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THE REPRESENTATION THEOREM AND THE H^p SPACE THEORY ASSOCIATED WITH SEMIGROUPS ON LIE GROUPS

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Introduction. In [7] Stein has considered one-parameter semigroups of operators $\{T^t\}_{t\geqq 0}$ defined simultaneously on all spaces $L^p(G)$, $1\leqq p\leqq\infty$, for a Lie group G, which satisfy the following properties:

(a) $\|T^tf\|_p\leqq\|f\|_p$.
(b) T^t is a self-adjoint operator on $L^2(G)$.
(c) $T^tf\geqq 0$ for $f\geqq 0$.
(d) $T^t1=1$.

Leading examples of such a semigroup are the heat diffusion semigroup and the Poisson semigroup. Our purpose is to develop the analogue of results in classical harmonic analysis in the context of these semigroups.

In Section 2, we state the known facts about these semigroups, which are due to [2], [3], [5] and [7], and we prove the convergence theorems by using these facts.

In Section 3, we shall obtain results analogous to the classical properties of harmonic functions and subharmonic functions. The main result in this section is the representation theorem for harmonic functions. The basic tool which is used there is the maximum (minimum) principle for the heat equation and the Laplace equation on Lie groups.

In Section 4, we study the H^p space theory in a noncompact Lie group, analogous to classical one, which has been developed in Coifman and Weiss [1] for the case of a compact Lie group. In this section, to obtain an extension of the theorem of F. and M. Riesz, we apply the idea of a theorem concerning nontangential boundedness.

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1. Notations. In this section we fix notations which will be used in Sections 2-4. If S is a topological space, $C_b(S)$ denotes the set of all bounded continuous real valued functions on S. The set of all f in $C_b(S)$ which vanish at infinity is denoted by $C_0(S)$. The set of all f in

$C_b(S)$ whose support is compact is denoted by $C_c(S)$. If S is a differentiable manifold, the set of all real valued m-times continuous differentiable functions on S is denoted by $C^m(S)$. $C^\infty(S)$ denotes the space of real valued indefinitely differentiable functions in S. The set of functions f in $C^\infty(S)$ of compact support is denoted by $C_c^\infty(S)$. If S is a locally compact Hausdorff space, $M(S)$ denotes the set of all real valued, bounded, regular Borel measures on S. We write the total variation measure and the total variation norm of μ in $M(S)$ by $|\mu|$ and $||\mu||$ respectively. If G is a locally compact group and dx is a right invariant Haar measure on G, then if $1 \leqq p < \infty$, $L^p(G)$ denotes the set of all real valued Borel functions f on G for which the norm

$$||f||_p = \left(\int_G |f(x)|^p dx\right)^{1/p}$$

is finite. $L^\infty(G)$ is the space of all essentially bounded real valued Borel functions on G, normed by $||f||_\infty = \text{ess. sup}\,|f(x)|$. If E is a measurable subset of G, we write

$$|E| = \int_E dx .$$

Let G be a connected Lie group with dimension n. By $\mathfrak{G}$ we denote its Lie algebra. A basis $\{X_1, \cdots, X_n\}$ in $\mathfrak{G}$ defines a Riemannian structure on G:

$$g_x(X_x, Y_x) = \sum_{j=1}^n a_j b_j \quad \text{if} \quad X = \sum_{j=1}^n a_j X_j , \quad Y = \sum_{j=1}^n b_j X_j .$$

This defines a Riemannian metric $d(x, y)$ on G by

$$d(x, y) = \inf_\gamma \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt ,$$

where the infimum is taken over all C^1-curve $\gamma: [0, 1] \rightarrow G$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma}(t_0)$ is the element of the tangent space at $\gamma(t_0)$ defined by

$$\dot{\gamma}(t_0)f = \frac{d}{dt} f(\gamma(t))\,|_{t=t_0} .$$

Then $d(xz, yz) = d(x, y)$ for all x, y, z in G,

$$d(x, y) \leqq d(x, z) + d(z, y) , \qquad x, y, z \text{ in } G .$$

Let $d(x) = d(x, e)$: then we have

$$d(xy) \leqq d(x) + d(y) \quad \text{and} \quad d(x) = d(x^{-1})$$

e means the identity of G throughout this paper.

