_n)\| = (\sum_{i=0}^{n} e^{4\pi q_i t} |Z_i|^2)^{1/2}$ for all $t \in (-\infty, \infty)$, we have $\|t(Z_0, \cdots, Z_n)\| \leqq K e^{4\pi t(q_0 + \cdots + q_n)}$ for all points in $S(\varepsilon)$, where K is a positive constant. This tells us that $\|t(Z_0, \cdots, Z_n)\| \to 0$ uniformly as $t \to -\infty$; therefore, for the given $\delta > 0$, there exists a real number t_0 such that $\tilde{F}([-\infty, t_0) \times \Sigma(\varepsilon)) \subset B(\delta)$. This shows that $\tilde{F}$ is continuous everywhere. Let F be the mapping from the cone onto V which is naturally induced from $\tilde{F}$. Then the following diagram commutes. Note that F is clearly continuous.

$$\begin{array}{ccc} [-\infty, \infty) \times \Sigma(\varepsilon) & \xrightarrow{\tilde{F}} & V \\ \Big\downarrow P & \nearrow F & \\ \text{the cone} & & \end{array}$$

Here P is the quotient mapping of the cone which is of course continuous. Next we show that F is one to one and onto, and F^{-1} is continuous. Again by the definition of $\tilde{F}$ (or F), it is clear that F is one to one. Let $(\omega_0, \cdots, \omega_n)$ be any point of V. If $(\omega_0, \cdots, \omega_n)$ is the origin, it is clear that $(\omega_0, \cdots, \omega_n)$ is the image of some point under F. Let $(\omega_0, \cdots, \omega_n)$ be a point in V_0. As before $t(\omega_0, \cdots, \omega_n) \to 0$ (or ∞) as $t \to -\infty$ (or ∞). Thus there must exist some $t_0 \in (-\infty, \infty)$ such that $t_0(\omega_0, \cdots, \omega_n)$ belongs to $S(\varepsilon)$; therefore, it belongs to $\Sigma(\varepsilon)$. Then $F(-t_0, t_0(\omega_0, \cdots, \omega_n)) = (-t_0 + t_0)(\omega_0, \cdots, \omega_n) = (\omega_0, \cdots, \omega_n)$. So we have shown that F is onto. It is easy to show that F^{-1} is continuous and the proof is left to the reader. This proves the first half of d). Now let 0 be either a regular or isolated singular point of V. It can be easily seen that F restricted to $(-\infty, \infty) \times \Sigma(\varepsilon)$ is a diffeomorphism as follows. Let $\bar{F}$ be the mapping from $(-\infty, \infty) \times S(\varepsilon)$ onto $\mathbb{C}^{n+1} - \{0\}$ defined by $(t, \omega_0, \cdots, \omega_n) \mapsto t(\omega_0, \cdots, \omega_n)$ for $t \in (-\infty, \infty)$ and $(\omega_0, \cdots, \omega_n) \in S(\varepsilon)$. Then clearly $\bar{F}$ is a diffeomorphism. Since $(-\infty, \infty) \times \Sigma(\varepsilon)$ is a regular submanifold of $(-\infty, \infty) \times S(\varepsilon)$ and $\bar{F}$ restricted to $(-\infty, \infty) \times \Sigma(\varepsilon)$ is F, F is a diffeomorphism. This completes the proof of Lemma 1. q.e.d.

EXAMPLE 1 (Brieskorn manifold). The following is the original polynomial studied by Brieskorn and others. Let $P(Z) = Z_0^{a_0} + \cdots + Z_n^{a_n}$ be a polynomial of n variables $Z_0, \cdots, Z_n$, where $a_0, \cdots, a_n$ are positive integers. Then it is well known that the origin is the only possible singular point of the locus of zeros of the polynomial, say V. Let S be the unit hypersphere of $\mathbb{C}^{n+1}$ at the origin. Then $\Sigma = V \cap S$ is a $(2n-1)$-dimensional, smooth manifold, and is called the Brieskorn manifold associated with the polynomial $P(Z)$. The topological aspects of this Σ has been studied thoroughly by many people, and have produced a great deal of stimulation in the related areas. For example, Σ is $(n-2)$-connected, and represent all the exotic spheres which bound a parallelizable manifold. For the fundamental information of Σ, see Milnor [17], and of course, the original papers by Brieskorn. Next, we show that V admits a $\mathbb{C}$-action such as described previously.

Let d denote the least common multiple of $a_0, \cdots, a_n$, which is sometimes denoted by $[a_0, \cdots, a_n]$. For any $Z = (Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}$, and for any complex number $t \in \mathbb{C}$, define the action by

$$t(Z) = t(Z_0, \cdots, Z_n) = (e^{2\pi d/a_0 t} Z_0, \cdots, e^{2\pi d/a_n t} Z_n) .$$

It is clear that this action leaves V invariant. Thus Σ is a generalized Brieskorn manifold.

EXAMPLE 2. Let $P_i(Z_0, \cdots, Z_n) = \sum_{j=0}^{\infty} \alpha_{ij} Z_j^{a_{ij}}$, $i = 1, \cdots, m$, be a set of m polynomials of $n+1$ variables, where α_{ij} $(1 \leqq i \leqq m, 0 \leqq j \leqq n)$ is a real number and a_{ij} $(1 \leqq i \leqq m, 0 \leqq j \leqq n)$ is a positive integer. Denote by V the locus of common zeros of P_i $(1 \leqq i \leqq m)$ in $\mathbb{C}^{n+1}$; i.e., $V = \{(Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1}: P_i(Z_0, \cdots, Z_n) = 0 \text{ for } 1 \leqq i \leqq m\}$. We define a $\mathbb{C}$-action on V. To this end, denote by d_i $(1 \leqq i \leqq m)$ the least common multiple of $a_{i0}, \cdots, a_{in}$, and set $q_{ij} = d_i/a_{ij}$ for $1 \leqq i \leqq m, 0 \leqq j \leqq n$. Furthermore, we assume that q_{ij} is independent of i. Let us denote $q_j = q_{1j} (= q_{2j} = \cdots = q_{mj})$, $j = 0, \cdots, n$. Define a $\mathbb{C}$-action on $\mathbb{C}^{n+1}$ by

$$t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t} Z_0, \cdots, e^{2\pi q_n t} Z_n) , \quad \text{for} \quad t \in \mathbb{C} .$$

Then this $\mathbb{C}$-action leaves V invariant. If we denote by $S(\varepsilon)$ a hypersphere of radius ε at the origin, $\Sigma(\varepsilon) = V \cap S(\varepsilon)$ is a generalized Brieskorn manifold. The topological aspects of this $\Sigma(\varepsilon)$ have been studied in [5] [19] [21].

EXAMPLE 3. (Weighted homogeneous madifolds). Let $(\omega_0, \cdots, \omega_n)$ be an $(n+1)$-tuple of positive rational numbers. A polynomial $P(Z_0, \cdots, Z_n)$ is said to be weighted homogeneous with weights $(\omega_0, \cdots, \omega_n)$ if $P(Z)$ is a linear combination of monomials $Z_0^{i_0} Z_1^{i_1} \cdots Z_n^{i_n}$ for which $i_0/\omega_0 + \cdots +$

$i_n/\omega_n = 1$. For example, any polynomial in Example 1 is weighted homogeneous with weights $(a_0, \cdots, a_n)$. Also, consider $P(Z_0, Z_1, Z_2) = Z_0Z_1^2 + Z_1Z_2^4 + Z_2Z_0^3$ is weighted homogeneous with weights $(25/7, 25/9, 25/4)$. For more examples, see [17]. Now write $\omega_j = u_j/v_j$, $j = 0, \cdots, n$, where u_j and v_j are relatively prime positive integers. Let d be the least common multiple of $u_0, \cdots, u_n$, and let $q_j = d/\omega_j = dv_j/u_j$, $0 \leqq j \leqq n$. Then $\mathbb{C}$ acts on $\mathbb{C}^{n+1}$ by $t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t}Z_0, \cdots, e^{2\pi q_n t}Z_n)$. It is easy to see that this $\mathbb{C}$-action leaves V invariant; therefore, $V \cap S(\varepsilon) = \Sigma(\varepsilon)$ is a generalized Brieskorn manifold.

3. Almost contact structures and contact structures on the generalized Brieskorn manifolds. First we recall some notions and notations on almost contact structures and contact structures. We follow Sasaki [22] for this purpose.

Let M be a $(2n+1)$-dimensional smooth manifold. A triple (ϕ, ξ, η) of smooth tensor fields of type (1, 1), (1, 0) and (0, 1) is called an almost contact structure on M, if the following two conditions are satisfied:

1) $\eta(\xi) = 1$ everywhere.

2) $\phi^2(X) = -X + \eta(X)\xi$ for all smooth vector fields X on M.

From 1), one sees that ξ is a nowhere vanishing vector field on M, and it generates a 1-dimensional foliation on M which we call the associated foliation. The almost contact structure (ϕ, ξ, η) is called regular if the associated foliation is regular in the sense of Palais [20], and otherwise called non-regular. To be more precise, a foliation is regular if for each point $x \in M$ there exists Fröbenius coordinates around x such that different slices belong to different leaves of the foliation.

Let M be the same as above. A contact structure on M is a smooth 1-form ω on M such that $\omega \wedge (d\omega)^n \neq 0$ everywhere on M. Then a distribution D on M is associated with ω as follows. Let

$$D_x = \{X \in TM_x : d\omega(X, Y) = 0 \text{ for all } Y \in TM_x\} .$$

Because of $\omega \wedge (d\omega)^n \neq 0$ everywhere, $\dim D_x = 1$. Thus D is integrable and determines a 1-dimensional foliation on M which we call the associated foliation with ω. In fact, it is easy to see that D is generated by a nowhere vanishing vector field. The contact structure ω is called regular if this associated foliation is regular, and otherwise non-regular.

Next we briefly mention that a contact structure on M gives rise to a natural almost contact structure on M under a certain Riemannian metric. For the details, see [22]. Let (ϕ, ξ, η) be an almost contact structure on M. Then it is known [22] that there exists a Riemannian metric g on M such that $\eta(X) = g(\xi, X)$ and $g(\phi X, \phi Y) = g(X, Y) -$

$\eta(X)\eta(Y)$ hold for all vector fields X and Y on M. The quadruple (ϕ, ξ, η, g) is called the almost contact Riemannian (or metric) structure on M associated with the almost contact structure (ϕ, ξ, η). Now let ω be a contact structure on M. Then there exists an almost contact metric structure (ϕ, ξ, η, g) such that $\eta(X) = \omega(X)$, $\eta(X) = g(\xi, X)$ and $d\eta(X, Y) = d\omega(X, Y) = g(\phi X, Y)$. This almost contact metric structure is called a contact metric structure associated with ω. A contact structure can be called regular if the associated almost contact metric structure is regular, otherwise non-regular.

As an almost complex structure has a torsion tensor whose vanishing is a necessary and sufficient condition for the almost complex structure to be a complex structure, there can be defined a torsion tensor T for an almost contact structure (ϕ, ξ, η) as follows.

$$T(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] \\ -(X\eta(Y) - Y\eta(X))\xi ,$$

where X and Y are any smooth vector fields on M and $[X, Y]$ denotes the Lie bracket between X and Y. (ϕ, ξ, η) is called normal if $T \equiv 0$ everywhere. A contact structure is called normal if the associated almost contact structure is normal.

Going back to the generalized Brieskorn manifolds, let V be an irreducible complex subvariety of $\mathbb{C}^{n+1}$ such as in § 2 which has a $\mathbb{C}$-action given by $t(Z_0, \cdots, Z_n) = (e^{2\pi q_0 t}Z_0, \cdots, e^{2\pi q_n t}Z_n)$, $t \in \mathbb{C}$. Let $S(\varepsilon)$ be the ellipsoid in $\mathbb{C}^{n+1}$ defined by the equation $r(Z) = b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon^2$ $(\varepsilon > 0)$. Note here $b_0 = \cdots = b_n = 1$ gives us the hypersphere of radius ε. As before, we denote by $\Sigma(\varepsilon)$ the intersection of V and $S(\varepsilon)$. In this section, we always assume that the origin of $\mathbb{C}^{n+1}$, say 0, is a regular or isolated singular point of V. We also denote $V -$ {the origin} $= V - \{0\}$ by V_0 for the sake of convenience. First we show that the $\mathbb{C}$-action on V induces a natural S^1-action on $\Sigma(\varepsilon)$ under certain conditions. Let $i\boldsymbol{R}$ be the subgroup of $\mathbb{C}$ represented by purely imaginary numbers. Then $i\boldsymbol{R}$ acts on V by the induced action from that of $\mathbb{C}$. The action leaves V invariant. We see that $i\boldsymbol{R}$-action leaves $S(\varepsilon)$ invariant. This implies the $i\boldsymbol{R}$-action leaves $\Sigma(\varepsilon) = V \cap S(\varepsilon)$ invariant. This can be considered as an $\boldsymbol{R}$-action on $\Sigma(\varepsilon)$. In particular, let all of $q_0, \cdots, q_n$ be all positive rational numbers. Put $q_0 = u_0/v_0, \cdots, q_n = u_n/v_n$, where u_i and v_i $(i = 0, \cdots, n)$ are mutually prime positive integers. Denote by d the least common multiple of $v_0, \cdots, v_n$. Then $q_0 d, \cdots, q_n d$ are positive integers. Therefore,

$$i(ld+r)(Z_0, \cdots, Z_n) = (e^{2\pi q_0(ld+r)i}Z_0, \cdots, e^{2\pi q_n(ld+r)i}Z_n) \\ = ir(Z_0, \cdots, Z_n) .$$

This tells us that the $i\boldsymbol{R}$-action on $\Sigma(\varepsilon)$ is periodic with period d, and it induces an S^1-action on $\Sigma(\varepsilon)$. We call the S^1-action the induced S^1-action. It is easy to see that this S^1-action is fixed point free, and that the S^1-orbits are all diffeomorphic to S^1. The $\boldsymbol{C}$-actions in Examples 1, 2 and 3 induce the natural S^1-actions.

Going back to the $\boldsymbol{C}$-action on V, it is well known that each element of the Lie algebra of a Lie transformation group generates a vector field in a natural way on the manifold on which it acts. In particular, 1 and $\sqrt{-1}$ considered as elements of the Lie algebra of $\boldsymbol{C}$ generate vector fields $\mathfrak{A}$ and $\mathfrak{B}$ on $V_0 = V - \{\text{the origin}\}$ as given below.

$$\mathfrak{A} = (2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n)$$
$$\mathfrak{B} = (2\pi q_0\sqrt{-1}Z_0, \cdots, 2\pi q_n\sqrt{-1}Z_n) \quad \text{for all} \quad (Z_0, \cdots, Z_n) \in V_0 .$$

Note here that $\mathfrak{A}$ and $\mathfrak{B}$ are nothing but the velocity vectors of the $\boldsymbol{R}$ and $i\boldsymbol{R}$-actions at the corresponding point, respectively. It is clear that $\mathfrak{A}$ and $\mathfrak{B}$ are nowhere vanishing vector fields on V_0, and they are tangent to the $\boldsymbol{C}$-orbit of $(Z_0, Z_1, \cdots, Z_n)$. Note here that if 0 is the only possible singular point, by Lemma 1, b), V_0 is a complex submanifold of C^{n+1}, and therefore, V_0 is a Kählerian submanifold of $\boldsymbol{C}^{n+1}$ with its induced metric from that of $\boldsymbol{C}^{n+1}$. Since $\mathfrak{B} = \sqrt{-1}\,\mathfrak{A}$ and since the complex structure J on V_0 is induced from that of $\boldsymbol{C}^{n+1}$, we see that the tangent spaces of the $\boldsymbol{C}$-orbits are J-invariant. In fact, each $\boldsymbol{C}$-orbit in V_0 is a complex curve. It is clear also that $\mathfrak{A}$ and $\mathfrak{B}$ are orthogonal to each other with respect to the induced metric.

