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SOME THEOREMS ON (*CA*) ANALYTIC GROUPS II

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Abstract. An analytic group G is called (CA) if the group of inner automorphisms of G is closed in the Lie group of all (bicontinuous) automorphisms of G. It has been previously proved by this author that each non-(CA) analytic group G can be densely immersed in a (CA) analytic group H, such that the center of G is closed in H. We now show that there is no (CA) analytic group "smaller" than H into which G can be densely immersed, but H, however, is not the "smallest" such (CA) analytic group. Furthermore, we will isolate those properties of H which determine it uniquely up to dimension, diffeomorphism, diffeomorphism together with local isomorphism, and finally isomorphism.

1. Introduction. By an analytic group and an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. If G and H are Lie groups and φ is a one-to-one (continuous) homomorphism from G into H, φ will be called an immersion. φ will be called closed or dense, as $\varphi(G)$ is closed or dense in H. G_0 and $Z(G)$ will denote the identity component group and center of G, respectively.

If G is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of G, topologized with the generalized compact-open topology. G will be called (CA) if $I(G)$, the Lie group of all inner automorphisms of G, is closed in $A(G)$. It is well known that G is (CA) if and only if its universal covering group is (CA).

If G is a normal analytic subgroup of an analytic group H, then each element h of H induces an automorphism of G, namely, $g \mapsto hgh^{-1}$. We will denote this homomorphism from H into $A(G)$ by ρ_{GH}. $I_H(h)$ will denote the inner automorphism of H determined by $h \in H$. More generally, if A is a subset of H, $I_H(A)$ will denote the set of all inner automorphisms of H determined by elements of A. $I_H(H)$ will be written as $I(H)$, and the mapping $h \mapsto I_H(h)$ of H onto $I(H)$ will be denoted by I_H.

If N is an analytic group and H is an analytic subgroup of $A(N)$, then $N \circledS H$ will denote the semidirect product of N and H. On the other hand, if G is an analytic group containing a closed normal analytic subgroup N and a closed analytic subgroup H, such that $G = NH$,

$N \cap H = \{e\}$, and such that the restriction of ρ_{NG} to H is one-to-one, we will frequently identify G with $N \circledS \rho_{NG}(H)$ and H with $\rho_{NG}(H)$, that is, we may write $G = N \circledS H$.

In Zerling [5] we proved the following theorem.

MAIN STRUCTURE THEOREM. *Let G be a non-(CA) analytic group. Then there exist a (CA) analytic group M, a toral group T in $A(M)$, and a dense vector subgroup V of T, such that:*

(i) *$H = M \circledS T$ is a (CA) analytic group.*

(ii) *G is isomorphic to the dense analytic subgroup $M \circledS V$ of H.*

(iii) *$Z(G)$ is contained in M.*

(iv) *$Z_0(G) = Z_0(H)$, and $\pi(Z(H))$ is finite, where π is the natural projection of H onto T. Moreover, if $G/Z(G)$ is homeomorphic to Euclidean space, then $Z(G) = Z(H)$.*

(v) *Each automorphism σ if G can be extended to an automorphism $\varepsilon(\sigma)$ of H, such that $\varepsilon: A(G) \rightarrow A(H)$ is a closed immersion.*

In Section 2 we show that there is no (CA) analytic group "smaller" than H into which G can be densely immersed, but H, however, is not the "smallest" such (CA) analytic group. In Section 3 we will isolate those properties of H which determine it uniquely up to dimension, diffeomorphism, diffeomorphism together with local isomorphism, and finally isomorphism.

2.