2. The convergence theorems. Let G be an (unimodular) connected Lie group with dimension n. By $\mathfrak{G}$ we denote its Lie algebra as the space of differential operators of the first order acting on $C_c^\infty(G)$ and commuting with right translations. We select the basis $X_1, \cdots, X_n$ of the Lie algebra $\mathfrak{G}$ and we write

$$\varDelta = X_1^2 + \cdots + X_n^2 .$$

We consider $\varDelta$ as an operator on $L^2(G)$ with the domain $C_c^\infty(G)$ which is dense in $L^2(G)$. Let $\bar{\varDelta}$ be the closure of $\varDelta$ and let $D(\bar{\varDelta})$ be the domain of $\bar{\varDelta}$. Then $\bar{\varDelta}$ is an elliptic, self-adjoint and negative operator, and $\bar{\varDelta}$ commutes with right translations (see [7], p. 35 and [2]). We can construct a semigroup $\{T^t\}_{t\geqq 0}$ corresponding to the operator $\varDelta$. This semigroup $\{T^t\}_{t\geqq 0}$ has the following properties (we refer the reder to [2], [3], [5] and [7] for these properties):

THEOREM 1. (i) *$\{T^t\}_{t\geqq 0}$ satisfies the semigroup axioms: $T^{t_1+t_2} = T^{t_1}T^{t_2}$ and $T^0 =$ the identity operator.*

(ii) *Each T^t is a bounded operator of norm less than 1 defined simultaneously on spaces $C_0(G)$, $L^p(G)(1 \leqq p \leqq \infty)$ and $M(G)$. Moreover, the norms of operators T^t are all 1 on the spaces $L^1(G)$, $L^2(G)$ and $L^\infty(G)$.*

(iii) *Each T^t is positive, that is, $f \geqq 0$ implies $T^t f \geqq 0$.*

(iv) *Each T^t is a self-adjoint operator on $L^2(G)$.*

(v) *$T^t 1 = 1$.*

(vi) *Each T^t commutes with right translations.*

(vii) *We put $R_\lambda = (\lambda - \bar{\varDelta})^{-1}$ for $\lambda > 0$;*
then the norm of the operator R_λ on $C_0(G)$ equals to $1/\lambda$, and for $\lambda > 0$ we have

$$R_\lambda f = \int_0^\infty e^{-\lambda t} T^t f dt$$

for all $f \in C_c^\infty(G)$. Moreover, we have

$$T^t f = \lim_{n\to\infty} \left(\frac{n}{t} R_{n/t}\right)^n f$$

for all $f \in C_c^\infty(G)$.

(viii) *If $f \in L^p(G)$, $1 \leqq p \leqq \infty$, then $(T^t f)(x) \in C^\infty(G \times (0, \infty))$ and*

$$\frac{\partial}{\partial t}(T^t f)(x) = \varDelta T^t f(x) .$$

If $\mu \in M(G)$, then $T^t \mu(x) \in C^\infty(G \times (0, \infty))$ and

$$\frac{\partial}{\partial t}(T^t \mu)(x) = \varDelta T^t \mu(x) .$$

(ix) *T^t may be written in the convolution form:*

$$T^t f(x) = K_t * f(x) = \int_G K_t(y) f(y^{-1}x) dy$$

for $0 < t < \infty$.

This kernel function $K_t(x)$ has the following properties:

(x) *Each K_t is a nonnegative function and symmetric, that is, $K_t(x) = K_t(x^{-1})$ for all x in G. Furthermore, we have*

$$K_{t_1} * K_{t_2}(x) = K_{t_1+t_2}(x) = K_{t_2} * K_{t_1}(x) .$$

(xi) *Each K_t belongs to $C_0(G) \cap L^p(G)$, $1 \leqq p \leqq \infty$. Moreover, we have*

$$\int_G K_t(y) dy = 1 .$$

(xii) *$K_t(x)$ is an analytic function of x for each $t > 0$, and*

$$K_t(x) \in C^\infty(G \times (0, \infty)) .$$

(xiii) *For each neighborhood U of the identity e in G,*

$$\lim_{t \to 0} \int_U K_t(y) dy = 1 .$$

We define the operator P^t by

$$P^t = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\lambda}}{\sqrt{\lambda}} T^{t^2/4\lambda} d\lambda \qquad (t > 0)$$

and P^0 = the identity operator. Then we have $P^t f = p_t * f$, for $t > 0$, where

$$\begin{aligned} p_t(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\lambda}}{\sqrt{\lambda}} K_{t^2/4\lambda} d\lambda \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty t e^{-t^2/4\lambda} \lambda^{-3/2} K_\lambda d\lambda . \end{aligned}$$

The operator P^t and the kernel function p_t satisfy the properties (i)-(vi) and (ix)-(xiii) in the above theorem. Instead of the property (viii), we have

$$\frac{\partial^2}{\partial t^2}(P^t f)(x) + \Delta P^t f(x) = 0 .$$

By using these properties we have the following convergence theorems for these semigroups.