Let TV_0 be the tangent bundle of V_0, and let A and B be the line subbundles of TV_0 which are generated by $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Next let Σ have the Riemannian metric induced from that of $S(\varepsilon)$ (or V_0), which is the same metric induced from the natural metric of $\boldsymbol{C}^{n+1}$; and let $\boldsymbol{R}$ have the natural metric. Then the tangent boundle $T(\boldsymbol{R} \times \Sigma(\varepsilon))$ of $\boldsymbol{R} \times \Sigma(\varepsilon)$ has the orthogonal direct sum decomposition:

$$T(\boldsymbol{R} \times \Sigma(\varepsilon)) = \widetilde{T\boldsymbol{R}} \oplus \widetilde{T\Sigma}(\varepsilon) ,$$

where $\widetilde{T\Sigma}(\varepsilon)$ is the vector bundle over $\boldsymbol{R} \times \Sigma(\varepsilon)$ which is induced from the tangent bundle $T\Sigma(\varepsilon)$ of $\Sigma(\varepsilon)$ via the natural projection from $\boldsymbol{R} \times \Sigma(\varepsilon)$ onto $\Sigma(\varepsilon)$, and $\widetilde{T\boldsymbol{R}}$ is the vector bundle over $\boldsymbol{R} \times \Sigma(\varepsilon)$ which is induced from the tangent bundle $T\boldsymbol{R}$ of $\boldsymbol{R}$ via the natural projection from $\boldsymbol{R} \times \Sigma(\varepsilon)$ onto $\boldsymbol{R}$. By Lemma 1, d), there is a global diffeomorphism F from

$\boldsymbol{R} \times \Sigma(\varepsilon)$ onto V_0. Therefore, there exists a smooth bundle isomorphism $F_*: T(\boldsymbol{R} \times \Sigma(\varepsilon)) \rightarrow TV_0$, which is nothing but the Jacobian transformation of F; therefore, the following diagram commutes:

$$\begin{array}{ccc} T(\boldsymbol{R} \times \Sigma(\varepsilon)) & \xrightarrow{F_*} & TV_0 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \boldsymbol{R} \times \Sigma(\varepsilon) & \xrightarrow{F} & V_0 . \end{array}$$

Here π_1 and π_2 are the bundle projections of the corresponding tangent bundles.

Let us denote by F^{-1} and F_*^{-1} the inverse mappings of F and F_*, respectively. By the definition of F, F_*^{-1} maps the line subbundle A of TV_0 onto $\widetilde{T}\widetilde{\boldsymbol{R}}$, and the line subbundle B of TV_0 into $\widetilde{T}\widetilde{\Sigma}(\varepsilon)$, respectively. If we denote by $\widetilde{B}$ the line subbundle over $\boldsymbol{R} \times \Sigma(\varepsilon)$ generated by $F_*^{-1}(\mathfrak{B})$, we have the following orthogonal decomposition of $T(\boldsymbol{R} \times \Sigma(\varepsilon))$ with respect to the product Riemannian metric:

$$T(\boldsymbol{R} \times \Sigma(\varepsilon)) = \widetilde{T}\widetilde{\boldsymbol{R}} \oplus \widetilde{T}\widetilde{\Sigma}(\varepsilon) = \widetilde{T}\widetilde{\boldsymbol{R}} \oplus \widetilde{B} \oplus (\widetilde{T}\widetilde{\boldsymbol{R}} \oplus \widetilde{B})^{\perp} .$$

Here the symbol $\perp$ denotes the orthogonal complement. Note that $(\widetilde{T}\widetilde{\boldsymbol{R}} \oplus \widetilde{B})^{\perp}$ is actually the orthogonal complement of $\widetilde{B}$ in $\widetilde{T}\widetilde{\Sigma}(\varepsilon)$.

As before, let k denote the complex dimension of V_0, and let Θ be a complex vector subbundle of TV_0 of complex dimension $k-1$ such that Θ is transversal to $A \oplus B$. This means that $A \oplus B$ and Θ span TV_0 and $(A \oplus B) \cap \Theta = \{0\}$.

LEMMA 2. *Let* $P: T(\boldsymbol{R} \times \Sigma(\varepsilon)) \rightarrow \widetilde{T}\widetilde{\Sigma}(\varepsilon)$ *be the natural orthogonal bundle projection map. Then* $P \circ F_*^{-1}$ *restricted to* Θ *is a bundle isomorphism such that* $P \circ F_*^{-1}(\Theta)$ *is a vector subbundle of* $\widetilde{T}\widetilde{\Sigma}(\varepsilon)$ *of real dimension* $2(k-1)$, *and such that* $P \circ F_*^{-1}(\Theta)$ *is transversal to* $\widetilde{B}$ *in* $\widetilde{T}\widetilde{\Sigma}(\varepsilon)$.

PROOF. A mere verification; and left to the reader.

THEOREM 1. *Let* $\Sigma(\varepsilon)$ *be a generalized Brieskorn manifold.*

a) $\Sigma(\varepsilon)$ *admits almost contact structures.*

b) *Let* Θ *be a complex* $(k-1)$*-dimensional subbundle of* TV_0 *which is transversal to* $A \oplus B$. *Then there is in general a* 1*-parameter family of almost contact structures* $(\phi_\varepsilon(t, \Theta), \xi_\varepsilon(t, \Theta), \eta_\varepsilon(t, \Theta))$, $-\infty < t < \infty$, *on* $\Sigma(\varepsilon)$ *associated to* Θ. *These structures are in general non-regular. If* $q_0, \cdots, q_n$ *are all rational, the associated foliations have closed curves as their leaves.*

PROOF. By Lemma 2, $P\circ F_*^{-1}$ establishes a bundle isomorphism G from Θ onto $P\circ F_*^{-1}(\Theta)$ which we denote by $\tilde{\Theta}$. Clearly, $\tilde{\Theta}$ and $\tilde{B}$ are transversal to each other in $\tilde{T}\tilde{\Sigma}(\varepsilon)$ and span $\tilde{T}\tilde{\Sigma}(\varepsilon)$. Define a bundle homomorphism $\tilde{\phi}_\varepsilon(t, \Theta)$: $\tilde{T}\tilde{\Sigma}(\varepsilon)\rightarrow\tilde{T}\tilde{\Sigma}(\varepsilon)$ as follows:

$$\tilde{\phi}_\varepsilon(t, \Theta)(X) = G\circ J\circ G^{-1}(X) \quad \text{if } X \text{ is a section of } \tilde{\Theta}\,.$$
$$\tilde{\phi}_\varepsilon(t, \Theta)(X) = 0 \quad \text{if } X \text{ is a section of } \tilde{B}\,.$$

Here J denotes the complex structure of Θ which is the induced complex structure from that of TV_0. Now extend $\tilde{\phi}_\varepsilon(t, \Theta)$ linearly to the bundle $\tilde{T}\tilde{\Sigma}(\varepsilon)$. It is clear that the resulting bundle homomorphism $\tilde{\phi}_\varepsilon(t, \Theta)$ is smooth. Next define a smooth section $\tilde{\xi}_\varepsilon(t, \Theta)$ of $\tilde{T}\tilde{\Sigma}(\varepsilon)$ by

$$\tilde{\xi}_\varepsilon(t, \Theta) = P\circ F_*^{-1}(\mathfrak{B})\,.$$

Finally, define a smooth section $\tilde{\eta}_\varepsilon(t, \Theta)$ of Hom $(\tilde{T}\tilde{\Sigma}(\varepsilon), \boldsymbol{R})$ by

$$\tilde{\eta}_\varepsilon(t, \Theta)(X) = 0 \quad \text{if } X \text{ is a section of } \tilde{\Theta}$$

and

$$\tilde{\eta}_\varepsilon(t, \Theta)(\tilde{\xi}_\varepsilon(t, \Theta)) = 1\,.$$

Then we have, for any section X of $\tilde{T}\tilde{\Sigma}(\varepsilon)$,

$$\tilde{\phi}_\varepsilon^2(t, \Theta)(X) = -X + \tilde{\eta}_\varepsilon(t, \Theta)(X)\tilde{\xi}_\varepsilon(t, \Theta)\,.$$

Recall that $\tilde{T}\tilde{\Sigma}(\varepsilon) = \boldsymbol{R}\times T\Sigma(\varepsilon)$, where $T\Sigma(\varepsilon)$ is the tangent bundle of $\Sigma(\varepsilon)$. Let Q: $\tilde{T}\tilde{\Sigma}(\varepsilon)\rightarrow T\Sigma(\varepsilon)$ be the natural projection of $\tilde{T}\tilde{\Sigma}(\varepsilon)$ onto $T\Sigma(\varepsilon)$, and let i_t: $T\Sigma(\varepsilon)\rightarrow\tilde{T}\tilde{\Sigma}(\varepsilon)$ be the natural injection of $T\Sigma(\varepsilon)$ onto $(t, T\Sigma(\varepsilon))$ in $\tilde{T}\tilde{\Sigma}(\varepsilon)$, $-\infty<t<\infty$. Now define $\xi_\varepsilon(t, \Theta)$, $\eta_\varepsilon(t, \Theta)$ and $\phi_\varepsilon(t, \Theta)$ by $Q(\tilde{\xi}_\varepsilon(t, \Theta))$, $\tilde{\eta}_\varepsilon(t, \Theta)\circ i_t$ and $Q\circ\tilde{\phi}_\varepsilon(t, \Theta)\circ i_t$ $(-\infty<t<\infty)$, respectively. Then for any smooth vector field X in $T\Sigma(\varepsilon)$, we have

$$\begin{aligned}\phi_\varepsilon^2(t, \Theta)(X) &= Q\circ\tilde{\phi}_\varepsilon(t, \tilde{\Theta}_\varepsilon)\circ i_t\circ Q\circ\tilde{\phi}_\varepsilon(t, \Theta)\circ i_t(X)\\ &= Q\circ\tilde{\phi}_\varepsilon^2(t, \Theta)\circ i_t(X) = Q(-i_t(X) + \tilde{\eta}_\varepsilon(t, \Theta)(i_t(X))\tilde{\xi}_\varepsilon(t, \Theta))\\ &= -X + \eta_\varepsilon(t, \Theta)(X)\xi_\varepsilon(t, \Theta).\end{aligned}$$

It is clear that $\eta_\varepsilon(t, \Theta)(\xi_\varepsilon(t, \Theta)) = 1$. Thus, the family of triple $(\phi_\varepsilon(t, \Theta)$, $\xi_\varepsilon(t, \Theta)$, $\eta_\varepsilon(t, \Theta))$ satisfy the two conditions to be an almost contact structure.

Next we see that these structures are in general non-regular. First, let us assume that there is at least one leaf of the associated foliation which is not closed. Call it L. Let $(Z_0, \cdots, Z_n)$ be a point of L. Then for $s = 0, 1, 2, \cdots$, $is(Z_0, \cdots, Z_n) = (e^{2\pi q_0 is}Z_0, \cdots, e^{2\pi q_n is}Z_n)$ lies in L. Since $\Sigma(\varepsilon)$ is compact, $\{is(Z_0, \cdots, Z_n)\}_{s=0,1,2,\cdots}$ converges to a point $(\omega_0, \cdots, \omega_n)$ in $\Sigma(\varepsilon)$. Now take any Frobenius coordinates neighborhood around

$(\omega_0, \cdots, \omega_n)$. Then this neighborhood contains more than one slice which belongs to L. In order to see this, it suffices to point out that the $i\boldsymbol{R}$-action preserves the Hermitian product of $\mathbb{C}^{n+1}$, i.e., it is an isometric action; therefore, it induces an isometric action on $\Sigma(\varepsilon)$ with respect to the induced metric on $\Sigma(\varepsilon)$. Such a leaf L as above occurs, except for rather special cases, if $q_0, \cdots, q_n$ contains irrational numbers.

If all the leaves of the associated foliation are closed curves, we can assume except for the above special cases that all $q_0, \cdots, q_n$ are rational. Now consider the induced S^1-action and its slice diagram. It is clear by the slice theorem that if the slice diagram contains more than two different slice types, the foliation is nonregular. The brief discussion of slice diagrams will be given later (see the paragraphs after Theorem 4). For details, see [13] and [19]. Obviously, most of $(n+1)$-tuples $(q_0, \cdots, q_n)$ of rational numbers give rise to more than two different slice types. This completes the proof of Theorem 1. q.e.d.

In Theorem 1 we assumed the existence of complex $(k-1)$-dimensional subbundle Θ. We now give some typical examples of such bundles.

EXAMPLE 4. Let $\Sigma(\varepsilon)$ and V_0 be given as in § 2. As mentioned before, V_0 is a Kählerian submanifold of $\mathbb{C}^{n+1}$ with respect to the induced metric, and $\Sigma(\varepsilon)$ is an orientable Riemannian submanifold of V_0 with codimension 1. Therefore, the normal bundle of $\Sigma(\varepsilon)$ in V_0 is the trivial line bundle over $\Sigma(\varepsilon)$. Let N be a unit normal vector field to $\Sigma(\varepsilon)$. Then JN is a unit tangent field to $\Sigma(\varepsilon)$; therefore, it generates a trivial line subbundle of $T\Sigma(\varepsilon)$. Denote by Θ_ε the orthocomplementary subbundle of $T\Sigma(\varepsilon)$ with respect to the induced Riemannian metric. Making use of Θ_ε, define a subbundle of $T(\boldsymbol{R} \times \Sigma(\varepsilon))$ to be the pullback Θ_ε^* of Θ_ε under the natural projection from $\boldsymbol{R} \times \Sigma(\varepsilon)$ onto the second factor $\Sigma(\varepsilon)$. Map Θ_ε^* into TV_0 under F_*, and denote the image $F_*(\Theta_\varepsilon^*)$ by Θ. Note here that Θ restricted to $\Sigma(\varepsilon)$ is exactly Θ_ε. Now it is easy to see that $F_*(\Theta_\varepsilon^* | (t, \Sigma(\varepsilon)))$, where $\Theta_\varepsilon^* | (t, \Sigma(\varepsilon))$ is the restriction of Θ_ε^* to $(t, \Sigma(\varepsilon))$ at $t(-\infty < t < \infty)$, is the image of Θ_ε under the Jacobian map of t considered as a transformation of the induced $\boldsymbol{R}$-action. Θ_ε is the orthogonal complement of the subbundle generated by N and JN in the restriction of TV_0 to $\Sigma(\varepsilon)$. Since the subbundle generated by N and JN is a complex line bundle, and since V_0 has the induced Kählerian metric, its orthogonal complement Θ_ε is invariant under the complex structure J on V_0, i.e., Θ_ε is a complex bundle. Next we show that Θ_ε is transversal to $A \oplus B$ on $\Sigma(\varepsilon)$. To this end, it suffices to show that the Hermitian inner product between $\mathfrak{A}$ and N is nowhere zero on $\Sigma(\varepsilon)$, because Θ_ε is a complex subbundle of complex codimension 1. Suppose that there is a point in $\Sigma(\varepsilon)$ where the inner product between $\mathfrak{A}$ and N fails to be non-zero. Then $\mathfrak{A}$ must be in the span of Θ_ε and JN. By Lemma 1, $\mathfrak{A}$ is transversal to $S(\varepsilon)$. This is a contradiction. Since $A \oplus B$ is invariant under the $\mathbb{C}$-action and since the $\mathbb{C}$-action is a holomorphic action, we immediately see that Θ is transversal to $A \oplus B$ and J-invariant everywhere in V_0. This Θ is the most important subbundle, and will be used later.

EXAMPLE 5. Let $(A \oplus B)^\perp$ be the orthogonal complement of $A \oplus B$ in TV_0 with respect to the induced Hermitian metric. Then $(A \oplus B)^\perp$ is a complex subbundle of complex $(k-1)$-dimension, and is transversal to $A \oplus B$. This $(A \oplus B)^\perp$ was used earlier to give an example of almost contact structure in [2].

EXAMPLE 6. Let Θ_ε $(0 < \varepsilon < \infty)$ be the complex $(k-1)$-dimensional subbundle of $T\Sigma(\varepsilon)$. Define Θ on V_0 by putting $\Theta = \bigcup_{0<\varepsilon<\infty} \Theta_\varepsilon$. It is not so hard to show that this Θ is a complex $(k-1)$-dimensional subbundle of TV_0 which is transversal to $A \oplus B$.