LEMMA 2.1. *Maintaining the notation in the Main Structure Theorem, we have that $Z(G)$ is of finite index in $Z(H)$.*

PROOF. Simple calculation reveals that $Z(M \circledS T) = \{(m, \tau): \tau = I_M(m^{-1}), \bar{\tau}(m) = m \text{ for all } \bar{\tau} \in T\}$. Now let $\tau_1, \tau_2, \cdots, \tau_k$ be the k distinct elements in $\pi(Z(H))$. Then there exist k distinct elements $m_1, m_2, \cdots, m_k$ in M, such that $\tau_i = I_M(m_i^{-1})$, and $(m_i, \tau_i) \in Z(H)$. Let $(m, \tau) \in Z(H)$. Then $\tau = I_M(m^{-1})$. Hence, $m = zm_1$, $z \in Z(M)$. Therefore, $z = m_1^{-1}m$ and $\bar{\tau}(z) = \bar{\tau}(m_1^{-1}m) = \bar{\tau}(m_1^{-1}) \cdot \bar{\tau}(m) = m_1^{-1}m = z$ for all $\bar{\tau} \in T$. So $z = (z, e) \in Z(G)$. Hence each $(m, \tau) \in Z(H)$ can be written as $(m, \tau) = (m_i, \tau_i) \cdot z$, $z \in Z(G)$. Letting $A = \{(m_i, \tau_i): i = 1, 2, \cdots, k\}$ we have $Z(H) = Z(G) \cdot A$, that is, $Z(G)$ is of finite index in $Z(H)$.

THEOREM 2.1. *Let G and H represent the groups in the Main Structure Theorem and let $\psi: G \rightarrow H$ be the given dense immersion. Suppose that L is a (CA) analytic group and $\alpha: G \rightarrow L$ is an immersion for which there exists a dense immersion $\varphi: L \rightarrow H$, such that $\psi = \varphi \circ \alpha$. Then H is isomorphic to L.*

PROOF. Since L is (CA), $H = \varphi(L)\cdot Z(H)$ from van Est [4, Theorem 2.2.1]. But $Z(H) = \psi(Z(G))\cdot A$, where A is a finite set, from Lemma 2.1. Therefore, $H = \varphi(L)\cdot A$. Since $\varphi(L)$ is of finite index in H, $H = \varphi(L)$, that is, H is isomorphic to L.

THEOREM 2.2. *Let us maintain the notation of the Main Structure Theorem and let $\psi: G \to H$ be the given dense immersion. Then there exist a (CA) analytic group P and a dense immersion $\beta: G \to P$ for which there is no homomorphism $\varphi: H \to P$, such that $\beta = \varphi \circ \psi$.*

PROOF. The construction of our (CA) analytic group P will be based on the proof of the proposition in Goto [1]. Let $T' = \rho_{GH}(T)$. Since $\tau(m, v)\tau^{-1} = (\tau(m), v)$ for $m \in M$, $v \in V$, $\tau \in T$, we see that ρ_{GH} is 1-1 on T. Let $S = G \circledS T'$. We first show that S is a (CA) group.

Simple calculation reveals that $Z(G \circledS T') = \{(g, \tau'): \tau' = I_G(g^{-1}), \tau''(g) = g$ for all $\tau'' \in T'\}$. However, if $I_G(g^{-1}) \in T'$, then $I_G(g^{-1})$ commutes with all elements of T'. Since T' keeps $Z(G)$ elementwise fixed, we have $\tau''(g) = g$ for all $\tau'' \in T'$ by Lemma 4 of Goto [1]. Therefore $Z(S) = \{(g, \tau'): \tau' = I_G(g^{-1})\}$.

Since $I_G(V)$ is contained in T', we see that $\{(v, I_G(v^{-1})): v \in V\} \subset Z(S)$. Therefore, $I_S(v) = I_S(I_G(v)) \in I_S(T')$ for all $v \in V$. Thus, $I_S(V) \subset I_S(T')$ and so $I(S) = I_S(M)\cdot I_S(V)\cdot I_S(T') = I_S(M)\cdot I_S(T')$. Hence S will be (CA) if we can show that $I_S(M)$ is closed in $I(S)$.