THEOREM 2. *Let f be a measurable function or a measure on G. Then $\lim_{t \to 0} T^t f = f$ holds in the following senses;*

(i) *in the L^p norm if $f \in L^p(G)$, $1 \leqq p < \infty$.*

(ii) *in the weak star topology of* $L^\infty(G)$ *if* $f \in L^\infty(G)$.
(iii) *uniformly on each compact subset of* G *if* $f \in C_b(G)$.
(iv) *uniformly if* $f \in C_0(G)$.
(v) *in the weak star topology of* $M(G)$ *if* $f \in M(G)$.
(vi) *almost everywhere if* $f \in L^p(G)$, $1 < p \leqq \infty$.

PROOF. By Minkowski's inequality for integrals, we have, for a neighborhood U of e,

$$\begin{aligned}\|T^t f - f\|_p &\leqq \int_G \left(\int_G |f(y^{-1}x) - f(x)|^p dx\right)^{1/p} K_t(y)dy \\ &\leqq \int_U \left(\int_G |f(y^{-1}x) - f(x)|^p dx\right)^{1/p} K_t(y)dy \\ &\quad + 2\|f\|_p \int_{G-U} K_t(y)dy\end{aligned}$$

for all $f \in L^p(G)$, $1 \leqq p \leqq \infty$. The map $x \to f(y^{-1}x)$ is a uniformly continuous map of G into $L^p(G)$, $1 \leqq p < \infty$, and the integral $\int_{G-U} K_t(y)dy$ tends to zero as $t \to 0$. This fact completes the proof of (i). In order to prove (ii), we see that for all $\varphi \in L^1(G)$ and all $f \in L^\infty(G)$,

$$\begin{aligned}&\left|\int_G T^t f(x)\varphi(x)dx - \int_G f(x)\varphi(x)dx\right| \\ &= \left|\int_G\int_G K_t(y)(\varphi(yx) - \varphi(x))f(x)dydx\right| \\ &\leqq \|f\|_\infty \int_G\int_G K_t(y)|\varphi(yx) - \varphi(x)|dydx \\ &= \|f\|_\infty \int_G\int_G K_t(y)|\varphi(y^{-1}x) - \varphi(x)|dydx .\end{aligned}$$

By the proof of (i), the integral $\int_G\int_G K_t(y)|\varphi(y^{-1}x) - \varphi(x)|dydx$ tends to zero as $t \to 0$. Therefore, (ii) has been proved. If $f \in C_b(G)$, then f is uniformly continuous on each compact subset F in G, that is, given $\varepsilon > 0$ there exists a neighborhood U of e such that

$$|f(y^{-1}x) - f(x)| < \varepsilon/2$$

whenever $x \in F$ and $y \in U$. Therefore, we have

$$\begin{aligned}|T^t f(x) - f(x)| &\leqq \int_U |f(y^{-1}x) - f(x)|K_t(y)dy \\ &\quad + \int_{G-U}|f(y^{-1}x) - f(x)|K_t(y)dy \\ &< \varepsilon/2 + 2\|f\|_\infty \int_{G-U} K_t(y)dy\end{aligned}$$

whenever $x \in F$. Since the integral $\int_{G-U} K_t(y)dy$ tends to zero as $t \to 0$,

$$|T^t f(x) - f(x)| < \varepsilon$$

for a sufficiently small $t > 0$. Hence (iii) has been proved. If $f \in C_0(G)$, then it is uniformly continuous on G. In the same way as above, we can prove (iv). If $\mu \in M(G)$ and $\varphi \in C_0(G)$, then we have

$$\begin{aligned} &\left|\int_G T^t\mu(x)\varphi(x)dx - \int_G \varphi(x)d\mu(x)\right| \\ &\quad = \left|\int_G\int_G K_t(y)(\varphi(yx) - \varphi(x))dyd\mu(x)\right| \\ &\quad \leqq \|\mu\| \, \|T^t\varphi - \varphi\|_\infty . \end{aligned}$$

By (iv), we get (v). For (vi), we show only in the case $p = \infty$. For other cases, see [7], p. 73. We fix compact neighborhoods V and U of e such that $U^{-1}U \subset V$. For any $f \in L^\infty(G)$, we write $f = f_1 + f_2$ where f_1 is the restriction of f to V and $f_2 = f - f_1$. Then for all $x \in U$ we have

$$\begin{aligned} |T^t f_2(x)| &= \left|\int_G f_2(y^{-1}x)K_t(y)dy\right| \\ &\leqq \|f\|_\infty \int_{G-U} K_t(y)dy \to 0 \quad \text{as} \quad t \to 0 . \end{aligned}$$

Since $f_1 \in L^p(G)$, $1 \leqq p < \infty$, $T^t f_1(x)$ converges to $f_1(x)$ for almost every $x \in G$. This shows that $\lim_{t\to 0} T^t f(x) = f(x)$ for almost every $x \in U$. Since G has the Lindelöf property, we have $\lim_{t\to 0} T^t f(x) = f(x)$ for almost every $x \in G$.