Erbacher and the author [2] have shown that a broad class of compact manifolds which are given as intersections of complex submanifolds in $\mathbb{C}^{n+1}$ and hyperspheres in $\mathbb{C}^{n+1}$ admit a contact structure. This class contains all the generalized Brieskorn manifolds. In what follows, we show that our generalized Brieskorn manifolds admit a contact structure which is slightly different from those of Erbacher and the author. Our structures, in a natural way, generalize the contact structures of the standard spheres which are given by the Hopf fibration. Indeed, our contact structures possess most of the properties which characterize the Hopf fibrations. These properties will be shown later. First we state existence of contact structures on $\Sigma(\varepsilon)$.

THEOREM 2. *Let $\Sigma(\varepsilon)$ be a generalized Brieskorn manifold. Then there is in general a* 1*-parameter family of normal contact Riemannian structures on $\Sigma(\varepsilon)$. These structures are connected to the structure in* [2] *through a* 1*-parameter family of contact structures. Most of these contact structures are non-regular. If $q_0, \cdots, q_n$ are rationals, the corresponding contact structures have closed curves as their leaves of the associated foliations.*

First of all, we show the following lemmas.

LEMMA 3. *Let $\Sigma(\varepsilon)$ $(0 < \varepsilon < \infty)$ be a generalized Brieskorn manifold associated with V. Then $\Sigma(\varepsilon)$ are diffeomorphic to each other for* *all ε and isotopic in V.*

PROOF. Let ε_1 and ε_2 be two positive numbers in $\boldsymbol{R}$, and let $\Sigma(\varepsilon_1)$ and $\Sigma(\varepsilon_2)$ be the corresponding generalized Brieskorn manifolds, i.e., $\Sigma(\varepsilon_1) = V \cap S(\varepsilon_1)$ and $\Sigma(\varepsilon_2) = V \cap S(\varepsilon_2)$. We can assume $\varepsilon_1 < \varepsilon_2$ without loss of generality. We define a mapping $h(\varepsilon_1, \varepsilon_2)$ from $\Sigma(\varepsilon_1)$ onto $\Sigma(\varepsilon_2)$ as follows. Let $(Z_0, \cdots, Z_n) = Z$ be a point in $\Sigma(\varepsilon_1)$. Consider the orbit of Z under the induced $\boldsymbol{R}$-action on V. The orbit meets $\Sigma(\varepsilon_2)$ once and only once at $w = (w_0, \cdots, w_n)$ by Lemma 1, d). Define $h(\varepsilon_1, \varepsilon_2)(Z) = w$ for all $Z \in \Sigma(\varepsilon_1)$, i.e., $w = h(\varepsilon_1, \varepsilon_2)(Z) = t(Z)$, where $t \in \boldsymbol{R}$ does depend upon Z. This mapping is clearly one to one and onto. Next we show that $h(\varepsilon_1, \varepsilon_2)$ is a diffeomorphism. Consider the foliation on V_0 generated by the $\boldsymbol{R}$-action, whose leaves are nothing but the $\boldsymbol{R}$-orbits. By the argument given in the proof of d), Lemma 1, it is obvious that this foliation is regular. Let $w = (w_0, \cdots, w_n)$ be a point in $\Sigma(\varepsilon_2)$ and let $(y_1, \cdots, y_{2k-1})$ be local coordinates in a neighborhood W around ω. By making use of the diffeomorphism F in Lemma 1, we know that $F|\boldsymbol{R} \times W$ gives rise to a Frobenius local coordinate system in the neighborhood $\boldsymbol{R} \times W$ of ω in V_0, which we denote by $(t, y_1, \cdots, y_{2k-1})$. Now let Z be a point of $\Sigma(\varepsilon_1)$ which is mapped into ω under $h(\varepsilon_1, \varepsilon_2)$, and let $((x_1, \cdots, x_{2k-1}), U)$ be a local coordinate system in a neighborhood U of Z in $\Sigma(\varepsilon_1)$. By taking U sufficiently small, we can consider U as a regular submanifold of $\boldsymbol{R} \times W$. Denote by P the natural projection of $\boldsymbol{R} \times W$ onto W, i.e., $P(t, y_1, \cdots, y_{2k-1}) = (y_1, \cdots, y_{2k-1})$. P is then a smooth map, and P restricted to the submanifold U is precisely $h(\varepsilon_1, \varepsilon_2)$ restricted to U by the definition of $h(\varepsilon_1, \varepsilon_2)$. By Lemma 1, the tangent space of U at Z is transversal to the orbit passing through Z, which is the first coordinate axis. Therefore, the Jacobian map of P maps isomorphically the tangent space of U at Z onto the tangent space of W at ω, which is nothing but the coordinate space $(y_1, \cdots, y_{2k-1})$. Thus, the Jacobian map of P restricted to U at Z is an isomorphism between the tangent space of U at Z and the tangent space of W at ω. Now by the inverse function theorem, P restricted to U is a local diffeomorphism, i.e., $h(\varepsilon_1, \varepsilon_2)$ is a local diffeomorphism. We showed earlier that $h(\varepsilon_1, \varepsilon_2)$ is one to one and onto, so $h(\varepsilon_1, \varepsilon_2)$ is a global diffeomorphism between $\Sigma(\varepsilon_1)$ and $\Sigma(\varepsilon_2)$. q.e.d.

Let $Z = (Z_0, \cdots, Z_n)$ be a point in $\Sigma(\varepsilon)$ and denote by $\mathfrak{A} = (2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n)$ the velocity of the induced $\boldsymbol{R}$-action on V_0, and by $\mathfrak{B}$ the velocity vectors of the induced $i\boldsymbol{R}$-action on $\Sigma(\varepsilon)$. We know that $J\mathfrak{A} = J(2\pi q_0 Z_0, \cdots, 2\pi q_n Z_n) = (2\pi i q_0 Z_0, \cdots, 2\pi i q_n Z_n) = \mathfrak{B}$, where J is the induced complex structure of V_0. Define a 1-form η_ε on $\Sigma(\varepsilon)$ as follows. Let Θ be the complex vector subbundle of TV_0 (or $\tilde{T}\tilde{\Sigma}(\varepsilon)$) given in Example 4, and let Θ_ε be its restriction to $\Sigma(\varepsilon)$. Then we know that Θ_ε is a J-invariant subbundle of $T\Sigma(\varepsilon)$, and $\mathfrak{V}$ is transversal to Θ_ε. Set $\eta_\varepsilon(\mathfrak{V}) = 1$ and $\eta_\varepsilon(\Theta_\varepsilon) = 0$. Then clearly η_ε is a C^∞ 1-form on $\Sigma(\varepsilon)$. In the sequel, Θ_ε will sometimes denote also the space of all cross sections of Θ_ε.

LEMMA 4. *Let us denote by $L_{\mathfrak{V}}$ the Lie derivative in $\mathfrak{V}$ direction on $\Sigma(\varepsilon)$. Then $L_{\mathfrak{V}}(\Theta_\varepsilon) \subset \Theta_\varepsilon$,* i.e., *$\Theta_\varepsilon$ is invariant under the Lie derivative.*

PROOF. Let Z be any point of $\Sigma(\varepsilon)$. Then the fiber of Θ_ε over Z is the only $2(k-1)$-dimensional subspace of the tangent space of $\Sigma(\varepsilon)$ at Z which is invariant under J. This can be easily seen by noticing that the fiber of Θ_ε at Z has real codimension 1 in the tangent space of $\Sigma(\varepsilon)$ at Z, and that its orthogonal complement with respect to the induced metric has its image under J outside $T\Sigma(\varepsilon)$. Since $\mathfrak{V}$ is the velocity vector fields of the $i\boldsymbol{R}$-action on $\Sigma(\varepsilon)$ induced from the $\boldsymbol{C}$-action on V_0, the local (global) transformations generated by $\mathfrak{V}$ are nothing but the transformations which belong to the $\boldsymbol{C}$-action ($i\boldsymbol{R}$-action). Since the $i\boldsymbol{R}$-action (or $\boldsymbol{C}$-action) is a holomorphic action, each element of $i\boldsymbol{R}$ is a holomorphic mapping of V_0; and therefore, it leaves Θ_ε invariant. This fact can be seen by noting that Θ_ε is the only J-invariant $2(k-1)$-dimensional subbundle of $T\Sigma(\varepsilon)$ and that the $i\boldsymbol{R}$-action leaves $T\Sigma(\varepsilon)$ invariant. Let us denote by ih $(-\infty < h < \infty)$ the global transformations generated by $\mathfrak{V}$ and let $(ih)_*$ denote their Jacobian maps. Then by the definition of the Lie derivative [15], for any vector field X in Θ_ε,

$$L_{\mathfrak{V}}X = \lim_{h\to 0} \frac{X - (i(-h))_* X}{h} .$$

Thus $L_{\mathfrak{V}}X$ is again in Θ_ε for X in Θ_ε. q.e.d.

As before, let $r(Z) = b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon^2$ $(b_0 > 0, \cdots, b_n > 0)$ give an ellipsoid $S(\varepsilon)$, and let $\operatorname{grad} r(Z)$ denote the gradient of $r(Z)$ at $r(Z) = \varepsilon^2$. Denote by $\langle \ \ \rangle$ and $\|\ \ \|$ the natural Hermitian product and its norm of $\boldsymbol{C}^{n+1}$. Finally, ∇ denotes the Riemannian connection of $\boldsymbol{C}^{n+1}$, and α denotes the second fundamental form of $S(\varepsilon)$ in $\boldsymbol{C}^{n+1}$.

LEMMA 5. *Let η denote η_ε as before,* i.e., *$\eta(\mathfrak{V}) = 1$ and $\eta(\Theta_\varepsilon) = 0$, where Θ_ε is given in Example 4. Then for any X and $Y \in \Theta_\varepsilon$,*

a) $2d\eta(\mathfrak{V}, X) = 0$.

b) $2d\eta(X, Y) = 1/\omega(\mathfrak{V})\langle \alpha(JY, X) - \alpha(JX, Y), N\rangle$, *where $\omega(\mathfrak{V})$ and N will be given below.*

PROOF. a) $2d\eta(\mathfrak{B}, X) = \mathfrak{B}\eta(X) - X\eta(\mathfrak{B}) - \eta([\mathfrak{B}, X]) = -\eta(L_{\mathfrak{B}}X) = 0$, since $\eta(X) = 0$, $\eta(\mathfrak{B}) = 1$ and $L_{\mathfrak{B}}X \in \Theta_\varepsilon$, i.e., $\eta(L_{\mathfrak{B}}X) = 0$ by Lemma 4.

b) Let N be the normalized gradient of $r(Z)$ at Z, i.e., $N = \operatorname{grad} r(Z)/\|\operatorname{grad} r(Z)\|$, where $\operatorname{grad} r(Z) = (2b_0Z_0, \cdots, 2b_nZ_n)$, and therefore, $\|\operatorname{grad} r(Z)\| = \sqrt{4(b_0^2|Z_0|^2 + \cdots + b_n^2|Z_n|^2)} > 0$ everywhere. Note here that N is a unit normal vector field to $S(\varepsilon)$. Define a new 1-form ω on $\Sigma(\varepsilon)$ by $\omega(X) = \langle X, JN\rangle$ for all $X \in T\Sigma(\varepsilon)$, where J is the complex structure of $\mathbb{C}^{n+1}$. Then for any $X \in \Theta_\varepsilon$, $\omega(X) = \langle X, JN\rangle = 0$, since Θ_ε is J-invariant and N is orthogonal to Θ_ε; therefore, JN is orthogonal to Θ_ε. For any $Z = (Z_0, \cdots, Z_n) \in \Sigma(\varepsilon)$, we have

$$\begin{aligned}\omega(\mathfrak{B}) &= \operatorname{Re}\langle(2\pi q_0 iZ_0, \cdots, 2\pi q_n iZ_n), JN\rangle \\ &= \operatorname{Re}\left\langle(2\pi q_0 iZ_0, \cdots, 2\pi q_n iZ_n), \frac{(2b_0iZ_0, \cdots, 2b_niZ_n)}{\|\operatorname{grad} r(Z)\|}\right\rangle \\ &= \frac{2\pi(q_0b_0|Z_0|^2 + \cdots + q_nb_n|Z_n|^2)}{\sqrt{b_0^2|Z_0|^2 + \cdots + b_n^2|Z_n|^2}} > 0\,.\end{aligned}$$

Thus $\omega(X) = \omega(\mathfrak{B})\eta(X)$ for all $X \in T\Sigma(\varepsilon)$, i.e., $\omega = \omega(\mathfrak{B})_\eta$ or $\eta = (1/\omega(\mathfrak{B}))\omega$. For any X and Y in Θ_ε, we have

$$\begin{aligned}2d\eta(X, Y) &= X\eta(Y) - Y\eta(X) - \eta([X, Y]) = -\eta([X, Y]) \\ &= -\frac{1}{\omega(\mathfrak{B})}\omega([X, Y]) \\ &= -\frac{1}{\omega(\mathfrak{B})}\langle[X, Y], JN\rangle = -\frac{1}{\omega(\mathfrak{B})}\langle\nabla_XY - \nabla_YX, JN\rangle \\ &= -\frac{1}{\omega(\mathfrak{B})}\{(X\langle Y, JN\rangle - \langle Y, \nabla_XJN\rangle) - (Y\langle X, JN\rangle - \langle X, \nabla_YJN\rangle)\} \\ &= \frac{1}{\omega(\mathfrak{B})}\{\langle Y, \nabla_XJN\rangle - \langle X, \nabla_YJN\rangle\}\,.\end{aligned}$$

Noting that ∇ is a Kählerian connection as well, we have $\nabla_XJN = J\nabla_XN$ and $\nabla_YJN = J\nabla_YN$. Therefore, the last expression $= (1/\omega(\mathfrak{B}))\{\langle Y, J\nabla_XN\rangle - \langle X, J\nabla_YN\rangle\} = (1/\omega(\mathfrak{B}))\{-\langle JY, \nabla_XN\rangle + \langle JX, \nabla_YN\rangle\} = (1/\omega(\mathfrak{B}))(\langle\alpha(JY, X), N\rangle - \langle\alpha(JX, Y), N\rangle)$. The last equality fallows from the relation between the second fundamental form and the shape operators. q.e.d.

The following lemma is, in a way, well known.