To this end let $\{I_S(m_n)\}$ converge to σ in $A(S)$, where m_n is in M for all n. Since $I_S(m_n)(G) = G$ for all n, $\sigma|_G \in A(G)$. Since $\{I_G(m_n)\}$ converges to $\sigma|_G$ in $A(G)$, and since $I_G(M)$ is closed in $A(G)$ from the proof of Theorem 2.1 in Zerling [5], we have $\sigma|_G = I_G(\bar{m})$ for some $\bar{m} \in M$.

We now want to show that $\sigma = I_S(\bar{m})$. Let $v \in V$ and let $v' = I_G(v) \in S$. Then $\{I_S(m_n)(v')\}$ converges to $\sigma(v')$ in S. But $I_S(m_n)(v') = (m_n v'(m_n^{-1}), v') = (m_n v m_n^{-1} v^{-1}, v')$, and $\{m_n v m_n^{-1}\}$ converges in G to $I_G(\bar{m})(v)$. Hence, $\{I_S(m_n)(v')\}$ converges to $(\bar{m} v \bar{m}^{-1} v^{-1}, v') = (\bar{m} v'(\bar{m}^{-1}), v') = I_S(\bar{m})(v')$. Therefore, $\sigma(v') = I_S(\bar{m})(v')$ and so $\sigma(\tau') = I_S(\bar{m})(\tau')$ for all $\tau' \in T'$. Thus, $\sigma = I_S(\bar{m})$ and $I_S(M)$ is closed in $A(S)$. This proves that S is (CA).

By Goto [3; p. 163] we can find some $v_0' \in V'$, such that v_0' generates a dense subgroup of T'. Let $I_G(v_0) = v_0'$. Let D denote the subgroup of S generated by $(v_0, v_0'^{-1})$. Since $\{v_0^n\}$ is free and discrete in V, D will be a free discrete central subgroup of S.

Let $P = (G \circledS T')/D$. Then the homomorphism $\beta: G \to P$ given by $g \mapsto (g, e)D$ is a proper dense immersion. Now suppose that there exists a homomorphism $\varphi: H \to P$, such that $\beta = \varphi \circ \psi$. Since $H \cong M \circledS T$ and $T \cong T'$, clearly $\dim H < \dim P$. We will now show that $\varphi(H)$ is closed in P, which leads to a contradiction.

Since $\varphi(H) = \varphi(M)\cdot\varphi(T)$, we need only show that $\varphi(M)$ is closed. However, $\varphi(M) = \beta(M) = \{(m, e)D: m \in M\}$, and $\varphi(M)$ is closed in P if and only if $\delta^{-1}(\beta(M))$ is closed in S, where $\delta: S \to P$ is the canonical homomorphism. But $\delta^{-1}(\beta(M)) = MD$ is closed in the topological space $M \times V \times T'$, since D is closed in $V \times T'$. Hence $\varphi(M)$ is closed in P and so $\varphi(H)$ is a proper closed subgroup of P. This completes the proof of our theorem.

3.

LEMMA 3.1. *Let L be an analytic group. Let M and H be a closed normal analytic subgroup and a closed abelian analytic subgroup of L, respectively, such that $L = MH$, $M \cap H = \{e\}$. Let G be a dense analytic subgroup of L and let S be a subset of H. Then $\rho_{ML}(S)$ is closed in $A(M)$ if and only if $\rho_{GL}(S)$ is closed in $A(G)$.*

PROOF. Let ψ and φ denote the respective restrictions of ρ_{ML} and ρ_{GL} to H. For each α in $\overline{\psi(H)}$ let $E\alpha$ denote the automorphism of L defined by $(E\alpha)(m, h) = \alpha(m)\cdot h$. Then $\alpha \mapsto E\alpha$ is a closed immersion of $\overline{\psi(H)}$ into $A(L)$.