THEOREM 2′. *Let f be a measurable function or a measure on G. Then $\lim_{t\to 0} P^t f = f$ holds in the following senses;*

(i) *in the L^p norm if $f \in L^p(G)$, $1 \leqq p < \infty$.*
(ii) *in the weak star topology of $L^\infty(G)$ if $f \in L^\infty(G)$.*
(iii) *uniformly on each compact subset of G if $f \in C_b(G)$.*
(iv) *uniformly if $f \in C_0(G)$.*
(v) *in the weak star topology of $M(G)$ if $f \in M(G)$.*
(vi) *almost everywhere if $f \in L^p(G)$, $1 \leqq p \leqq \infty$.*

PROOF. To show (vi), we need two following lemmata.

LEMMA 3 ([7], p. 48). *For $f \in L^p(G)$, $1 \leqq p \leqq \infty$, we define the maximal function Mf as*

$$Mf(x) = \sup_{t>0}\left(\frac{1}{t}\int_0^t T^s f(x)ds\right).$$

Then Mf satisfies the inequalities:

(i) $\|Mf\|_p \leqq C\|f\|_p$ *for each* p *with* $1 < p \leqq \infty$, *where* C *is a constant independent of* f.

(ii) $|\{x \in G: Mf(x) > \alpha\}| \leqq (C/\alpha)\|f\|_1$ *for each* $\alpha > 0$ *and any* $f \in L^1(G)$, *where* C *is a constant independent of* f *and* α.

LEMMA 4 ([7], p. 49). *For all* $f \in L^p(G)$, $1 \leqq p \leqq \infty$, *and all* $f \in M(G)$, *there exists a constant* C *such that* $P^t f(x) \leqq CMf(x)$ *for each* $t > 0$.

PROOF OF (vi). This is in the line of classical limit process (see [6], p. 64).

REMARK. Let $f \in C_b(G)$; then extensions $F(x, t)$, of the functions $T^t f(x)$ and $P^t f(x)$, equal to $f(x)$ for $t = 0$ are continuous on $G \times [0, \infty)$. This can be showed in the same way as the proof of (iii) in the above theorems.

3. The representation theorem. Suppose that a (real valued) function u on an (unimodular) connected Lie group G with dimension n is of class $C^2(G)$. For a fixed basis $\{X_1, \cdots, X_n\}$ of the Lie algebra of G we write $\Delta u = \sum_{i=1}^n X_i^2 u$. The mapping

$$\exp(x_1X_1 + \cdots + x_nX_n)x \longrightarrow (x_1, \cdots, x_n)$$

gives a coordinate system on a neighborhood of each element x in G. Then we have, in a local coordinate,

$$\Delta u = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u .$$

We can easily prove the following lemmata because these are local versions. We refer the reader to [4] and [8] for the proof.

LEMMA 5. (i) (Mean value property) *Suppose a function* u *is of class* $C^2(G)$. *If the function* u *on* G *satisfies the equality* $\Delta u = 0$, *then for any ball* B *centered at* x *in* G *contained in a local coordinate neighborhood*

$$u(x) = \frac{1}{|B|}\int_B u(y)dy .$$

The converse is also valid.

(ii) *Suppose a function* u *is of class* $C^2(G)$. *If the function* u *on* G *satisfies the inequality* $\Delta u \geqq 0$, *then for any ball* B *centered at* x *in* G *contained in a local coordinate neighborhood,*

$$u(x) \leqq \frac{1}{|B|}\int_B u(y)dy .$$

The converse is also valid.

(iii) *If* $u \in C^2(G)$ *satisfies the inequality* $\Delta u \geqq 0$ *and* φ *is a non-decreasing convex* C^2*-function defined on an interval containing the range of* u, *then the composition* $s = \varphi \circ u$ *satisfies the inequality* $\Delta s \geqq 0$.

(iv) (Maximum and minimum principle) *Suppose a function* u, *defined in a open domain* D *contained in a local coordinate neighborhood, satisfies the equality* $\Delta u = 0$. *If* u *satisfies* $\sup u(x) = C < \infty$ ($\inf u(x) = C > -\infty$); *then* $u(x) < C$ *for all* $x \in D$ ($u(x) > C$ *for all* $x \in D$ *respectively*), *provided* u *is not a constant function on* D.