LEMMA 6. *Let $S(\varepsilon)$ be the ellipsoid in $\mathbb{C}^{n+1}$ given by the equation $r(Z) = b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon^2$. Then the second fundamental form is (strictly) negative definite with respect to N. If we let $X = (x_0, \cdots, x_n, x_0^*, \cdots, x_n^*)$ and $Y = (y_0, \cdots, y_n, y_0^*, \cdots, y_n^*)$, then*

$$\alpha(X, Y) = -\frac{b_0x_0y_0 + \cdots + b_nx_ny_n + b_0x_0^*y_0^* + \cdots + b_nx_n^*y_n^*}{\sqrt{b_0^2|Z_0|^2 + \cdots + b_n^2|Z_n|^2}} \cdot N .$$

PROOF OF THEOREM 2. First of all, we show that $\eta\Lambda(d\eta)^{n-1} \neq 0$ everywhere. Let us set $\beta(X, Y) = -\langle\alpha(X, Y), N\rangle$ for all X and Y in $TS(\varepsilon)$. By Lemma 6, β is strictly positive definite and symmetric. Thus β gives rise to an inner product of $TS(\varepsilon)$. Next let X and Y be two vectors in Θ_ε. Since Θ_ε is J-invariant, JX and JY are in Θ_ε. We show that $\beta(JX, JY) = \beta(X, Y)$ in Θ_ε. As before, let $X = (x_0, \cdots, x_n, x_0^*, \cdots, x_n^*)$ and $Y = (y_0, \cdots, y_n, y_0^*, \cdots, y_n^*)$. Then $JX = (-x_0^*, \cdots, -x_n^*, x_0, \cdots, x_n)$ and $JY = (-y_0^*, \cdots, -y_n^*, y_0, \cdots, y_n)$.

$$\begin{aligned}\beta(JX, JY) &= -\langle\alpha(JX, JY), N\rangle \\ &= \frac{(2b_0x_0^*y_0^* + \cdots + 2b_nx_n^*y_n^* + 2b_0x_0y_0 + \cdots + 2b_nx_ny_n)}{2\sqrt{b_0^2|Z_0|^2 + \cdots + b_n^2|Z_n|^2}} \\ &= \beta(X, Y) .\end{aligned}$$

This tells us that β restricted to Θ_ε is a Hermitian metric with respect to the induced metric. It is well known, then, that there exists an orthonormal basis for Θ_ε of the form $\{X_1, \cdots, X_{n-1}, JX_1, \cdots, JX_{n-1}\}$ with respect to β at every point of $S(\varepsilon)$. For the sake of convenience, let us denote $\mathfrak{B} = e_0$, $X_1 = e_1, \cdots, X_{n-1} = e_{n-1}$, $JX_1 = e_n, \cdots, JX_{n-1} = e_{2n-2}$. Then $\{e_0, e_1, \cdots, e_{2n-2}\}$ forms a basis for $TS(\varepsilon)$ at the point Z. Up to a positive constant k,

$$\begin{aligned}\eta\Lambda(d\eta)^{n-1}(\mathfrak{B}, X_1, \cdots, X_{n-1}, JX_1, \cdots, JX_{n-1}) &= \eta\Lambda(d\eta)^{n-1}(e_0, \cdots, e_{2n-2}) \\ &= k\sum_{\sigma\in\mathfrak{S}}(\operatorname{sgn}\sigma)\eta(e_{\sigma(0)})d\eta(e_{\sigma(1)}, e_{\sigma(2)}) \cdots d\eta(e_{\sigma(2n-3)}, e_{\sigma(2n-2)}) .\end{aligned}$$

Here $\mathfrak{S}$ is the symmetric group of letters $\{0, 1, \cdots, 2n-2\}$, and $\operatorname{sgn}\sigma = 1$ if σ is an even permutation, and $\operatorname{sgn}\sigma = -1$ if σ is an odd permutation. By the definition of η, $\eta(e_i) = 0$ for $1 \leqq i \leqq 2n-2$, and $\eta(e_0) = \eta(\mathfrak{B}) = 1$. Therefore, $\eta(e_{\sigma(0)})d\eta(e_{\sigma(1)}, e_{\sigma(2)}) \cdots d\eta(e_{\sigma(2n-3)}, e_{\sigma(2n-2)}) \neq 0$ only if $\sigma(0) = 0$, and it equals $d\eta(e_{\sigma(1)}, e_{\sigma(2)}) \cdots d\eta(e_{\sigma(2n-3)}, e_{\sigma(2n-2)})$. By Lemma 5,

$$\begin{aligned}d\eta(e_i, e_j) &= \frac{1}{\omega(\mathfrak{B})}(\langle\alpha(Je_j, e_i), N\rangle - \langle\alpha(Je_i, e_j), N\rangle) \\ &= \frac{-1}{\omega(\mathfrak{B})}(\beta(Je_j, e_i) - \beta(Je_i, e_j)) .\end{aligned}$$

Therefore, $d\eta(e_i, e_j) = 0$ unless $e_i = Je_j$ or $e_j = Je_i$ by the choice of $e_1, \cdots, e_{2n-2}$. Now let τ be a permutation such that

$$(\operatorname{sgn}\tau)\eta(e_{\tau(0)})d\eta(e_{\tau(1)}, e_{\tau(2)}) \cdots d\eta(e_{\tau(2n-3)}, e_{\tau(2n-2)}) \neq 0 ,$$

i.e.,

$$\tau = \begin{pmatrix} 0, 1, 2, \cdots, 2n-1 \\ 0, \tau(1), \tau(2), \cdots, \tau(2n-2) \end{pmatrix} \quad \text{where} \quad e_{\tau(2i-1)} = \pm Je_{\tau(2i)} , \quad i = 1, \cdots, (n-1) .$$

Note here that

$$d\eta(e_i, Je_i) = \frac{-1}{\omega(\mathfrak{B})}(\beta(-e_i, e_i) - \beta(Je_i, Je_i)) = \frac{+2}{\omega(\mathfrak{B})}\beta(e_i, e_i) = \frac{+2}{\omega(\mathfrak{B})}$$

and $d\eta(e_i, Je_i) = -d\eta(Je_i, e_i)$. Therefore, if we denote by μ the permutation τ followed by the transposition of $e_{\tau(2i-1)}$ and $e_{\tau(2i)}$ $(1 \leqq i \leqq n-1)$, $\operatorname{sgn}\tau = -\operatorname{sgn}\mu$ and

$$\begin{aligned}
&\operatorname{sgn}\tau d\eta(e_{\tau(1)}, e_{\tau(2)}) \cdots d\eta(e_{\tau(2i-1)}, e_{\tau(2i)}) \cdots d\eta(e_{\tau(2n-3)}, e_{\tau(2n-2)}) \\
&\quad = -\operatorname{sgn}\tau d\eta(e_{\tau(1)}, e_{\tau(2)}) \cdots d\eta(e_{\tau(2i)}, e_{\tau(2i-1)}) \cdots d\eta(e_{\tau(2n-3)}, e_{\tau(2n-2)}) \\
&\quad = \operatorname{sgn}\mu d\eta(e_{\mu(1)}, e_{\mu(2)}) \cdots d\eta(e_{\mu(2i-1)}, e_{\mu(2i)}) \cdots d\eta(e_{\mu(2n-3)}, e_{\mu(2n-2)}) .
\end{aligned}$$

Next let ρ be the permutation τ followed by two transpositions between $\tau(2i-1)$ and $\tau(2j-1)$ and between $\tau(2i)$ and $\tau(2j)$ for $i < j$, i.e.,

$$\rho = \begin{pmatrix} 0 & 1 & \cdots & (2i-1) & 2i & \cdots & (2j-1) & 2j & \cdots & 2n-2 \\ 0 & \tau(1) & \cdots & \tau(2j-1) & \tau(2j) & \cdots & \tau(2i-1) & \tau(2i) & \cdots & \tau(2n-2) \end{pmatrix} .$$

Then $\operatorname{sgn}\tau = \operatorname{sgn}\rho$ and

$$d\eta(e_{\tau(1)}, e_{\tau(2)}) \cdots d\eta(e_{\tau(2n-3)}, e_{\tau(2n-2)}) = d\eta(e_{\rho(1)}, e_{\rho(2)}) \cdots d\eta(e_{\rho(2n-3)}, e_{\rho(2n-2)}) .$$

Thus, by these two observations, we can conclude that

$$\begin{aligned}
&(\operatorname{sgn}\tau)\eta(e_{\tau(0)})d\eta(e_{\tau(1)}, e_{\tau(2)}) \cdots d\eta(e_{\tau(2n-3)}, e_{\tau(2n-2)}) \\
&\quad = (-1)^{(n-1)(n-2)/2}\eta(e_0)d\eta(e_1, e_n) \cdots d\eta(e_{n-1}, e_{2n-2}) \\
&\quad = (-1)^{(n-1)(n-2)/2}\eta(\mathfrak{B})d\eta(X_1, JX_1) \cdots d\eta(X_{n-1}, JX_{n-1})
\end{aligned}$$

for all τ such as described above. Therefore, up to a non-zero constant $\bar{k}$,

$$\begin{aligned}
&\eta \Lambda (d\eta)^{n-1}(\mathfrak{B}, X_1, \cdots, X_{n-1}, JX_1, \cdots, JX_{n-1}) \\
&\quad = \bar{k}\eta(\mathfrak{B})d\eta(X_1, JX_1) \cdots d\eta(X_{n-1}, JX_{n-1}) \\
&\quad = \bar{k}\left(\frac{+2}{\omega(\mathfrak{B})}\right)^{n-1} \neq 0 .
\end{aligned}$$

Hence, η is a contact form on $\Sigma(\varepsilon)$.

Our next aim is to show that η is normal. To this end, we show that there is a Riemannian metric on $\Sigma(\varepsilon)$ with which the almost contact structure $(\phi_\varepsilon(0, \Theta), \xi_\varepsilon(0, \Theta), \eta_\varepsilon(0, \Theta))$ in Theorem 1, which is associated with the vector subbundle of Example 4, is exactly the associated almost contact Riemannian structure on $\Sigma(\varepsilon)$. By the definition of β as above and by Lemma 5, b), for any X and $Y \in \Theta_\varepsilon$,

$$d\eta(X, Y) = \frac{1}{2}\frac{1}{\omega(\mathfrak{B})}(\langle\alpha(JY, X), N\rangle - \langle\alpha(JX, Y), N\rangle)$$
$$= \frac{1}{2\omega(\mathfrak{B})}(-\beta(JY, X) + \beta(JX, Y)$$
$$= \frac{1}{\omega(\mathfrak{B})}\beta(JX, Y)\,, \quad \text{since } \beta \text{ is Hermitian in } \Theta_\varepsilon.$$

As we know,

$$\omega(\mathfrak{B}) = \frac{2\pi(q_0b_0|Z_0|^2 + \cdots + q_nb_n|Z_n|^2)}{\sqrt{b_0^2|Z_0|^2 + \cdots + b_n^2|Z_n|^2}} > 0\,.$$

Now define an inner product g in $T\Sigma(\varepsilon) = B \oplus \Theta_\varepsilon$ as follows.

$$g(\mathfrak{B}, \mathfrak{B}) = \eta(\mathfrak{B}) = 1\,.$$
$$g(\mathfrak{B}, X) = g(X, \mathfrak{B}) = 0 \quad \text{for all } X \in \Theta_\varepsilon\,.$$
$$g(X, Y) = \frac{1}{\omega(\mathfrak{B})}\beta(X, Y) \quad \text{for all } X \text{ and } Y \in \Theta_\varepsilon\,.$$

It is clear that the above g extends linearly on $T\Sigma(\varepsilon)$, and it is smooth. If we define a type (1, 1) tensor ϕ on $T\Sigma(\varepsilon)$ by $d\eta(X, Y) = g(\phi X, Y)$, then $\phi X = JX$ on Θ_ε and $\phi\mathfrak{B} = 0$ by the above definition of g and Lemma 5, a). Also $\eta(X) = g(\mathfrak{B}, X)$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ are clear from the definition of g. Hence, $(\phi, \mathfrak{B}, \eta, g)$ is an associated almost contact Riemannian structure, and $(\phi, \mathfrak{B}, \eta)$ coincides with $(\phi(\Theta, \varepsilon), \xi(\Theta, \varepsilon), \eta(\Theta, \varepsilon))$, as is mentioned above.

The contact structure η being normal will be shown via the following convenient lemma. Let (ϕ, ξ, η) be an almost contact structure on M of odd dimension. Then $\boldsymbol{R} \times M$ admits an almost complex structure $\tilde{J}$ naturally induced from (ϕ, ξ, η) in the following sense. Let d/dt be the unit coordinate vector field of $\boldsymbol{R}$. Define $\tilde{J}$ by

$$\tilde{J}X = \phi X \quad \text{if} \quad X \in T\Sigma(\varepsilon) \quad \text{and} \quad \eta(X) = 0$$
$$\tilde{J}\xi = -\frac{d}{dt}\,, \quad \text{and} \quad \tilde{J}\left(\frac{d}{dt}\right) = \xi\,.$$

It is easy to see that $(\tilde{J})^2 = -I$; therefore, $\tilde{J}$ is an almost complex structure on $\boldsymbol{R} \times \Sigma(\varepsilon)$.

LEMMA 7. *The almost complex structure $\tilde{J}$ on $\boldsymbol{R} \times M$ reduces to a complex structure if and only if (ϕ, ξ, η) on M is normal.*

PROOF. See [22].

Next we will show that $\tilde{J}$ on $\boldsymbol{R} \times \Sigma(\varepsilon)$ induced from $(\phi, \mathfrak{B}, \eta)$ is a complex structure.

According to d) of Lemma 1, $F: \boldsymbol{R} \times \Sigma(\varepsilon) \to V_0$ is a diffeomorphism. From the construction of Θ in Example 4, it is clear that $F_* \circ \tilde{J} = J \circ F_*$, where F_* is the Jacobian mapping of F from $T(\boldsymbol{R} \times \Sigma(\varepsilon))$ onto TV_0, and J is the induced complex structure on V_0. The torsion tensor $T\tilde{J}$ of $\tilde{J}$ on $\boldsymbol{R} \times \Sigma(\varepsilon)$ is given as, for any X and Y in $T(\boldsymbol{R} \times \Sigma(\varepsilon))$,

$$\begin{aligned} T\tilde{J}(X, Y) &= [X, Y] + \tilde{J}[\tilde{J}X, Y] + \tilde{J}[X, \tilde{J}Y] - [\tilde{J}X, \tilde{J}Y] \\ &= F_*^{-1} \circ F_*([X, Y] + \tilde{J}[\tilde{J}X, Y] + \tilde{J}[X, \tilde{J}Y] - [\tilde{J}X, \tilde{J}Y]) \\ &= F_*^{-1}(TJ(F_*X, F_*Y)) = 0 , \end{aligned}$$

since the torsion of J on $V_0 = TJ = 0$. Thus $\tilde{J}$ is a complex structure on $\boldsymbol{R} \times \Sigma(\varepsilon)$; therefore, η on $\Sigma(\varepsilon)$ is normal. The structure η on $\Sigma(\varepsilon)$ is usually non-regular. As we have seen, the associated vector field of η is $\mathfrak{B}$ which is the velocity vector field of $i\boldsymbol{R}$-action on $\Sigma(\varepsilon)$. As in the proof of Theorem 1, we can show that if $q_0, \cdots, q_n$ are all rational, the associated foliation has closed curves as its leaves.

Finally, for any δ $(0 < \delta < \infty)$, let η_δ denote the normal contact structure on $\Sigma(\delta)$. By Lemma 3, $\Sigma(\varepsilon)$ is diffeomorphic to $\Sigma(\delta)$. Let $h(\varepsilon, \delta)$ be the diffeomorphic between them. Define $\eta(\delta)$ $(0 < \delta < \infty)$ on $\Sigma(\varepsilon)$ as follows.

$$\eta(\delta) = h^*(\varepsilon, \delta)\eta_\delta \qquad \text{if} \quad \varepsilon < \delta$$

and

$$\eta(\delta) = (h^{-1}(\delta, \varepsilon))^*\eta_\delta \quad \text{if} \quad \delta < \varepsilon ,$$

where the superscript $*$ denotes the pullback of the forms. Clearly, $\eta(\delta)$ $(0 < \delta < \infty)$ is the desired 1-parameter family in Theorem 2.

REMARK 1. The form ω in the proof of Lemma 5 is a contact form, which coincides with the contact form of Erbacher-Author [2]. Recently, Hsu and Sasaki [23] have constructed a contact form on Brieskorn manifolds. Their method is quite different from ours; however, the form itself coincides with our ω on original Brieskorn manifolds.

This contact form ω is actually connected to our η through a 1-parameter family of contact forms. To see this, put $\omega(t) = (1 - t)\eta + t\omega$ for $0 \leqq t \leqq 1$. Then it is easy to see that $\omega(t) \wedge (d\omega(t))^{n-1}$ $(\mathfrak{B}, X_1, \cdots, X_{2(n-1)}) \neq 0$ for all t, where $X_1, \cdots, X_{2(n-1)}$ are vectors in Θ_ε as before. Therefore, $\omega(t)$ is a 1-parameter family of contact forms such that $\omega(0) = \eta$ and $\omega(1) = \omega$. This completes the proof of Theorem 2. q.e.d.

LEMMA 8. *Let V be an irreducible analytic subvariety of $\boldsymbol{C}^{n+1}$ which has the origin as only possible singular point. Let $b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon^2$ and $\bar{b}_0|Z_0|^2 + \cdots + \bar{b}_n|Z_n|^2 = \bar{\varepsilon}^2$ be two ellipsoids in*

$\mathbb{C}^{n+1}$. *Then* $\Sigma(\bar{\varepsilon}) = V \cap S(\bar{\varepsilon})$ *and* $\Sigma(\varepsilon) = V \cap S(\varepsilon)$ *are diffeomorphic to each other, and isotopic in* V.