Let $\tilde{L}$ and $\tilde{G}$ be the universal covering groups of L and G, respectively, and let $\pi: \tilde{L} \to L$ be the natural projection. For $\alpha \in \overline{\psi(H)}$ let $(E\alpha)'$ denote the unique automorphism of $\tilde{L}$, such that $\pi \circ (E\alpha)' = (E\alpha) \circ \pi$. Since $\tilde{G}$ is closed and normal in $\tilde{L}$, each $(E\alpha)'$ keeps $\tilde{G}$ invariant. Therefore, each $E\alpha$ keeps G invariant.

Hence $\alpha \mapsto (E\alpha)|_G$ is a closed immersion of $\overline{\psi(H)}$ into $A(G)$. Since $\varphi(h) = (E(\psi(h)))|_G$, $\psi(S)$ is closed in $A(M) \Leftrightarrow \psi(S)$ is closed in $\overline{\psi(H)} \Leftrightarrow (E(\psi(S)))|_G$ is closed in $A(G) \Leftrightarrow \varphi(S)$ is closed in $A(G)$.

LEMMA 3.2. *Let G be a dense analytic subgroup of a (CA) analytic group L. Then $\rho_{GL}(L) = \overline{I(G)}$.*

PROOF. Suppose that $\rho_{GL}(L)$ is not closed in $A(G)$. We may then appeal to [2]: Let N be a maximal analytic subgroup of $\rho_{GL}(L)$, which contains the commutator subgroup of $\rho_{GL}(L)$ and is closed in $A(G)$. Then there is a closed vector subgroup V' of $\rho_{GL}(L)$ such that $\rho_{GL}(L) = N\cdot V'$, $N \cap V' = \{e\}$, and $\overline{\rho_{GL}(L)} = N\cdot \bar{V}'$, where $\bar{V}'$ is toral group. Hence, each one dimensional vector subgroup of V' is not closed in $A(G)$. Let $V' = V'_q\cdot V'_{q-1} \cdots V'_1$ be a direct product decomposition of V' into one dimensional subgroups: $\rho_{GL}(L) = N\cdot V'_q\cdot V'_{q-1} \cdots V'_1$.

For $\rho_{GL}: L \to \rho_{GL}(L)$ let M and H_i, $1 \leqq i \leqq q$, denote the identity component groups of the complete inverse images of N and V'_i, respectively. M is closed and normal in L, and each H_i is closed in L. Moreover, $L = M \cdot H_q \cdot H_{q-1} \cdots H_1$, where $M \cap H_i$ is contained in $Z(L)$ for each i. The restriction of ρ_{GL} to H_i is a homomorphism of H_i onto V_i' having kernel $Z(L) \cap H_i$. Therefore, $Z(L) \cap H_i$ is connected, and so it is contained in M. Also

$$H_i = (Z(L) \cap H_i) \cdot V_i , \quad Z(L) \cap H_i \cap V_i = Z(L) \cap V_i = \{e\} ,$$

where V_i is a one dimensional closed vector subgroup of H_i, such that $\rho_{GL}(V_i) = V_i'$. Therefore,

$$L = M(Z(L) \cap H_q) \cdot V_q \cdots (Z(L) \cap H_1) \cdot V_1 = M \cdot V_q \cdot V_{q-1} \cdots V_1 .$$

If $\rho_{GL}(mv_q \cdots v_1) = e$, then $\rho_{GL}(m) \cdot \rho_{GL}(v_q) \cdots \rho_{GL}(v_1) = e$. Therefore

$$\rho_{GL}(m) = \rho_{GL}(v_q) = \cdots = \rho_{GL}(v_1) = e .$$

Since $Z(L) \cap V_i = \{e\}$, we have $v_q = \cdots = v_1 = e$. Hence, $Z(L)$ is contained in M. In the same way we see that each element x in L can be written uniquely in the form $x = mv_qv_{q-1} \cdots v_1$, $m \in M$, $v_i \in V_i$. Therefore, L is homeomorphic to $M \times V_q \times V_{q-1} \times \cdots \times V_1$.