If a (real valued) function $u(x, t)$ is of class $C^2(G \times (0, \infty))$, we write, from now on,

$$L(u) = \sum_{i=1}^{n} X_i^2 u + \frac{\partial^2}{\partial t^2} u \quad \text{and} \quad H(u) = \sum_{i=1}^{n} X_i^2 u - \frac{\partial}{\partial t} u \,.$$

Let U be any open subset of G such that closure $\bar{U}$ of the set U is compact. The product $U \times (0, t_0)$, $t_0 > 0$, in $G \times (0, \infty)$ will be written as D. The boundary of U will be denoted by ∂U. The closure $\bar{D}$ of D is the product $\bar{U} \times [0, t_0]$. Γ denotes the union of $\bar{U} \times \{0\}$ and $\partial U \times [0, t_0]$.

LEMMA 6 (Maximum and minimum principle for the heat equation and the Laplace equation) *Suppose that a function* $u(x, t)$ *is of class* $C^2(G \times (0, \infty))$ *and continuous on* $\bar{D}$.

(i) *If the function* $u(x, t)$ *satisfies the inequalities* $H(u) \leqq 0$ *in* $\bar{D} - \Gamma$ *and* $u(x, t) \geqq 0$ *on* Γ, *then* $u(x, t) \geqq 0$ *in* $\bar{D}$.

(ii) *If the function* $u(x, t)$ *satisfies the inequalities* $H(u) \geqq 0$ *in* $\bar{D} - \Gamma$ *and* $u(x, t) \leqq 0$ *on* Γ, *then* $u(x, t) \leqq 0$ *in* $\bar{D}$.

(iii) *If the function* $u(x, t)$ *satisfies the inequalities* $L(u) \leqq 0$ *in* $\bar{D} - \Gamma$ *and* $u(x, t) \geqq 0$ *on* Γ, *then* $u(x, t) \geqq 0$ *in* $\bar{D}$.

(iv) *If the function* $u(x, t)$ *satisfies the inequalities* $L(u) \geqq 0$ *in* $\bar{D} - \Gamma$ *and* $u(x, t) \leqq 0$ *on* Γ, *then* $u(x, t) \leqq 0$ *in* $\bar{D}$.

PROPOSITION 7. *Suppose that a function* $u(x, t)$ *is of class* $C^2(G \times (0, \infty))$ *and continuous on* $G \times [0, \infty)$.

(i) *If the function* $u(x, t)$ *is bounded below in* $G \times [0, \infty)$ *and it satisfies the inequalities* $H(u) \leqq 0$ *in* $G \times (0, \infty)$ *and* $u(x, 0) \geqq 0$ *for all* $x \in G$, *then* $u(x, t) \geqq 0$ *in* $G \times [0, \infty)$.

(ii) *If the function* $u(x, t)$ *is bounded above in* $G \times [0, \infty)$ *and it satisfies the inequalities* $H(u) \geqq 0$ *in* $G \times (0, \infty)$ *and* $u(x, 0) \leqq 0$ *for all* $x \in G$, *then* $u(x, t) \leqq 0$ *in* $G \times [0, \infty)$.

(iii) *If the function* $u(x, t)$ *is bounded below in* $G \times [0, \infty)$ *and it satisfies the inequalities* $L(u) \leqq 0$ *in* $G \times (0, \infty)$ *and* $u(x, 0) \geqq 0$ *for all* $x \in G$, *then* $u(x, t) \geqq 0$ *in* $G \times [0, \infty)$.

(iv) *If the function* $u(x, t)$ *is bounded above in* $G \times [0, \infty)$ *and it satisfies the inequalities* $L(u) \geqq 0$ *in* $G \times (0, \infty)$ *and* $u(x, 0) \leqq 0$ *for all* $x \in G$, *then* $u(x, t) \leqq 0$ *in* $G \times [0, \infty)$.

PROOF. Let B_r be the ball of radius r centered at e in G, that is, $B_r = \{x \in G: d(x) < r\}$, and let f be a nonnegative function in $C_c^\infty(G)$ such that support of f is contained in B_1 and $\int_{B_1} f(x)dx = 1$. Then there exists a constant M such that

$$\left|\sum_{i=1}^{n} X_i^2(f * d)(x)\right| \leqq M$$

for all x in G (see [2]). In order to prove (i), suppose $u(x, t) > -m$; $m > 0$, on $G \times [0, \infty)$, and r_0 any positive number. We consider the auxiliary function

$$v(x, t) = \frac{m}{r_0}((f * d)(x) + 1 + Kt) + u(x, t) .$$

The constant $K > 0$ can be chosen so that for all $r_0 > 0$ the quantity $H(v)$ is negative. In fact,

$$\begin{aligned} H(v) &= \frac{m}{r_0}\Big(\sum_{i=1}^{n} X_i^2(f * d) - K\Big) + H(u) \\ &\leqq \frac{m}{r_0}(M - K) < 0 \quad \text{if} \quad M < K . \end{aligned}$$