PROOF. Consider the family of ellipsoids in $\mathbb{C}^{n+1}$ given by, for $0 \leqq t \leqq 1$,

$$((1-t)b_0 + t\bar{b}_0)|Z_0|^2 + \cdots + ((1-t)b_n + t\bar{b}_n)|Z_n|^2 = (1-t)\varepsilon^2 + t\bar{\varepsilon}^2 .$$

The rest of the proof follows from the argument used to prove Lemma 3. q.e.d.

LEMMA 9. *Let* $f_1(Z, t), \cdots, f_m(Z, t)$ $(t \in \boldsymbol{R})$ *be* m $(m \leqq n+1)$ *1-parameter families of holomorphic functions of variables* $Z_0, \cdots, Z_n$. *Assume that* $f_1(Z, t), \cdots, f_m(Z, t)$ *define an irreducible subvariety for all* t *such that the origin of* $\mathbb{C}^{n+1}$ *is the only possible singular point. Let* $S(t, \varepsilon)$ *be the 1-parameter family of ellipsoids defined by* $g_0(t)|Z_0|^2 + \cdots + g_n(t)|Z_n|^2 - \varepsilon^2 = 0$ *for* $t \in \boldsymbol{R}$, *where* $g_i(t) > 0$ *for* $0 \leqq i \leqq n$, *and let* $\Sigma(t, \varepsilon)$ *be the corresponding generalized Brieskorn manifolds for* $t \in \boldsymbol{R}$. *Then* $\Sigma(t, \varepsilon)$ *are diffeomorphic to each other for all* $t \in \boldsymbol{R}$, *and they are isotopic in* $\mathbb{C}^{n+1}$.

PROOF. The agument given in [11] works in this case. The proof is left to the reader.

The following are some examples for Lemma 9.

EXAMPLE 7. Let $f(Z, t) = \alpha_0(t)Z_0^{a_0} + \cdots + \alpha_n(t)Z_n^{a_n}$, where $\alpha_i(t) = (1-t) + t\alpha_i$ $(0 \leqq i \leqq n)$, α_i $(0 \leqq i \leqq n) > 0$ and a_i $(0 \leqq i \leqq n)$ are positive integers. Let $S(t, \varepsilon)$ be the ellipsoids defined by $g_0(t)|Z_0|^2 + \cdots + g_n(t)|Z_n|^2 - \varepsilon^2 = 0$, where $g_i(t) = (1-t) + tg_i$ and $g_i > 0$ for all $0 \leqq i \leqq n$. Then $\Sigma(0, \varepsilon)$ is the original Brieskorn manifold with the polynomial $f(Z, 0) = Z_0^{a_0} + \cdots + Z_n^{a_n}$ and the sphere defined by $|Z_0|^2 + \cdots + |Z_n|^2 = \varepsilon^2$; and $\Sigma(1, \varepsilon)$ is the generalized Brieskorn manifold associated with $f(Z, 1)$ and the ellipsoid $g_0|Z_0|^2 + \cdots + g_n|Z_n|^2 = \varepsilon^2$. The same kind of deformations can be constructed for the generalized Brieskorn manifolds.

LEMMA 10. *Let* $V(t)$ *be a 1-parameter family of irreducible varieties and let* $S(t)$ *be a 1-parameter family of ellipsoids. Then the contact forms* $\omega(t)$ *on* $\Sigma(t)$ *introduced in the proof of Theorem 2 form a 1-parameter family. Here* $V(t)$ *is invariant under a fixed natural* $\mathbb{C}$-*action on* C^{n+1}.

PROOF. Let us denote by $N(t)$ the normalized gradients of those ellipsoids. Then $\omega(t)(X(t)) = \langle X(t), JN(t)\rangle$ for all $t \in \boldsymbol{R}$, where $X(t)$ is tangent vector to $\Sigma(t)$ and J is the complex structure of $\mathbb{C}^{n+1}$. It is evident that $\omega(t)$ form a 1-parameter family from the expression. q.e.d.

THEOREM 3. *Let $f_i(Z, t)$ $(1 \leqq i \leqq m)$ be given as in Lemma 9, and let $V(t)$ be the corresponding irreducible varieties. Furthermore, assume that there is a 1-parameter family of $\mathbf{C}$-actions on $V(t)$ of the form $(Z_0, \cdots, Z_n) \mapsto (e^{2\pi q_0(t)s}Z_0, \cdots, e^{2\pi q_n(t)s}Z_n)$ $(s \in \mathbf{C})$, where $q_i(t)$ $(0 \leqq i \leqq n)$ is a 1-parameter family of positive real numbers. For any $t_0 \in \boldsymbol{R}$, let $S(t_0, \varepsilon)$ be an ellipsoid given by the equation $b_0|Z_0|^2 + \cdots + b_n|Z_n|^2 = \varepsilon$, and let $S(0, 1)$ be the unit sphere defined by the equation $|Z_0|^2 + \cdots + |Z_n|^2 = 1$. Then the contact forms $\eta(t_0, \varepsilon)$ and $\eta(0, 1)$ on $\Sigma(t_0, \varepsilon) = S(t_0, \varepsilon) \cap V(t_0)$ and $\Sigma(0, 1) = S(0, 1) \cap V(0)$ constructed in Theorem 2 are connected by a 1-parameter family of normal contact forms on $\Sigma(0, 1)$ up to diffeomorphisms.*

PROOF. First, we define a 1-parameter family of ellipsoids connecting $S(0, 1)$ and $S(t_0, \varepsilon)$. Consider the 1-parameter family of equations given by

$$((1 - t) + tb_0)|Z_0|^2 + \cdots + ((1 - t) + tb_n)|Z_n|^2 - \varepsilon^2 = 0 \quad \text{for} \quad 0 \leqq t \leqq 1 .$$

Clearly the ellipsoids $S(t)$ defined by these equations form a 1-parameter family connecting $S(0, \varepsilon)$ and $S(t_0, \varepsilon)$. Denote by $\Sigma(t)$ the corresponding generalized Brieskorn manifold $\Sigma(t) = S(t) \cap V(t, t_0)$ for $0 \leqq t \leqq 1$. Then by Lemma 9, $\Sigma(t)$ is diffeomorphic to $\Sigma(0, \varepsilon) = \Sigma(0)$. Now by Lemma 3, $\Sigma(0, \varepsilon) = \Sigma(0)$ is diffeomorphic to $\Sigma(0, 1)$. Denote this composition of diffeomorphism from $\Sigma(t)$ onto $\Sigma(0, 1)$ by $h(t)$, $0 \leqq t \leqq 1$. Let $\eta(t)$ be the normal contact form on $\Sigma(t)$ given in Theorem 2. Then the pullback of $\eta(t)$ by $h(t)^{-1}$, i.e., $(h^{-1}(t))^*(\eta(t))$ is a 1-parameter family of contact forms on $\Sigma(0, 1)$ which connects $\eta(0, 1)$ and $\eta(t_0, \varepsilon)$ up to the diffeomorphism $h(t)$. q.e.d.

Roughly speaking, Theorem 3 tells us that isotopic deformations of varieties and ellipsoids give nothing new. For example, the generalized Brieskorn manifold associated with a polynomial of the form $P(Z) = \alpha_0 Z_0^{a_0} + \cdots + \alpha_n Z_n^{a_n}$ $(\alpha_0 > 0, \cdots, \alpha_n > 0)$ is essentially the same as the original Brieskorn manifold associated with $P(Z) = Z_0^{a_0} + \cdots + Z_n^{a_n}$. From this point of view, we will only treat, in what follows, the generalized Brieskorn manifolds given as intersections of varieties and the unit sphere of $\mathbf{C}^{n+1}$.

The following corollaries will be obtained from our theorems and known results. It is a well known fact [5] that every odd dimensional exotic sphere bounding a parallelizable manifold can be represented as a Brieskorn manifold. In fact, it is pointed out [5] that such an exotic sphere has infinitely many representations as a Brieskorn manifold. As for the standard spheres of odd dimension, there are clearly infinitely many representations as a Brieskorn manifold. The latter can be easily seen by finding that the intersection of an algebraic subvariety of $\mathbb{C}^{n+1}$ and a hypersphere around a regular point is always diffeomorphic to standard sphere. Of course, a generalized Brieskorn manifold around an isolated singular point can be a standard sphere. Indeed, there are infinitely many such examples. In order to determine whether or not a given generalized Brieskorn manifold is an exotic sphere, one essentially wants to show that the given manifold is a homology sphere. For dimension $\geqq 5$, it is then homeomorphic to a sphere by the well known theorem of Smale. Note that $\pi_1(\Sigma) = 0$ in general. Let Σ be an original Brieskorn manifold. It bounds an even dimensional manifold M which is a fiber of the Milnor fibration [17] and which has the homotopy type of a bouquet $S^n \vee \cdots \vee S^n$ of spheres. Let us assume that the Brieskorn manifold is given by polynomial $P(Z) = Z_0^{a_0} + \cdots + Z_n^{a_n}$, and let $H_n M$ be the n-th homology group of M. Then

THEOREM (Brieskorn-Pham). *The group $H_n M$ is free abelian of rank $(a_0 - 1) \cdots (a_n - 1)$. The characteristic roots of the characteristic homeomorphism h [17] are the products $r_0 r_1 \cdots r_n$ where each r_j ranges over all a_j-th roots of unity other than 1. Hence the characteristic polynomial is given by $\Delta(t) = \Pi(t - r_0 r_1 \cdots r_n)$.*

THEOREM ([5] [17]). *Σ is a topological sphere if and only if $\Delta(1) = \pm 1$.*

Using these results, Brieskorn showed [5] that every Brieskorn exotic sphere has infinitely many representations as a Brieskorn manifold. Thus we have

COROLLARY 1. *Every Brieskorn sphere (exotic or standard) admits infinitely many seemingly different almost contact structures such as in Theorem 1 and normal contact structures such as in Theorem 2.*

The following example shows how to determine whether or not the given Brieskorn manifold is a Brieskorn sphere.

EXAMPLE 8. First, let $P(Z) = Z_0^2 + \cdots + Z_{n-1}^2 + Z_n^l$, where l is an odd number $\geqq 3$. Then $r_0 = \cdots = r_{n-1} = -1$, and $r_n = \omega_j$ $(1 \leqq j \leqq l - 1)$, where ω is an l-th root of unity different from 1. If $n =$ odd, $\Delta(1) = 1$; therefore, Σ is a topological sphere. Next, let $P(Z) = Z_0^2 + \cdots + Z_{n-2}^2 + Z_{n-1}^3 + Z_n^q$, where q is odd and 3 and q are relatively prime. Then $r_0 = r_1 = \cdots = r_{n-2} = -1$ and $r_{n-1} = \omega$ or ω^2 and $r_n = \rho^j$ $(1 \leqq j \leqq q - 1)$, where ω is a 3rd root of unity $\neq 1$, and ρ is a q-th root of unity $\neq 1$. If $n = 2m$ and $q = 6k - 1$ $(m \geqq 4, k = 1, 2, \cdots)$, $\Delta(1) = \omega_2 \omega_1 = 1$. Thus Σ corresponding to $P(Z)$ is homeomorphic to a standard sphere. Going back to the first polynomial, if $n = \text{odd}$, then $\Delta(-1) = l$. By a well known theorem of Levine [16], the Arf-invariant $C(M) = 0$ if $\Delta(-1) \equiv \pm 1 \pmod 8$ and $C(M) = 1$ if $\Delta(-1) \equiv \pm 3 \pmod 8$. Hence if $l = +3 \pmod 8$, the above Σ corresponding to the first polynomial is an exotic sphere; and if $l = \pm 1 \pmod 8$, Σ is a standard sphere. For the 2nd polynomial, as $n = 2m$, the signature of the intersection pairing $H_n M \otimes H_n M \to \boldsymbol{Z}$ completely determines the diffeomorphism class of Σ, and it is given by $(-1)^m 8k$. In particular, $P(Z) = Z_0^2 + Z_1^2 + Z_2^2 + Z_3^3 + Z_4^{6k-1}$ $(k = 1, 2, \cdots, 28)$ represents all the 28 exotic spheres of dimension 7; and a similar polynomial with $k > 28$, also represents one of these spheres.

COROLLARY 2. *For even* $n \geqq 2$, $S^n \times S^{n+1}$ *admit infinitely many seemingly different normal contact structures which are non-regular and have closed curves as the leaves of the associated foliations.*

PROOF. Recently, L. Kauffman [14] showed based on the work of Durfee that the Brieskorn manifold associated with the polynomial (n odd) $P(Z) = Z_0^2 + \cdots + Z_{n-1}^2 + Z_n^k$ $(n \geqq 3)$ has a certain periodicity. If we denote by Σ_k the above Brieskorn manifold, then Σ_k is diffeomorphic to Σ_{k+8} $(k = 1, 2, 3, \cdots)$, and furthermore $\Sigma_1 \cong S^{2n-1}$, $\Sigma_2 \cong T$, $\Sigma_3 \cong \Sigma$, $\Sigma_4 \cong \Sigma \# S^{n-1} \times S^n$, $\Sigma_5 \cong \Sigma$, $\Sigma_6 \cong T$, $\Sigma_7 \cong S^{2n-1}$ and $\Sigma_8 \cong S^{n-1} \times S^n$, where T is the tangent sphere bundle of S^n, Σ is the Kervaire sphere of dimension $2n-1$, and $\#$ denotes connected sum. Thus applying Theorem 2, we have the desired result. Note here T also has non-regular normal contact structure. It is well known that any sphere bundle over a smooth manifold admits a regular contact structure [4]. q.e.d.

COROLLARY 3. *Let* B_p *denote the* p-th *Betti number of a generalized Brieskorn manifold* Σ. *Then*

$$B_p \equiv 0 \pmod 2 \quad \textit{if} \quad p \equiv 1 \pmod 2 \quad \textit{and} \quad 1 \leqq p \leqq \left[\frac{\dim \Sigma}{2}\right], \quad \textit{and}$$

$$B_p \equiv 0 \pmod 2 \quad \textit{if} \quad p \equiv 0 \pmod 2 \quad \textit{and} \quad \left[\frac{\dim \Sigma}{2}\right] + 1 < p \leqq 2\left[\frac{\dim \Sigma}{2}\right],$$

where [ ] *denotes the greatest integer function.*

PROOF. As is shown in Theorem 2, Σ is a normal contact Riemannian manifold. Thus, by making use of harmonic p-forms on Σ, one can show that the set of harmonic p-forms is even dimensional; for the details, see Theorem 33.5 [22]. Now by Poincaré duality, we get the desired result. Brieskorn [5] gives an expression of the $(n-1)$-st Betti number of a Brieskorn manifold in terms of the powers of the polynomials. That is given as follows. Let $P(Z) = Z_0^{a_0} + \cdots + Z_n^{a_n}$ be a

Brieskorn polynomial and let Σ be the Brieskorn manifold. Then

$$B_{n-1}(\Sigma) = \frac{a_0 \cdots a_n}{[a_0, \cdots, a_n]} + (-1)^1 \sum_i \frac{a_0 \cdots \hat{a}_i \cdots a_n}{[a_0, \cdots, \hat{a}_i, \cdots, a_n]}$$
$$+ \sum_{i<j} \frac{a_0 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_n}{[a_0, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots, a_n]}$$
$$+ \cdots + (-1)^n \Sigma \frac{a_i}{a_i} + (-1)^{n+1} ,$$

where $[a_0, \cdots, a_n]$ is the least common multiple of $a_0, \cdots, a_n$, and $\hat{a}_i$ means to delete a_i. He also showed that $B_{n-1}(\Sigma)$ is even under a certain condition. Our result generalizes this aspect of his result. q.e.d.