Let $M_2 = MV_q \cdot V_{q-1} \cdots V_2$. M_2 is closed and normal in L, and $L = M_2 V_1$, $M_2 \cap V_1 = \{e\}$. Let $\psi_1 \colon V_1 \to A(M_2)$ be given by $\psi_1(v_1)(m_2) = v_1 m_2 v_1^{-1}$. Since $Z(L)$ is contained in M_2, and since V_1 is abelian, we see that ψ_1 is an immersion. From Lemma 3.1 we see that $\psi_1(V_1)$ is not closed in $A(M_2)$, since $\rho_{GL}(V_1) = V_1'$ is not closed in $A(G)$. Consider $M_2 \circledS \overline{\psi_1(V_1)}$, where $\overline{\psi_1(V_1)}$ is the closure of $\psi_1(V_1)$ in $A(M_2)$. L is properly dense in $M_2 \circledS \overline{\psi_1(V_1)}$. Since $Z(L)$ is contained in M_2, and since L is (CA), we have a contradiction by van Est [4, Theorem 2.2.1]. Hence $\rho_{GL}(L) = \overline{I(G)}$.

COROLLARY. *Let us maintain the notation of the Main Structure Theorem and let L be a (CA) analytic group containing G as a dense analytic subgroup.* Then $\dim L = \dim H + \dim Z(L) - \dim Z(G) \geqq \dim H$.

PROOF. Since $H/Z(H) \cong \overline{I(G)} \cong L/Z(L)$, and $\dim Z(H) = \dim Z(G) \leqq \dim Z(L)$, we have our result.

THEOREM 3.1. *Let us main the notation of the Main Structure Theorem and let L be an analytic group with the following properties, which we know to be exhibited by H.*

(i) *L is (CA).*

(ii) *There is a dense immersion $f \colon G \to L$.*

(iii) *$Z(f(G))$ is of finite index in $Z(L)$.*

Then L is diffeomorphic to H, and $Z(f(G))$ is closed in L.

PROOF. Since G is non-(CA) we can appeal to Goto [2]: Let N be a maximal analytic subgroup of $I(G)$, which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$. Then there is a closed vector subgroup V' of $I(G)$, such that $I(G) = N \cdot V'$, $N \cap V' = \{e\}$, and $\overline{I(G)} = N \cdot \bar{V}'$, where $\bar{V}'$ is a toral group. Moreover, $N \cap \bar{V}'$ is finite, and the space of $\overline{I(G)}$ is diffeomorphic to the product space $N \times \bar{V}'$. In the proof of the Main Structure Theorem in Zerling [5], H is constructed in such a way that $\rho_{GH}(M) = N$, $\rho_{GH}(V) = V'$, and $\rho_{GH}(T) = T' = \bar{V}'$. Moreover, ρ_{GH} is 1-1 on T.

Therefore, since $\rho_{GL}^{-1}(N) = f(M) \cdot Z(L)$, and $Z(L) = Z(G) \cdot F$, where F is a finite set, we have $\rho_{GL}^{-1}(N) = f(M) \cdot F$ because $Z(G)$ is contained in M from the Main Structure Theorem. Hence $f(M)$ is the identity component group of $\rho_{GL}^{-1}(N)$, and so it is closed in L. Thus, $Z(f(G))$ is closed in L.

Since $\rho_{GL}(L) = \overline{I(G)}$ from Lemma 3.2, there is a unique closed immersion $\varepsilon' \colon \overline{I(G)} \to A(L)$ such that the following diagram commutes:

$$\begin{array}{ccccc} H & \xrightarrow{\rho_{GH}} & \overline{I(G)} & \xrightarrow{\varepsilon'} & A(L) \\ \psi \uparrow & & \uparrow \rho_{GL} & \nearrow I_L & \\ G & \xrightarrow{f} & L & & \end{array}$$

Because $f(M)$ is closed in L, and $f(G)$ and L have the same commutator subgroup, there exists a maximal analytic subgroup J of $f(G)$, which contains the commutator subgroup of $f(G)$ and is closed in L, so that from Goto [2] we have $L = J \cdot T''$, where T'' is a toral group, and $J \cap T''$ is finite. Moreover, the space of L is diffeomorphic to the space of $J \times T''$. We will show that J may be taken to be $f(M)$.