We have $v(x, 0) \geqq u(x, 0) \geqq 0$ and

$$\begin{aligned} f * d(x) &= \int_G d(y^{-1}x)f(y)dy \\ &\geqq \int_G d(x)f(y)dy - \int_G d(y)f(y)dy \\ &\geqq r_0\int_G f(y)dy - \int_G f(y)dy = r_0 - 1 \end{aligned}$$

whenever $d(x) = r_0$. Thus

$$\begin{aligned} v(x, t) &\geqq \frac{m}{r_0}(r_0 + Kt) + u(x, t) \\ &\geqq m + u(x, t) > 0 \end{aligned}$$

whenever $d(x) = r_0$ and $0 \leqq t < t_0$. By Lemma 6, we have $v(x, t) \geqq 0$

on $\bar{D} = \bar{B}_{r_0} \times [0, t_0]$. Any fixed point (x, t) in $G \times [0, \infty)$ lies in some $\bar{D}$ for sufficiently large r_0 and t_0. Hence, at such a point, we have $v(x, t) \geqq 0$. On letting r_0 tend to ∞, we get $u(x, t) \geqq 0$ in $G \times [0, \infty)$.

Similarly we can prove (ii).

We can also prove (iii) and (iv) by considering the auxiliary function

$$v(x, t) = \frac{m}{r_0}\left((f * d)(x) + 1 + K \cos\frac{\pi t}{4t_0}\right) + u(x, t)$$

instead of that in the proof of (i).

COROLLARY 8. *Suppose that a C^2-function $u(x, t)$ is a bounded continuous function on $G \times [0, \infty)$.*

(i) *If the function $u(x, t)$ satisfies the heat equation $H(u) = 0$ on $G \times (0, \infty)$ and $u(x, 0) = 0$ for all x in G, then $u(x, t) = 0$ on $G \times [0, \infty)$.*

(ii) *If the function $u(x, t)$ satisfies the Laplace equation $L(u) = 0$ on $G \times (0, \infty)$ and $u(x, 0) = 0$ for all x in G, then $u(x, t) = 0$ on $G \times [0, \infty)$.*

LEMMA 9. *Suppose that a function $u(x, t)$ is of class $C^2(G \times (0, \infty))$ such that $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$ for some $1 \leqq p \leqq \infty$, and $u(x, t)$ converges to zero as $t \to 0$ in the weak star topology of $L^p(G)$ when $1 < p \leqq \infty$ and in the sense*

$$\int_G u(y, t)\varphi(y)dy \longrightarrow 0 \quad \text{for all} \quad \varphi \in C_0(G)\ .$$

when $p = 1$.

(i) *If the function $u(x, t)$ satisfies the inequality $H(u) \leqq 0$ on $G \times (0, \infty)$, then we have $u(x, t) \geqq 0$ on $G \times (0, \infty)$.*

(ii) *If the function $u(x, t)$ satisfies the inequality $H(u) \geqq 0$ on $G \times (0, \infty)$, then we have $u(x, t) \leqq 0$ on $G \times (0, \infty)$.*

(iii) *If the function $u(x, t)$ satisfies the inequality $L(u) \leqq 0$ on $G \times (0, \infty)$, then we have $u(x, t) \geqq 0$ on $G \times (0, \infty)$.*

(iv) *If the function $u(x, t)$ satisfies the inequality $L(u) \geqq 0$ on $G \times (0, \infty)$, then we have $u(x, t) \leqq 0$ on $G \times (0, \infty)$.*

PROOF. For any nonnegative function $\varphi \in C_c^\infty(G)$, a function $v = u*\varphi$ belongs to $C_b(G \times [0, \infty))$ if we define $v(x, 0) = 0$. The function v satisfies the inequality $H(v) \leqq 0$ when $H(u) \leqq 0$. By Proposition 7, we get $v(x, t) \geqq 0$ on $G \times [0, \infty)$. Since φ is an auxiliary nonnegative function in $C_c^\infty(G)$, $u(x, t) \geqq 0$ on $G \times (0, \infty)$. This completes the proof of (i). Another parts of the lemma are proved in the same way.

COROLLARY 10. *Suppose that a function* $u(x, t)$ *is of class* $C^2(G \times (0, \infty))$ *such that* $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$, $1 \leqq p \leqq \infty$, *and* $u(x, t)$ *converges to zero as* $t \to 0$ *in the weak star topology of* $L^p(G)$ *when* $1 < p \leqq \infty$ *and in the sense* $\int_G u(y, t)\varphi(y)dy \to 0$ *for all* $\varphi \in C_0(G)$ *when* $p = 1$. *If the function* $u(x, t)$ *satisfies the heat equation* $H(u) = 0$ *or the Laplace equation* $L(u) = 0$ *on* $G \times (0, \infty)$, *then* $u(x, t) = 0$ *on* $G \times (0, \infty)$.