COROLLARY 4. *For any pair of positive integers k and n (odd), there exists an n-dimansional compact manifold which admits infinitely many seemingly different contact structures and whose fundamental group is $\boldsymbol{Z}_k$. Furthermore, these structures are normal and they have closed curves as the leaves of the associated foliations.*

PROOF. Let Σ be a $(2m-1)$-dimensional generalized Brieskorn manifold. We know that S^1 acts on Σ isometrically with respect to the induced Riemannian metric from that of $\mathbb{C}^{m+1}$. Now let $\boldsymbol{Z}_k$ be the cyclic subgroup of S^1 consisting of the k-th roots of unity. Then $\boldsymbol{Z}_k$ acts on Σ isometrically. The action is the induced action from that of S^1. The orbit space $M = \Sigma/\boldsymbol{Z}_k$ is in general not a manifold. However, in the case of the generalized Brieskorn manifolds given in Examples 1, 2 and 3, a necessary and sufficient condition for M to be manifold is given [5] [19]. In particular, if Σ is three dimensional, $M = \Sigma/\boldsymbol{Z}_k$ is always a manifold. In any case, Σ is, in general, a ramified covering manifold of M. Thus $\pi_1(M)$ may not be easy to compute. Next consider whether or not the contact form η on Σ can give rise to a contact structure on M. One of such cases occurs when Σ becomes a covering space of M. It is well known that a finite group action gives rise to a covering space if the action is fixed point free. Now let $P(Z) = Z_0 + \cdots + Z_{m-1} + Z_m^l$ be a Brieskorn polynomial. As in Example 1, $P(Z)$ gives a Brieskorn manifold Σ. Since the variety V associated with $P(Z)$ has no singular point, Σ is diffeomorphic to S^{2m-1}. Again as in Example 1, the action on Σ is given by

$$t(Z_0, \cdots, Z_m) = (e^{2\pi l t i} Z_0, \cdots, e^{2\pi l t i} Z_{m-1}, e^{2\pi t i} Z_m) .$$

It is easy to see that this S^1-action on Σ has only one isotropy group $\boldsymbol{Z}_l$ other than the identity $\{e\}$. Thus if k and l are mutually prime, $\boldsymbol{Z}_k \cap$

$Z_l = \{e\}$; therefore, Z_k-action on Σ does not have any fixed point. Hence $\Sigma/Z_k = M$ is the base space of the covering triple (Σ, P, M), where P is the natural quotient map. Now let η be the normal contact structure on Σ as in Theorem 2, and let (ϕ, ξ, η, g) be the almost contact Riemannian structure associated with η. Later, we will show in the proof of Theorem 7 that $L_\xi\phi = 0$, $L_\xi\xi = 0$, $L_\xi\eta = 0$ and $L_\xi g = 0$; i.e., (ϕ, ξ, η, g) is invariant under the S^1-action. Therefore, as the covering projection P is locally diffeomorphic, we can uniquely define an almost contact Riemannian structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ on M such that $P_* \circ \phi = \bar{\phi} \circ P_*$, $\bar{\xi} = P_*(\xi)$, $\eta = p^*\bar{\eta}$ and $g = p^*\bar{g}$. It is clear that $\bar{\eta}$ is a contact structure on M and $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an associated almost contact Riemannian structure with $\bar{\eta}$. Next since a contact structure being normal is a local property given by vanishing of the torsion tensor, and since the torsion tensors of (ϕ, ξ, η) and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ coincide locally, $\bar{\eta}$ is a normal contact structure on M. It is clear that $\pi_1(M) = Z_k$. It is not so hard to see that the orbits generated by $\bar{\xi}$ are precisely the images of the orbits of ξ under P, so they are closed curves. Thus by varying m and l, we have the desired result. q.e.d.

REMARK 2. It is easy to see that $\bar{\eta}$ is non-regular unless $l = 1$. When $l = 1$, η on Σ is the Hopf fibration, and M has a regular contact structure and $\pi_1(M) = Z_k$ for any k. In particular, if $k = 2$, we have real projective space as M. Some of this kind of examples were given by Tanno [25]. We can actually give more examples than are given in the proof of Corollary 4. Now let $P(Z) = Z_0^2 + \cdots + Z_{n-1}^2 + Z_n^l$ ($l =$ odd) be a Brieskorn polynomial. As was described previously, if $l = \pm 3$ (mod 8) and $n =$ even, Σ is an exotic sphere of dimension $2n - 1$. Now the S^1-action on Σ in this case is given by

$$t(Z_0, \cdots, Z_n) = (e^{2\pi lti}Z_0, \cdots, e^{2\pi lti}Z_{n-1}, e^{4\pi ti}Z_n) .$$

The only non-trivial isotropy subgroup of this S^1-action is Z_l. Therefore, for any k which is relatively prime to $l = \pm 3$ (mod 8), the orbit space $\bar{M} = \bar{\Sigma}/Z_k$ is a compact manifold such that $\pi_1(\bar{M}) = Z_k$, and the triple $(\bar{\Sigma}, \bar{P}, \bar{M})$ is a covering space. Now let $n = m$, then Σ in the proof of Corollary 4 and $\bar{\Sigma}$ have different differentiable structures, but they are homeomorphic. Then M and $\bar{M}$ can be homeomorphic to each other, but they can never be diffeomorphic. Suppose that there is a diffeomorphism $\bar{f}$ from M onto $\bar{M}$. Then by the unique lifting property of the covering space $(\bar{\Sigma}, \bar{P}, \bar{M})$, there exists a diffeomorphism f such that $\bar{P} \circ f = \bar{f} \circ P$. This is a contradiction. Clearly the above construction works for more general situations, and we can get more examples of compact manifolds which admit normal contact structures and which are not simply connected.

In the next place, we will refine the argument in the proof of Corollary 4 to show that the generalized lens space admits a contact structure. First, we review the definition of lens spaces. For the details, see Spanier [24]. Let p and $q_1, \cdots, q_{n-1}$ be positive integers such that p and q_i $(1 \leqq i \leqq n-1)$ are relatively prime. Let $S^{2n-1} = \{(Z_0, \cdots, Z_{n-1}) \in \mathbb{C}^n : |Z_0|^2 + \cdots + |Z_{n-1}|^2 = 1\}$. Define an S^1-action on S^{2n-1} by

$$t(Z_0, \cdots, Z_{n-1}) = (e^{2\pi ti} Z_0, e^{2\pi q_1 ti} Z_1, \cdots, e^{2\pi q_{n-1} ti} Z_{n-1}) .$$

Here S^1 is identified with $[0, 1)$; therefore, t is a real number such that $0 \leqq t < 1$. Now let us identify the group of the p-th roots of unity with $\boldsymbol{Z}_p$. Let ω be $e^{2\pi(1/p)i}$ which is a generator of the p-th roots of unity. Then ω^l is identified with l in $\boldsymbol{Z}_p$, i.e., $\omega^l = e^{2\pi(l/p)i}$ $(0 \leqq l \leqq p-1)$. Consider $\boldsymbol{Z}_p$ as a subgroup of S^1 in this way, and restrict the above S^1-action to $\boldsymbol{Z}_p$. Then we have a fixed point free action of $\boldsymbol{Z}_p$ on S^{2n-1}. The orbit space of this action is a manifold which is covered by S^{2n-1}. We call this orbit space the lens space $L(p, q_1, \cdots, q_{n-1})$. It is clear that the fundamental group of $L(p, q_1, \cdots, q_{n-1}) = \boldsymbol{Z}_p$. Now we have

COROLLARY 5. *$L(p, q_1, \cdots, q_{n-1})$ admits a normal contact structure whose leavcs of the associated foliation are closed curves.*

PROOF. Let us denote by $q_1 \cdots \hat{q}_j \cdots q_{n-1}$ the product of q_1 through q_{n-1} divided by q_j $(j = 1, \cdots, n-1)$.

Let

$$P(Z) = Z_0^{q_1 \cdots q_{n-1}} + Z_1^{\hat{q}_1 q_2 \cdots q_{n-1}} + \cdots + Z_j^{q_1 \cdots \hat{q}_j \cdots q_{n-1}} + \cdots + Z_{n-1}^{q_1 \cdots q_{n-1} \hat{q}_{n-1}} + Z_n$$

be a Brieskorn polynomial. As before, we denote by Σ, V and S the Brieskorn manifold, the variety and the unit sphere in $\mathbb{C}^{n+1}$. The S^1-action on Σ is given by

$$t(Z_0, \cdots, Z_n) = (e^{2\pi ti} Z_0, e^{2\pi q_1 ti} Z_1, \cdots, e^{2\pi q_{n-1} ti} Z_{n-1}, e^{2\pi q_1 \cdots q_{n-1} ti} Z_n) .$$

Here, as before, S^1 is identified with $[0, 1)$. Now we wish to establish a diffeomorphism between $\Sigma/\boldsymbol{Z}_p$ and $L(p, q_1, \cdots, q_{n-1})$. To this end, we first establish a diffeomorphism between Σ and the image of Σ under the projection of $\mathbb{C}^{n+1}$ onto $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$. Since V is given as the locus of zeros of $P(Z) = 0$, it is easy to see that the correspondence h between $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$ and V given by

$$(Z_0, \cdots, Z_{n-1}) \overset{h}{\longleftarrow} (Z_0, \cdots, Z_{n-1}, -(Z_0^{q_1 \cdots q_{n-1}} + Z_1^{\hat{q}_1 q_2 \cdots q_{n-1}} + \cdots + Z_{n-1}^{q_1 \cdots q_{n-2} \hat{q}_{n-1}}))$$

is a holomorphic diffeomorphism. Now restrict this correspondence to $\Sigma = V \cap S$. Then we have a diffeomorphism h from Σ onto $h(\Sigma)$ in

$\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$. Of course, $h(\Sigma)$ is a submanifold of $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$. It is obvious that $h(\Sigma)$ invariant under the S^1-action induced from the $\mathbb{C}$-action on $\mathbb{C}^{n+1}$, and that h commutes with these actions. Hence, h is an equivariant diffeomorphism between Σ and $h(\Sigma)$. Now let S^{2n-1} be the unit sphere of $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$. Note that $h(\Sigma)$ lies in the unit ball of $\mathbb{C}^n$. Next we will establish an equivariant diffeomorphism between $h(\Sigma)$ and S^{2n-1}. As is described in § 1, the $\mathbb{C}$-action on V induces the $\boldsymbol{R}$-action given by, for any $s \in \boldsymbol{R}$,

$$s(Z_0, \cdots, Z_n) = (e^{2\pi s} Z_0, e^{2\pi q_1 s} Z_1, \cdots, e^{2\pi q_{n-1} s} Z_{n-1}, e^{2\pi q_1 \cdots q_{n-1} s} Z_n) .$$

The orbits of this $\boldsymbol{R}$-action are diffeomorphic to $\boldsymbol{R}$ and they intersect transversally every sphere of $\mathbb{C}^{n+1}$ with the origin as its center once and only once. As h is a diffeomorphism, the h-images of these orbits intersect $h(\Sigma(\varepsilon))$ transversally once and only once. Of course, they are diffeomorphic to $\boldsymbol{R}$, and nothing but the orbits of $\boldsymbol{R}$-action on $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$ induced from the $\boldsymbol{R}$-action on $\mathbb{C}^{n+1}$ restricted to the first n coordinates. Thus, it is easy to see that these orbits intersect the unit sphere S^{2n-1} of $\mathbb{C}^n$ transversally and once and only once. We define a correspondence g from $h(\Sigma)$ onto S^{2n-1} as follows. Let $(Z_0, \cdots, Z_{n-1})$ be a point in $h(\Sigma)$. Define $g(Z_0, \cdots, Z_{n-1})$ to be the point of intersection of S^{2n-1} and the $\boldsymbol{R}$-orbit passing through $(Z_0, \cdots, Z_{n-1})$. We may write it as follows.

$$g(Z_0, \cdots, Z_{n-1}) = (e^{2\pi s} Z_0, e^{2\pi q_1 s} Z_1, \cdots, e^{2\pi q_{n-1} s} Z_{n-1}) .$$

Here s depends on $(Z_0, \cdots, Z_{n-1})$, and it is uniquely determined for each $(Z_0, \cdots, Z_{n-1})$ in such a way that $|e^{2\pi s} Z_0|^2 + |e^{2\pi q_1 s} Z_1|^2 + \cdots + |e^{2\pi q_{n-1} s} Z_{n-1}|^2 = 1$. Obviously, this mapping g is one-to-one and onto. By the similar argument used in the proof of Lemma 3, we can show that g is indeed a diffeomorphism. This part of the proof is left to the reader. We next show that g is equivariant. Note that the following diagram commutes.

$$\begin{array}{ccc}
h(\Sigma) \ni (Z_0, \cdots, Z_{n-1}) & \xrightarrow{g} & (e^{2\pi s} Z_0, \cdots, e^{2\pi q_{n-1} s} Z_{n-1}) \in S^{2n-1} \\
\updownarrow S^1\text{-action} & & \updownarrow S^1\text{-action} \\
h(\Sigma) \ni (e^{2\pi t i} Z_0, \cdots, e^{2\pi q_{n-1} t i} Z_{n-1}) & \xrightarrow{g} & (e^{2\pi (ti+s)} Z_0, \cdots, e^{2\pi q_{n-1}(ti+s)} Z_{n-1}) \in S^{2n-1}
\end{array}$$

Here $s \in \boldsymbol{R}$ is a real number given as above, and it depends upon $(Z_0, \cdots, Z_{n-1})$. The fact that we can use the same $s \in \boldsymbol{R}$ for the right hand side follows from the fact that each $\boldsymbol{R}$-orbit on $\mathbb{C}^n(Z_0, \cdots, Z_{n-1})$ is mapped onto an $\boldsymbol{R}$-orbit under the S^1-action, and that it intersects S^{2n-1} only once. Now we have established an equivariant diffeomorphism $g \circ h$ between Σ and S^{2n-1}. It is clear that $g\circ h$ is also $\boldsymbol{Z}_p$-equivariant, since the $\boldsymbol{Z}_p$-actions on Σ and S^{2n-1} are induced from these S^1-actions. Passing to the orbit spaces of these $\boldsymbol{Z}_p$-actions, we have established a diffeomorphism $\overline{g\circ h}$ between $\Sigma/\boldsymbol{Z}_p = B$ and $L(p, q_1, \cdots, q_{n-1})$ such that the following diagram commutes.

$$\begin{array}{ccc} \Sigma & \xrightarrow{g\circ h} & S^{2n-1} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\overline{g\circ h}} & L(p, q_1, \cdots, q_{n-1}) \end{array}$$

Indeed, the pair $(g\circ h, \overline{g\circ h})$ gives rise to a covering isomorphism. As in Corollary 4, B admits a normal contact structure, so does $L(p, q_1, \cdots, q_{n-1})$. This structure clearly has the properties mentioned in Corollary 5. Furthermore, they are in general non-regular. q.e.d.

As was pointed out earlier, the $\mathbb{C}$-action on the irreducible variety V induces the $i\boldsymbol{R}$-action on $\Sigma(\varepsilon)$, and if all of $q_0, \cdots, q_n$ are rational numbers, then this $i\boldsymbol{R}$-action reduces to S^1-action. Recall the following. Let $q_0 = u_0/v_0, \cdots, q_n = u_n/v_n$, where u_i and v_i $(i = 0, \cdots, n)$ are mutually prime positive integers and let d be the least common multiple of $v_0, \cdots, v_n$. Then the S^1 should be identified with $[0, d)$, or the closed interval $[0, d]$ whose endpoints are identified. Following Brieskorn-Van de Ven [6], let Γ be the discrete subgroup of $\mathbb{C}$ generated by 1 and $id = \sqrt{-1}d$. Then the $\mathbb{C}$-action on V_0 induces a proper holomorphic action of the complex 1-torus $T = \mathbb{C}/\Gamma$ on $H = V_0/\Gamma$. Note here that the $\mathbb{C}$-action on V_0 induces a proper discontinuous Γ-action on V_0; and therefore, the quotient H is a complex manifold. Now using a theorem of Holman [12], the quotient H/T is in a natural way a normal complex space, and the canonical projection

$$\pi\colon H \longrightarrow H/T$$

is holomorphic in the sense of complex space. Furthermore, $(H, \pi, H/T)$ is a holomorphic Seifert fiber space with elliptic curves as its fibers.