There exists such a group J containing $f(M)$; assume that this containment is proper. Since $f(M)$ is the identity component group of $\rho_{GL}^{-1}(N)$, we see that $\rho_{GL}(J)$ properly contains N. Hence, $\rho_{GL}(J)$ is not closed in $A(G)$ by the maximality of N. N is also the maximal analytic subgroup of $\rho_{GL}(J)$, which contains the commutator of $\rho_{GL}(J)$ and is closed in $A(G)$.

Following Goto [2] there exists a closed vector subgroup W' of $\rho_{GL}(J)$ so that $\rho_{GL}(J) = N \cdot W'$, $N \cap W' = \{e\}$, and $\mathrm{Cl}_{A(G)} W'$ is a toral group. Let $W' = W'_q \cdots W'_1$ be a direct product decomposition of W' into one dimensional subgroups:

$$\rho_{GL}(J) = N \cdot W'_q \cdot W'_{q-1} \cdots W'_1 \, .$$

Since $\ker \rho_{GL}|_J = J \cap Z(L)$, we may repeat the technique of Lemma 3.2 in order to construct closed one dimensional vector subgroups $W_1, W_2, \cdots, W_q$ of J, such that $J = f(M) \cdot W_q \cdot W_{q-1} \cdots W_1$, where $\rho_{GL}(W_i) = W'_i$ and $J \cap Z(L)$ is contained in $f(M)$. Moreover, each element $x \in J$ can be written uniquely in the form $x = f(m)\cdot w_q \cdots w_1$, $m \in M$, $w_i \in W_i$. Therefore, J is homeomorphic to $f(M) \times W_q \times \cdots \times W_1$. In particular $W_q \cdot W_{q-1} \cdots W_1$ is closed in J, and so it is closed in L. We will now show that $W = W_q \cdots W_1$ is actually a closed vector subgroup of L.

We have $L = J\cdot T'' = (f(M)\cdot W)\cdot T'' = (f(M)\cdot T'')\cdot W$ where $f(M)\cdot T''$ is a closed analytic subgroup of L. Since $T'' \cap J$ is finite and contained in $f(G)$, it is contained in $f(M)$. Hence, if $(f(M)\cdot T'') \cap W \neq \{e\}$, then $w = f(m)\cdot \tau''$, and so $\tau'' = f(m)^{-1}\cdot w$. Hence $\tau'' \in T'' \cap J$, which is contained in $f(M)$. By the uniqueness of the decomposition in J, we have $w = e$. So $L = (f(M)\cdot T'')\cdot W$, $(f(M)\cdot T'') \cap W = \{e\}$. Moreover, $W \cap Z(L) = \{e\}$, since $J \cap Z(L)$ is contained in $f(M)$, and $W \cap f(M) = \{e\}$.

Let $Y_2 = f(M)\cdot T''\cdot W_q \cdots W_2$. Y_2 is closed and normal in L, since $f(M)\cdot W_q \cdots W_2$ is closed in J, and $L = Y_2\cdot W_1$, $Y_2 \cap W_1 = \{e\}$. Let φ_1: $W_1 \to A(Y_2)$ be given by $\varphi_1(w_1)(y_2) = w_1 y_2 w_1^{-1}$. Since $W \cap Z(L) = \{e\}$ and since W_1 is abelian, we see that φ_1 is an immersion.