THEOREM 11 (The representation theorem). (i) *A function* $u(x, t)$ *is of class* $C^\infty(G \times (0, \infty))$ *such that* $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$ *for some* $1 \leqq p \leqq \infty$ *and it satisfies the heat equation* $H(u) = 0$ *on* $G \times (0, \infty)$ *if and only if it is of the form* $u(x, t) = T^t f(x)$ *such that when* $1 < p \leqq \infty$, $f \in L^p(G)$ *and when* $p = 1$, $f \in M(G)$.

Moreover, this representation is unique and $\|f\| = \sup_{t>0} \|u(\cdot, t)\|_p$, *where* $\|f\|$ *means the* L^p *norm for* $1 < p \leqq \infty$ *and the total variation norm for* $p = 1$.

(ii) *A function* $u(x, t)$ *is of class* $C^\infty(G \times (0, \infty))$ *such that* $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$ *for some* $1 \leqq p \leqq \infty$ *and it satisfies the Laplace equation* $L(u) = 0$ *on* $G \times (0, \infty)$ *if and only if it is of the form* $u(x, t) = P^t f(x)$ *such that if* $1 < p \leqq \infty$, $f \in L^p(G)$ *and if* $p = 1$, $f \in M(G)$.

Moreover, this representation is unique and $\|f\| = \sup_{t>0} \|u(\cdot, t)\|_p$ *where* $\|f\|$ *means the* L^p *norm for* $1 < p \leqq \infty$ *and the total variation norm for* $p = 1$.

PROOF. We will show (i). Let $u_\varepsilon(x) = u(x, \varepsilon)$ where $\varepsilon > 0$. We put $v_\varepsilon(x, t) = u(x, \varepsilon + t) - T^t u_\varepsilon(x)$. Then from Theorem 1(ii), (viii) and Theorem 2(i), (ii), $v_\varepsilon(x, t)$ is of class $C^\infty(G \times (0, \infty))$ such that $\sup_{t>0} \|v_\varepsilon(\cdot, t)\|_p < \infty$, and $v_\varepsilon(x, t)$ converges to zero as $t \to 0$ in the weak star topology for $1 < p \leqq \infty$ and in the sense $\int_G v_\varepsilon(y, t)\varphi(y)dy \to 0$ for all $\varphi \in C_0(G)$ when $p = 1$. Moreover, we have $H(v_\varepsilon) = 0$. By Corollary 10, $v_\varepsilon = 0$, that is, $u(x, \varepsilon + t) = T^t u_\varepsilon(x)$ for all (x, t) in $G \times (0, \infty)$. Since u_ε is uniformly L^p bounded, there exists a subsequence $\{u_{\varepsilon'}\}$ such that $u_{\varepsilon'}$ converges to $f \in L^p(G)$ when $1 < p \leqq \infty$ and $f \in M(G)$ when $p = 1$ in the weak star topology as $\varepsilon' \to 0$. Hence $T^t u_{\varepsilon'}(x)$ converges to $T^t f(x)$ pointwise for each $t > 0$. On the other hand, $u(x, \varepsilon' + t)$ converges to $u(x, t)$ pointwise as $\varepsilon' \to 0$. Therefore we obtain $u(x, t) = T^t f(x)$ for all (x, t) in $G \times (0, \infty)$.

Since f is a weak star limit, we have $\|f\| \leqq \sup_{t>0} \|u(\cdot, t)\|_p$. On the other hand, by Theorem 2(ii), $\sup_{t>0} \|u(\cdot, t)\|_p = \sup_{t>0} \|T^t f\|_p \leqq \|f\|$. Hence $\|f\| = \sup \|u(\cdot, t)\|_p$. Uniqueness of this representation is an immediate consequence of Theorem 2.

The converse argument of (i) is easy.

We can also prove (ii) in the same way.

PROPOSITION 12. *Suppose a function* $s(x, t)$ *is of class* $C^\infty(G \times (0, \infty))$ *satisfying* $\sup_{t>0} \|s(\cdot, t)\|_p < \infty$ *for some* $1 \leqq p \leqq \infty$.

(i) *If the function* $s(x, t)$ *satisfies the inequality* $H(s) \geqq 0$, *then there exists a function* $u(x, t)$ *in* $C^\infty(G \times (0, \infty))$ *which is a minimal majorant of* $s(x, t)$ *on* $G \times (0, \infty)$ *satisfying* $H(u) = 0$.

If $s(x, t)$ *is, in addition, nonnegative, then* $\sup_{t>0} \|u(\cdot, t)\|_p = \sup_{t>0} \|s(\cdot, t)\|_p$.