In what follows, we will try to characterize this complex structure on H/T restricted to a dense open subset in connection with the contact structures in Theorem 2. Included will be some kind of Boothby-Wang fibration concerning the contact structures. Now let us assume that all $q_0, \cdots, q_n$ are rationals. Then, it is easy to see that the quotient Σ/S^1 is the same as H/T. Denote $\Sigma(\varepsilon)/S^1$ by $B(\varepsilon)$. By the definition of this S^1-action, it is easy to see that the S^1-orbits are either principal or exceptional orbits, and the isotropy groups are finite cyclic groups Z_p. If we denote by π the projection of $\Sigma(\varepsilon)$ onto $B(\varepsilon)$ in the natural way, π is continuous. By the well known slice theorem, π is smooth in a neighborhood of each principal orbit. If we call the π-images of exceptional orbits the singular points of $B(\varepsilon)$, π is smooth outside exceptional orbits with respect to the naturally induced differentiable structure on $B(\varepsilon)-\{$the singular points$\}$. In fact, $B(\varepsilon)$ as a whole can be given naturally a differentiable structure under certain conditions in such a way that π is smooth. Such conditions are given in [5] [19] [21]. They showed that $B(\varepsilon)$ is a topological manifold if and only if $B(\varepsilon)$ is a complex manifold with the quotient complex structure and π is holomorphic in the usual sense. By the definition of the $\mathbb{C}$-action, it is easy to see that $B(\varepsilon)-\{$singular points$\}$ is an open and dense subset of $B(\varepsilon)$, and we denote it by $U(\varepsilon)$. We also denote by $\pi^{-1}(U(\varepsilon))$ the set of principal orbits. Let $(\phi(\varepsilon), \xi(\varepsilon), \eta(\varepsilon), g(\varepsilon))$ be the associated almost contact Riemannian structure with the contact structure $\eta(\varepsilon)$. As long as there is no fear of confusion, we will denote them without ε. The restriction of (ϕ, ξ, η, g) to $\pi^{-1}(U)$ will be denoted by the same symbols for obvious reasons. The following arguments are routine.

LEMMA 11. *$(\pi^{-1}(U), \pi, U)$ is a principal circle bundle and the contact form η gives rise to a connection on $(\pi^{-1}(U), \pi, U)$ whose horizontal space is Θ, and whose vertical space is the orbit.*

PROOF. The first half of the statement is clear since the S^1-action on $\pi^{-1}(U)$ is free. Now let $\mathfrak{S}^1$ be the Lie algebra of S^1, and let d/dt be the left invariant basis for $\mathfrak{S}^1$. Define a $\mathfrak{S}^1$-valued 1-form $\bar{\eta}$ on $T(\pi^{-1}(U))$ by

$$\bar{\eta}(\xi) = \eta(\xi)d/dt = d/dt$$

and

$$\bar{\eta}(\Theta)) = 0 \, .$$

To show that $\bar{\eta}$ is a connection form, it suffices to show that a) $\bar{\eta}(\xi) = d/dt$ and b) $R_t^*\bar{\eta} = (\mathrm{ad}\, t^{-1})\bar{\eta}$, where t^{-1} is the inverse of t in S^1. a) is obvious because $\bar{\eta}(\xi) = \eta(\xi)$ and $d/dt = d/dt$ by the definition of $\bar{\eta}$. To show b), we first note $L_\xi\eta = 0$, where L_ξ is the Lie derivative in ξ direction. This follows easily from $\eta(\xi) = 1$ and a) in Lemma 5. This tells that η is invariant under the group S^1. Since the right translation R_t is exactly the group of transformation generated by ξ, we have $R_t^*\bar{\eta} = \bar{\eta}$. On the other hand, S^1 being commutative implies $\mathrm{ad}\, t^{-1} =$ identity. Thus b) has been shown. The fact Θ being horizontal is clear from the definition of $\bar{\eta}$ and from Lemma 4. q.e.d.

LEMMA 12. *U is a Kählerian manifold, and π_* restricted to Θ; i.e., $\pi_*|\Theta: \Theta \to TU$ is a complex linear map.*

PROOF. Since Θ is the horizontal space, for any vector field Y in TU, there exists a unique horizontal lift of Y in Θ, say X, and of course $\pi_*(X) = Y$ and $L_\xi X = 0$. Now define an almost complex structure $\bar{J}$ in U by $\bar{J}Y = \pi_*(\phi X)$, where X is the horizontal lift of Y. Clearly $(\bar{J})^2 Y = \bar{J}(\pi_*\phi X) = \pi_*\phi(\phi X) = -Y$. This follows from the fact that $L_\xi \phi = 0$ which will be shown in the proof of Theorem 7. Next we show that this $\bar{J}$ is a complex structure on U. Let Y_1 and Y_2 be any two vector fields on U and let X_1 and X_2 be their horizontal lifts, respectively. Since (ϕ, ξ, η) is normal in $\pi^{-1}(U)$, the torsion tensor $T\phi(X_1, X_2) = [X_1, X_2] + \phi[\phi X_1, X_2] + \phi[X_1, \phi X_2] - [\phi X_1, \phi X_2] - (X_1\eta(X_2) - X_2\eta(X_1))\xi = 0$. Since X_1 and X_2 are in Θ, the last two terms vanish. It is clear that $\pi_*[X_1, X_2] = [\pi_* X_1, \pi_* X_2] = [Y_1, Y_2]$. Thus we have the torsion tensor of $\bar{J} = [Y_1, Y_2] + \bar{J}[\bar{J}Y_1, Y_2] + \bar{J}[Y_1, JY_2] - [\bar{J}Y_1, JY_2] = \pi_*\{[X_1, X_2] + \phi[\phi X_1, X_2] + \phi[X_1, \phi X_2] - [\phi X_1, \phi X_2]\} = \pi_* T\phi(X_1, X_2) = 0$. Therefore, by the Newlander-Nirenberg theorem, $\bar{J}$ is a complex structure on U. Finally, we show that U admits a Kählerian structure. Let g be the Riemannian metric given in the proof of Theorem 2. Recall that $\eta(X) = g(\xi, X)$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ hold for all vector fields X, Y on Σ. Now $(L_\xi g)(X, Y) = L_\xi(g(X, Y)) - g(L_\xi X, Y) - g(X, L_\xi Y) = 0$ if X and Y are invariant under the S^1-action. Thus g is invariant under the S^1-action. In fact, this fact can be easily seen from the fact that the S^1-action on Σ is an isometric action with respect to the induced metric on Σ. Combining this fact and that the horizontal space Θ is S^1-invariant, we can define a Riemannian metric $\bar{g}$ on U as follows. Let Y_1 and Y_2 be any two vector fields on U and let X_1 and X_2 be their horizontal lifts. Define $\bar{g}(Y_1, Y_2) = g(X_1, X_2)$. Notice that X_1 and X_2 are S^1-invariant; therefore, $\bar{g}$ is well-defined. Clearly $\bar{g}$ is an inner product on U. Next $\bar{g}(\bar{J}Y_1, \bar{J}Y_2) = g(\phi X_1, \phi X_2) = g(X_1, X_2) = \bar{g}(Y_1, Y_2)$. Hence, $\bar{g}$ is a Hermitian metric on TU with respect to $\bar{J}$. Define a 2-form $\bar{\Omega}$ on U by $\bar{\Omega}(Y_1, Y_2) = \bar{g}(\bar{J}Y_1, Y_2)$. Consider $d\eta$. η is invariant under the S^1-action, i.e., $L_\xi \eta = 0$. Since L_ξ commutes with d, $L_\xi d\eta = dL_\xi \eta = 0$, i.e., $d\eta$ is invariant under the S^1-action. It is easy to see from Lemma 5 that $\pi^*\bar{\Omega} = d\eta$. Thus $\pi^* d\bar{\Omega} = d\pi^*\bar{\Omega} = dd\eta = 0$. This observation tells us that $\pi^* d\bar{\Omega}(X_1, X_2, X_3) = d\bar{\Omega}(\pi_* X_1, \pi_* X_2, \pi_* X_3) = d\bar{\Omega}(Y_1, Y_2, Y_3) = 0$, where X_1, X_2, X_3 are the horizontal lifts of Y_1, Y_2 and Y_3. Therefore $\bar{\Omega}$ is a closed form, and $(\bar{J}, \bar{g})$ gives rise to a Kählerian structure on U. q.e.d.

REMARK 3. The following observation concerning $\bar{\Omega}$ may be of interest. In Lemma 11, we have shown that $\bar{\eta}$ is a connection form of $(\pi^{-1}(U), \pi, U)$. Let us denote the curvature form of η by Ω. Then for any X and $Y \in T\pi^{-1}(U)$, $d\bar{\eta}(X, Y) = (-1/2)[\bar{\eta}(X), \bar{\eta}(Y)] + \Omega(X, Y)$. It is well known that $\Omega(X, Y)$ is a horizontal 2-form, and $[\bar{\eta}(X), \bar{\eta}(Y)] = 0$ since S^1 is abelian. Therefore, $d\bar{\eta}(X, Y) = \Omega(X, Y)$. Hence our $\bar{\Omega}$ on U is a 2-form such that $\pi^*\bar{\Omega} = \Omega$.

Next, by the definition of $\bar{g}$, one can easily see that Θ is mapped under π_* isometrically onto TU with respect to the metric g in $\pi^{-1}(U)$. This tells us that the triple $(\pi^{-1}(U), \pi, U)$ is a Riemannian submersion.

As we pointed out before, there are many cases where the whole quotient space B becomes a complex manifold. For example, if $\dim \Sigma = 3$, the examples 1, 2 and 3 all give compact Riemann surfaces as B. One can here ask the question whether or not B in these cases, is a Kählerian manifold as a whole. The answer is affirmative in most cases as Brieskorn pointed out that B in general is projective as a complex space. Also see Mumford [18] for the projective imbedding of B. We now see that these B have a Kählerian metric which is induced from the Fubini-Study metric of the ambient projective space. However, our metric $\bar{g}$ cannot be extended to the whole B if Σ has an exceptional orbit. It is quite easy to see that $\bar{g}$ blows up at the singular points. Even if B does not have a singular point, the metric $\bar{g}$ may not coincide with the one induced from the projective space. Of course, if the variety V is given by homogeneous polynomials, B is naturally projective algebraic variety and $\bar{g}$ coincides with the induced metric up to a constant.

Finally, let X be a differentiable manifold in general and denote by $H^l(X)$, $H^l(X: R)$ and $H^l(X: Z)$ the l-th de Rham cohomology group, the l-th singular cohomology group with real coefficient and the l-th singular cohomology group with the integer coefficient, respectively. Then there is the de Rham isomorphism $di: H^l(X) \to H^l(X: \boldsymbol{R})$. On the other hand, the natural imbedding j of the coefficient groups from $\boldsymbol{Z}$ into $\boldsymbol{R}$, say $j: \boldsymbol{Z} \to \boldsymbol{R}$, induces the homomorphism $j^*: H^l(X: \boldsymbol{Z}) \to H^l(X: \boldsymbol{R})$. We say an element α of $H^l(X)$ is integral if $di(\alpha)$ is contained in $j^*(H^l(X: \boldsymbol{Z}))$. A compact Kählerian manifold X is called a Hodge manifold if the Kählerian form is integral in $H^2(X)$. For convenience, even if X is not compact, we say X is of Hodge type if its Kählerian form is integral.

Going back to our fibration $(\pi^{-1}(U), \pi, U)$, we can in fact show that our Kählerian form $\bar{\Omega}$ is integral. The proof is in a way well known. We just mention that one usually makes use of the isomorphism between the singular cohomology and Čech cohomology groups. For a typical proof of this type, see Boothby-Wang [4], or Hatakeyama [10]. Now summarizing our observations, we have the following theorem which is the mixture of Boothby-Wang fibration theorem and a theorem of Hatakeyama [10].

THEOREM 4. *Let* (Σ, π, B) *be a fibration as before, and let* $(\pi^{-1}(U), \pi, U)$ *be the restricted fibration to the non-singular set* U *of* B. *Then*

a) $\pi^{-1}(U)$ *is a circle bundle over* U *with* π *as its projection.*

b) η *defines a connection over the bundle* $(\pi^{-1}(U), \pi, U)$.

c) U *is a Kählerian manifold and its Kählerian form* Ω *is the curvature form of the above connection* η, i.e., $d\eta = \pi^*\Omega$ *is the structure equation of the connection. Furthermore,* U *is of Hodge type. In other words,* Ω *is an integral cocycle.*

In fact we can describe the fibration (Σ, π, B) a little more precisely by decomposing (Σ, π, B) into the disjoint union of orbit bundles. This will be given below as a remark after we introduce relevant definitions.

In general let G be a compact Lie group and let G act on a smooth manifold M smoothly. We call such a manifold M a G-manifold. Let $x \in M$ be a point in the G-manifold M and let G_x be the isotropy group of x. Denote by $G(x)$ the orbit of x with respect to G. Let $N_x = TM_x/TG(x)_x$ be the normal space to the orbit $G(x)$ at x, where TM_x and $TG(x)_x$ denote the tangent spaces of M and $G(x)$ at x. Now let $\sigma_x\colon G_x \to GL(N_x)$ be the slice representation of G_x. Then the pair $[G_x, \sigma_x]_G$ is called the slice type at the point x. The slice type is constant along orbits in M, for if $g \in G$, then $G_{gx} = gG_xg^{-1}$ and $\sigma_{gx} \sim \sigma_x \circ (g * g^{-1})$; therefore, $[G_{gx}, \sigma_{gx}]_G = [G_x, \sigma_x]_G$. By the slice theorem, $[G_x, \sigma_x]$ completely determines the local structure of M at x. The set of all slice types of a G-manifold M can be given a partial order in a natural way, and the set with this natural partial order is called the slice diagram of M. Furthermore, if the orbit space M/G is connected, the slice diagram of M, say $\Delta(G, M)$, has the unique largest element $[H, \theta]$ called the principal type. The principal type is characterized by the fact that the representation is trivial. If M is compact, $\Delta(G, M)$ has a finite number of slice types. Let H be a closed subgroup of G, and M be the G-manifold. Then the orbit bundle M_H is defined as follows:

$$M_H = \{x \in M \mid G_x \text{ is conjugate to } H \text{ in } G\}\,.$$

M_H is an invariant submanifold, and has a natural structure of fiber bundle over M_H/G with the orbits as fibers. This M_H can be further partitioned as follows: Let $[H, \sigma]$ be a slice type of M. Then the set $M_{[H,\sigma]} = \{x \in M \mid [G_x, \sigma_x] = [H, \sigma]\}$ is known to be an invariant open and closed submanifold of M_H. $M_{[H,\sigma]}$ also has the natural fiber bundle structure. In fact, M_H is given as the disjoint union of $M_{[H,\sigma]}$, where $[H, \sigma]$ runs over all the slice types of M as H is fixed. Call $M_{[H,\sigma]}$ also an orbit bundle for convenience. For the details, see Jänich [13]. Now we have a remark,

REMARK 4. Let $\Sigma_{[H,\sigma]}$ be an orbit bundle of the S^1-manifold Σ. Then the triple $(\Sigma_{[H,\sigma]}, \pi, \pi(\Sigma_{[H,\sigma]}) = \Sigma_{[H,\sigma]}/S^1)$ has the similar structures to those described in Theorem 4. Of course, we should realize that the S^1-action on $\Sigma_{[H,\sigma]}$ is in general not effective; therefore, we have to consider the S^1-action as the quotient action.

As we have learned that some generalized Brieskorn manifolds may admit infinitely many contact structures as given in Theorem 2, it would be of natural interest to ask whether or not these structures can be classified in a certain way. In what follows, we give a simple classification of these structures based on the S^1-action on Σ.

Let M and N be contact manifolds, and let $f: M \to N$ be a diffeomorphism from M onto N. Let ω_M and ω_N be the contact forms on M and N, respectively. We say that f is a contact transformation from M onto N if $f^*\omega_N = \rho\omega_M$, where f^* is the pullback homomorphism of f and ρ is a nowhere vanishing smooth function on M. If $\rho \equiv 1$ on M, we say f is a strict contact transformation. Let ω_1 and ω_2 be two contact structures on M. Then we say $\omega_1 = \omega_2$ if there exists a diffeomorphism f of M and a nowhere vanishing smooth function ρ such that $f^*\omega_2 = \rho\omega_1$. Similarly, $\omega_1 = \omega_2$ in the strict sense if $f^*\omega_2 = \omega_1$. Our criterion to be used for the classification is the contact transformation in the strict sense. Obviously, there are two more natural ways to classify the contact structures. One is the contact transformation in the usual sense, and the other is the deformation of contact structures as defined in Gray [8]. Classification with respect to these two criteria seems to be much more difficult, and except for some special cases nothing has yet been known to the author.