Since W_1' is not closed in $\overline{I(G)}$, $\varepsilon'(W_1') = I_L(W_1)$ is not closed in $A(L)$. Hence, $\varphi_1(W_1)$ is not closed in $A(Y_2)$ by Lemma 3.1. Consider $Y_2 \circledS \overline{\varphi_1(W_1)}$. Let $w_1 \in W_1$ and $w_j \in W_j$, $2 \leqq j \leqq q$. Then $I_L(\varphi_1(w_1)(w_j)) = I_L(w_1 w_j w_1^{-1}) = \varepsilon'(\rho_{GL}(w_1 w_j w_1^{-1})) = I_L(w_j)$, since W' is abelian. Therefore, $(\varphi_1(w_1)(w_j))\cdot w_j^{-1} \in Z(L)$. Hence, $\sigma(w_j)\cdot w_j^{-1}$ is in $Z(L) \cap Y_2$ for all $\sigma \in \overline{\varphi_1(W_1)}$.

Since $Z(L) \cap Y_2$ is a closed central subgroup of Y_2, and each element of $\overline{\varphi_1(W_1)}$ keeps $Z(L) \cap Y_2$ elementwise fixed, we see by Lemma 2.2 of Zerling [5] that $\sigma(w_j) = w_j$ for each $\sigma \in \overline{\varphi_1(W_1)}$ and each w_j in W_j, $2 \leqq j \leqq q$; in particular, $w_1 w_j = w_j w_1$.

Since $L = f(M)\cdot T'' W_{\pi(q)} \cdots W_{\pi(1)}$ for each permutation π on $\{1, 2, \cdots, q\}$, we can show that $w_i w_j = w_j w_i$ for all $w_i \in W_i$, $w_j \in W_j$, $1 \leqq i, j \leqq q$. Hence $W = W_q \cdots W_1$ is a closed vector subgroup of L, which is isomorphic to W' under ρ_{GL}. Hence, $L = (f(M)\cdot T'') \circledS W$.

Let φ: $W \to A(f(M)\cdot T'')$ be given by $\varphi(w)(y) = wyw^{-1}$. φ is an immersion. Since W' is not closed in $\overline{I(G)}$, we see as before that $\varphi(W)$ is not closed in $A(f(M)\cdot T'')$. In fact, each one parameter subgroup of $\varphi(W)$ is not closed in $A(f(M)\cdot T'')$; therefore, $\overline{\varphi(W)}$ is a toral group.

Next let $z \in Z(L)$. Then $z = z'\cdot b$, $z' \in Z(G)$, $b \in F$. Therefore, $z = z'\cdot f(m)\cdot \tau''\cdot w$, $f(m) \in f(M)$, $\tau'' \in T''$, $w \in W$. But $z' f(m) = f(m_1)$ for some $m_1 \in M$. So $z = f(m_1)\cdot \tau''\cdot w$. Since F is *finite*, the projection of $Z(L)$ into W is finite, and, therefore, trivial. So $w = e$ and we have that $Z(L)$ is contained in $f(M)\cdot T''$.

Therefore, L is properly dense in $L' = (f(M)\cdot T'') \circledS \overline{\varphi(W)}$, and $Z(L)$ is closed in L'. This contradicts the fact that L is (CA) by van Est [4; Theorem 2.2.1]. Hence $J = f(M)$ and so $L = f(M)\cdot T''$, and $f(M) \cap T''$ is finite. Therefore, the space of L is diffeomorphic to the space of $f(M) \times T''$ by Goto [2]. However, $\dim \overline{I(G)} = \dim H - \dim Z(H) = \dim H - \dim Z(G) = \dim M + \dim T - \dim Z(G)$, and $\dim \overline{I(G)} = \dim L - \dim Z(L) = \dim L - \dim Z(G) = \dim f(M) + \dim T'' - \dim Z(G)$. Thus, $\dim T = \dim T''$, and H is then diffeomorphic to L.

REMARK. We have actually proved more than what was stated in Theorem 3.1. If $Z(f(G))$ is of countably infinite index in $Z(L)$, then $f(M)$ is still closed in L (see the proof of Theorem 3.4) and we still have $L = (f(M) \cdot T'') \circledS W$. To show that $W = \{e\}$, however, requires that "countably infinite" be replaced by "finite".