Our theorem states as follows:

THEOREM 5. *Let Σ_1 and Σ_2 be two generalized Brieskorn manifolds with the normal contact structures η_1 and η_2, respectively. Assume that the induced $i\boldsymbol{R}$-actions on Σ_1 and Σ_2 give rise to S^1-actions. If the slice diagrams $\Delta(S^1, \Sigma_1)$ and $\Delta(S^1, \Sigma_2)$ are not isomorphic to each other, then there is no strict contact transformation between η_1 and η_2.*

PROOF. Let $f: \Sigma_1 \to \Sigma_2$ be a strict contact transformation between (Σ_1, η_1) and (Σ_2, η_2), i.e., $f^*\eta_2 = \eta_1$. As before, let us denote by ξ_i and Θ_i $(i = 1, 2)$ the velocity vector fields of the S^1-orbits in Σ_i and the kernel of η_i.

We need the following lemma.

LEMMA 13. *Let* (Σ_i, η_i) $(i = 1, 2)$ *and* f *be as above. Then we have*
a) $f_*(\xi_1) = \xi_2$.
b) $f_*(\Theta_1) = \Theta_2$.

PROOF. Let X be an element of Θ_1. Then $\eta_2(f_*(X)) = f^*\eta_2(X) = \eta_1(X) = 0$. Therefore, $f_*(\Theta_1) \subset \Theta_2$. Since the dimensions of Θ_1 and Θ_2 are equal, we have $f_*(\Theta_1) = \Theta_2$. Tis proves b). Next $f^*\eta_2 = \eta_1$ implies that $d(f^*\eta_2) = d\eta_1$; therefore, $f^*d\eta_2 = df^*\eta_2 = d\eta_1$. As is shown in Lemma 5, $d\eta_i(\xi_i, X) = 0$ for all $X \in T\Sigma_i$ $(i = 1, 2)$, and ξ_i is unique $(i = 1, 2)$. Now let X be an element of $T\Sigma_1$. Then $0 = d\eta_1(\xi_1, X) = d(f^*\eta_2)(\xi_1, X) = f^*d\eta_2(\xi_1, X) = d\eta_2(f_*\xi_1, f_*X)$. Since f_* is an isomorphism, we have that $f_*\xi_1 = k\xi_2$, where k is a non-zero function. On the other hand, $1 = \eta_1(\xi_1) = f^*\eta_2(\xi_1) = \eta_2(f_*\xi_1) = \eta_2(k\xi_2) = k\eta_2(\xi_2) = k$. Thus $f_*\xi_1 = \xi_2$. This proves a). q.e.d.

We continue with the proof of Theorem 5. Let $\psi_1(t)$ and $\psi_2(t)$ be the 1-parameter groups of transformations generated by ξ_1 and ξ_2 on Σ_1 and Σ_2, respectively. Let $x \in \Sigma_1$ be a point in Σ_1, and let $f(x)$ be the image of x in Σ_2. Then $\psi_1(t)x$ is the orbit through x under the S^1-action by the choice of ξ_1 and its velocity vector is ξ_1. By a) of Lemma 13, we see that the image curve $f \circ \psi_1(t)x$ has ξ_2 as the velocity vector at each point. Thus by uniqueness of solutions of ordinary differential equations, the integral curve of ξ_2 through $f(x)$ must be the curve $f \circ \psi_1(t)(x)$. On the other hand, $\psi_2(t)(f(x))$ is the integral curve of ξ_2; therefore, we have $f \circ \psi_1(t)(x) = \psi_2(t) \circ f(x)$. As x is an arbitrary point in Σ_1, we have shown that $f \circ \psi_1(t) = \psi_2(t) \circ f$. In other words, the 1-parameter groups of transformations commute with f. These 1-parameter groups are precisely the S^1-actions on Σ_1 and Σ_2 again by the choice of ξ_1 and ξ_2 and by uniqueness of solutions of ordinary differential equations. Thus f commutes with these S^1-actions on Σ_1 and Σ_2. In other words, f is an equivariant diffeomorphism. Then it is well known that the corresponding slice diagrams are isomorphic to each other. This contradicts the assumption of Theorem 5; so there cannot exist a strict contact transformation. q.e.d.

Theorem 5 can have more precise forms in the cases where Σ_1 and

Σ_2 are given as in Examples 1, 2 and 3. The following observations which we will make for original Brieskorn manifolds can be carried out for Σ's in Examples 2 and 3; however, most interesting examples arise as a Brieskorn manifold.

Let $P(Z) = Z_0^{a_0} + \cdots + Z_0^{a_n}$ be a Brieskorn polynomial as in Example 1. Then the $\mathbb{C}$-action is given by $t(Z_0, \cdots, Z_n) = (e^{2\pi a_0' t}Z_0, \cdots, e^{2\pi a_n' t}Z_n)$, where $a_i' = d/a_i$ $(i = 0, \cdots, n)$ and $d =$ the least common multiple of $(a_0, \cdots, a_n)$. Let $\Sigma(a_i)$ be the corresponding Brieskorn manifold. Then Neumann [19] showed that the slice diagram $\Delta(S^1, \Sigma(a_i))$ is given by a slice type $[H, \sigma]$ which has the form $[\boldsymbol{Z}_{\gcd(a_0', \cdots, a_k')}; \sigma_{a_{k+1}'} \oplus \sigma_{a_{k+2}'} \oplus \cdots \oplus \sigma_{a_n'} \oplus (2k-1)]$ $(0 \leqq k \leqq n)$, or can be obtained from such a slice type by permuting the indices. The corresponding orbit bundle in Σ is given by the set $\{(Z_0, \cdots, Z_k, 0, \cdots, 0) \in \Sigma \subset \mathbb{C}^{n+1} \mid Z_i \neq 0\ (0 \leqq i \leqq k)\}$, or the set given by permuting the indices. Here $\gcd(a_0', \cdots, a_k')$ is the greatest common divisor of $a_0', \cdots, a_k'$, and $(2k-1)$ is the trivial representation of $(2k-1)$-dimensional Euclidean space. The representation σ_p is the representation of $\boldsymbol{Z}_q$ on $\mathbb{C} = \boldsymbol{R}^2$ given by $(e^{2\pi i t}, Z) = e^{2\pi p i t}Z$ for $Z \in \mathbb{C}$. For more details, see Neumann [19]. Thus we have

COROLLARY 6. *Let $\Sigma(a_0, \cdots, a_n)$ and $\Sigma(b_0, \cdots, b_n)$ be two Brieskorn manifolds, and let η_a and η_b be the corresponding normal contact structures on $\Sigma(a)$ and $\Sigma(b)$. Then η_a is not equivalent to η_b in the strict sense if their slce diagrams do not coincide.*

Making use of Corollary 6, we now show by examples that every odd dimensional standard sphere and some exotic spheres admit infinitely many distinct normal contact structures given as in Theorem 2. Here "distinct" means not equivalent in the strict sence unless otherwise mentioned.

EXAMPLE 9. Let $P_q(Z) = Z_0 + Z_1 + Z_2^q + \cdots + Z_n^q$ $(q > 0)$ be a Brieskorn polynomial. Since the origin of $\mathbb{C}^{n+1}$ is a regular point of the variety defined as the locus of zeros of $P_q(Z)$, the Brieskorn manifold associated with $P_q(Z)$, say Σ_q, is diffeomorphic to the standard unit sphere of dimension $2n-1$. It is easy to show that the $\mathbb{C}$-action is given by $t(Z_0, \cdots, Z_n) = (e^{2\pi q t}Z_0, e^{2\pi q t}Z_1, e^{2\pi t}Z_2, \cdots, e^{2\pi t}Z_n)$. Thus the slice diagram of this S^1-action on Σ_q contains the slice type given by $[Z_q: \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{(n-1)\text{ times}}]$, where σ_1 is the representation of $\boldsymbol{Z}_q$ on $\mathbb{C} = \boldsymbol{R}^2$ defined by $(e^{2\pi i t}, Z) \mapsto e^{2\pi i t}Z$, $Z \in \mathbb{C}$ and $t \in Z_q$. The corresponding orbit bundle is given by $\{(Z_0, Z_1, 0, \cdots, 0) \in \Sigma_q \subset \mathbb{C}^{n+1} \mid Z_0 \neq 0, Z_1 \neq 0\}$. It is clear that $[Z_q: \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{(n-1)\text{-times}}] \neq [Z_r: \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{(n-1)\text{-times}}]$ and $\Delta(S^1, \Sigma_q) \neq \Delta(S^1, \Sigma_r)$ if $q \neq r$. Thus the normal contact structures η_q $(q = 1, 2, \cdots)$ are all distinct on S^{2n-1}. In particular, η_1 is the normal contact structure on S^{2n-1} given by the Hopf fibration. It is obvious that we can have more distinct normal contact structures on S^{2n-1} by manipulating the powers of polynomials.

EXAMPLE 10. Consider the polynomial $P_l(Z) = Z_0^2 + \cdots + Z_{2m}^2 + Z_{2m+1}^l$ (l: odd) as given in Example 8. As was mentioned in Example 8, if $l = \pm 3 \pmod 8$, the corresponding Brieskorn manifold Σ_l is an exotic sphere of dimension $4m+1$ $(m \geqq 3)$. The induced S^1-action on Σ_l is given as follows:

$$t(Z_0, \cdots, Z_{2m+1}) = (e^{2\pi lit}Z_0, \cdots, e^{2\pi lit}Z_{2m}, e^{4\pi it}Z_{2m+1}) .$$

This action contains a slice type of the form $[Z_l: \underbrace{\sigma_l \oplus \cdots \oplus \sigma_l}_{2(m-1)} \oplus \sigma_2]$ and its orbit bundle is given by $\{(Z_0, Z_1, 0, \cdots, 0) \in \Sigma_l \subset \mathbb{C}^{2(m+1)} \mid Z_0 \neq 0$ and $Z_1 \neq 0\}$. As before, we can show that $\Delta(S^1, \Sigma_m)$ cannot contain the slice type $[Z_l: \sigma_l \oplus \cdots \oplus \sigma_l \oplus \sigma_2]$ if $l \neq m$. Thus the corresponding contact structures η_l and η_m are distinct if $l \neq m$. Since there are infinitely many $l = \pm 3 \pmod 8$ and there are a finite number of exotic spheres bounding a parallelizable manifold in general, we notice at least some of them must have infinitely many distinct contact structures. By making use of the second type of Brieskorn polynomial in Example 8, we can show that more exotic spheres have infinitely many normal contact structures which are distinct. In particular, a 7-dimensional exotic sphere has infinitely many distinct normal contact structures as given in Theorem 2. More precise computation is left to the reader. Within Raymond's scheme, Neumann [19] has obtained more precise classification of 3-dimensional Brieskorn manifolds. Therefore, according to his classification we can obtain the topological characterization of Brieskorn 3-manifolds, and the minimum number of distinct normal contact structures on each are as in Corollary 6 and Examples 9 and 10. Among these, it is perhaps of greatest interest to describe the situation about the Brieskorn manifolds which are diffeomorphic to S^3. Let $P(Z) = Z_0^{a_0} + Z_1^{a_1} + Z_2^{a_2}$ be a Brieskorn polynomial, and let $\Sigma(a_0, a_1, a_2)$ be the corresponding Brieskorn manifold. Then it follows from Neumann's classification that $\Sigma(a_0, a_1, a_2)$ is diffeomorphic to S^3 if and only if at least one of a_0, a_1 and a_2 equals 1. Thus we can assume that the polynomial has the form $P(Z) = Z_0 + Z_1^a + Z_2^b$. Then we have, THEOREM 6. *Let $P_i(Z) = Z_0 + Z_1^{a_i} + Z_2^{b_i}$ $(i = 1, 2)$ be two polynomial, and let Σ_i and η_i $(i = 1, 2)$ be the Brieskorn manifolds and its normal contact structure. Then Σ_i is diffeomorphic to S^3 and η_i $(i = 1, 2)$ considered as contact structures on S^3 are distinct if either*

$$\frac{a_1}{\gcd(a_1, b_1)} \neq \frac{a_2}{\gcd(a_2, b_2)} \quad \textit{and} \quad \frac{a_1}{\gcd(a_1, b_1)} \neq \frac{b_2}{\gcd(a_2, b_2)}, \quad \textit{or}$$

$$\frac{a_1}{\gcd(a_1, b_1)} \neq \frac{a_2}{\gcd(a_2, b_2)} \quad \textit{and} \quad \frac{b_1}{\gcd(a_1, b_1)} \neq \frac{a_2}{\gcd(a_2, b_2)}, \quad \textit{or}$$

$$\frac{b_1}{\gcd(a_1, b_1)} \neq \frac{b_2}{\gcd(a_2, b_2)} \quad \textit{and} \quad \frac{a_1}{\gcd(a_1, b_1)} \neq \frac{b_2}{\gcd(a_2, b_2)}, \quad \textit{or}$$

$$\frac{b_1}{\gcd(a_1, b_1)} \neq \frac{b_2}{\gcd(a_2, b_2)} \quad \textit{and} \quad \frac{b_1}{\gcd(a_1, b_1)} \neq \frac{a_2}{\gcd(a_2, b_2)}.$$

PROOF. Let h_j be $\gcd(a_j, b_j)$ $(j = 1, 2)$, and let $a_j = h_j c_j$ and $b_j = h_j d_j$ $(j = 1, 2)$. Then the triple of integers $(1, a_j, b_j)$ can be written as $(1, h_j c_j, h_j d_j)$ $(j = 1, 2)$. Denote by Σ_j the corresponding Brieskorn manifold to $P_j(Z) = 0$. The S^1-actions on Σ_j is given by $t(Z_0, Z_1, Z_2) = (e^{2\pi h_j c_j d_j t i} Z_0, e^{2\pi d_j t i} Z_1, e^{2\pi c_j t i} Z_2)$ $(j = 1, 2)$. It is easy to see that the exceptional orbits of Σ_j occur among $Z_1 = 0$ or $Z_2 = 0$, and they have $\boldsymbol{Z}_{c_j}$ and $\boldsymbol{Z}_{d_j}$ $(j = 1, 2)$ as their isotropy groups respectively. There is precisely one orbit in each case. Now if there exists a diffeomorphism $f: \Sigma_1 \to \Sigma_2$ such that $f^*\eta_2 = \eta_1$, the slice diagrams $\Delta(S^1, \Sigma_1)$ and $\Delta(S^1, \Sigma_2)$ must coincide. $\Delta(S^1, \Sigma_j)$ contains the slice types $\{[\boldsymbol{Z}_{c_j}; \sigma_{d_j}]$ and $[\boldsymbol{Z}_{d_j}; \sigma_{c_j}]\}$ $(j = 1, 2)$. $\Delta(S^1, \Sigma_1) = \Delta(S^1, \Sigma_2)$ if and only if either $[\boldsymbol{Z}_{c_1}; \sigma_{d_1}] = [\boldsymbol{Z}_{c_2}; \sigma_{d_2}]$ and $[\boldsymbol{Z}_{d_1}; \sigma_{c_1}] = [\boldsymbol{Z}_{d_2}; \sigma_{c_2}]$ or $[\boldsymbol{Z}_{c_1}; \sigma_{d_1}] = [\boldsymbol{Z}_{d_2}; \sigma_{c_2}]$ and $[\boldsymbol{Z}_{d_1}; \sigma_{c_1}] = [\boldsymbol{Z}_{c_2}; \sigma_{d_2}]$ holds. Since c_i and d_i are relatively prime $(i = 1, 2)$, the above is equivalent to either $c_1 = c_2$ and $d_1 = d_2$ or $c_1 = d_2$ or $d_1 = c_2$ holds. Thus if either $c_1 \neq c_2$ and $c_1 \neq d_2$, or $c_1 \neq c_2$ and $d_1 \neq c_2$, or $d_1 \neq d_2$ and $c_1 \neq d_2$ or $d_1 \neq d_2$ and $d_1 \neq c_2$, then η_1 and η_2 cannot be strictly equivalent. This completes the proof.

q.e.d.

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