If the index of $Z(f(G))$ in $Z(L)$ is not at most countably infinite, then the dimension of L may actually exceed the dimension of H, as is seen in the construction of P in Theorem 2.2.

THEOREM 3.2. *Let us maintain the hypothesis and notation of Theorem* 3.1, *and let $Z(G)$ be compact. Then L is also locally isomorphic with H.*

PROOF. $\rho_{GL}: L \to \overline{I(G)}$ is now a closed mapping. Therefore, $\overline{f(V)}$ is a toral group, since each one parameter subgroup of $\overline{f(V)}$ is not closed in L because each one parameter subgroup of $\rho_{GL}(f(V)) = V'$ is not closed in $\overline{I(G)}$. Hence, $L = f(M) \cdot \overline{f(V)}$. Let T_1 denote the identity component group of $f(M) \cap \overline{f(V)}$. Then there is a toral subgroup T_2 of $\overline{f(V)}$ so that $\overline{f(V)} = T_1 \cdot T_2$, $T_1 \cap T_2 = \{e\}$. Therefore, $L = f(M) \cdot T_2$ and $f(M) \cap T_2$ is finite. Now $T' = \overline{\rho_{GH}(V)} = \rho_{GL}(\overline{f(V)}) = \rho_{GL}(T_1 \cdot T_2) = \rho_{GL}(T_1) \cdot \rho_{GL}(T_2)$. But $\rho_{GL}(T_1)$ is contained in the finite group $N \cap T'$. Therefore $\rho_{GL}(T_1) = \{e\}$ and so $T' = \rho_{GL}(T_2)$. Since $\dim Z(L) = \dim Z(G)$, we see that $\dim T_2 = \dim T'$. Hence $T_2 \cap Z(L)$ must be discrete and, therefore, finite.

Since $f(M) \cap T_2$ is finite, we can find neighborhoods A of e in $f(M)$ and B of e in T_2 so that $A \cap B = \{e\}$ and $U = AB$ is open in L. Moreover, each $u \in U$ can be written uniquely as $u = a \cdot b$, $a \in A$, $b \in B$. U can be assumed symmetric and since $T_2 \cap Z(L)$ is finite, U can be selected so that $U^2 \cap T_2 \cap Z(L) = \{e\}$.

Since ρ_{GH} is 1-1 on T, for $f(m) \in A$ and $b \in B$ we can define $\beta: U \to H$ as follows:

$$\beta(f(m), b) = (\psi(m), \rho_{GH}^{-1}(\rho_{GL}(b))) .$$

Hence L is diffeomorphic and locally isomorphic with H.

THEOREM 3.3. *Let us maintain the hypothesis and notation of Theorem* 3.1 *and let G have trivial center and be homeomorphic to*

Euclidean space. If L possesses the property (possessed by H) that $Z(L)$ is trivial, then $H \cong L$.

PROOF. $H \cong \overline{I(G)} \cong L$ from Lemma 3.2.

THEOREM 3.4. *Let us maintain the hypothesis and notation of Theorem* 3.1, *except let $Z(f(G))$ be of countably infinite index in $Z(L)$. Then* $\dim L = \dim H$ *and $Z(f(G))$ is closed in L.*

PROOF. Let Q denote the identity component group of $\rho_{GL}^{-1}(N)$. Then since $Z(f(G))$ is of countably infinite index in $Z(L)$, and $f(M)$ contains $Z(f(G))$, we see that $Q = f(M) \cdot C$, where C is a countable set. By going to the universal covering group of Q, where analytic normal subgroups are closed, we see that $Q = f(M)$. Hence $f(M)$ and, therefore, $Z(f(G))$ are closed in L. Since $Z(L)$ is now a countable union of closed subsets, we see that $\dim Z(L) = \dim Z(G)$. Thus, $\dim L = \dim H$ by the corollary to Lemma 3.2.

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