(ii) *If the function* $s(x, t)$ *satisfies the inequality* $H(s) \leqq 0$, *then there exists a function* $u(x, t)$ *in* $C^\infty(G \times (0, \infty))$ *which is a maximal function in all of functions satisfying the heat equation and less than* $s(x, t)$ *on* $G \times (0, \infty)$.

(iii) *If the function* $s(x, t)$ *satisfies the inequality* $L(s) \geqq 0$, *then there exists a function* $u(x, t)$ *in* $C^\infty(G \times (0, \infty))$ *which is a minimal majorant of* $s(x, t)$ *on* $G \times (0, \infty)$ *satisfying* $L(u) = 0$.

If $s(x, t)$ *is, in addition, nonnegative, then* $\sup_{t>0} \|u(\cdot, t)\|_p = \sup_{t>0} \|s(\cdot, t)\|_p$.

(iv) *If the function* $s(x, t)$ *satisfies the inequality* $L(s) \leqq 0$, *then there exists a function* $u(x, t)$ *in* $C^\infty(G \times (0, \infty))$ *which is a maximal function in all of functions satisfying the Laplace equation and less than* $s(x, t)$ *on* $G \times (0, \infty)$.

PROOF. In order to prove (i), we put $s_\varepsilon(x) = s(x, \varepsilon)$ and $u_\varepsilon(x, t) = T^t s_\varepsilon(x)$ where $\varepsilon > 0$. Let $v_\varepsilon(x, t) = s(x, \varepsilon + t) - u_\varepsilon(x, t)$. Then $H(v_\varepsilon) \geqq 0$ when $H(s) \geqq 0$. Since $v_\varepsilon(x, t)$ converges to zero as $t \to 0$ in the weak star topology for $1 < p \leqq \infty$ and in the sense $\int_G v_\varepsilon(y, t)\varphi(y)dy \to 0$ for all $\varphi \in C_0(G)$ when $p = 1$. By Lemma 9, $v_\varepsilon(x, t) \leqq 0$, that is, $s(x, \varepsilon + t) \leqq u_\varepsilon(x, t)$ for all (x, t) in $G \times (0, \infty)$. Since $s_\varepsilon(x)$ is uniformly bounded in $L^p(G)$, there exists a subsequence $\{s_{\varepsilon'}(x)\}$ such that $s_{\varepsilon'}(x)$ converges to $f \in L^p(G)$ for $1 < p \leqq \infty$ and $f \in M(G)$ for $p = 1$ in the weak star topology as $\varepsilon' \to 0$. Then $u_{\varepsilon'}(x, t)$ converges to $T^t f(x)$ pointwise for each $t > 0$. On the other hand, $s(x, \varepsilon' + t)$ converges to $s(x, t)$ pointwise. Therefore $T^t f(x) \geqq s(x, t)$ and $u(x, t) = T^t f(x)$ satisfies the heat equation.

In order to prove the minimality of $u(x, t)$, we choose any function $h(x, t)$ satisfying $H(h) = 0$ and $u(x, t) \geqq h(x, t) \geqq s(x, t)$ on $G \times (0, \infty)$. By Theorem 11, h may be represented as the form $h(x, t) = T^t g(x)$ where $g \in L^p(G)$ for $1 < p \leqq \infty$ and $g \in M(G)$ for $p = 1$; then we have

$$\begin{aligned} u_\varepsilon(x, t) &= T^t s_\varepsilon(x) \leqq T^t T^\varepsilon g(x) \\ &= T^{t+\varepsilon} g(x) = h(x, t + \varepsilon) . \end{aligned}$$

Since $h(x, t+\varepsilon') \to h(x, t)$ as $\varepsilon' \to 0$, $u(x, t) \leqq h(x, t)$ on $G \times (0, \infty)$. Hence $u(x, t)$ is a minimal majorant of $s(x, t)$.

Next suppose s is a nonnegative function. Since the representing function (measure) f of u is a weak star limit,

$$\|f\| \leqq \sup_{t>0} \|s(\cdot, t)\|_p$$

where $\|f\|$ means the L^p norm if $1 < p \leqq \infty$ and the total variation norm if $p = 1$. On the other hand, s being nonnegative,

$$\sup_{t>0} \|s(\cdot, t)\|_p \leqq \sup_{t>0} \|u(\cdot, t)\|_p .$$

By Theorem 11, $\sup_{t>0} \|s(\cdot, t)\|_p = \sup_{t>0} \|u(\cdot, t)\|_p = \|f\|$. These conclude the proof of (i).

In the same way we can prove (ii), (iii) and (iv).

4. **The H^p space theory.** We assume that G is a semisimple connected noncompact Lie group with dimension n, K is a maximal compact subgroup and (G, K) is a Riemannian symmetric pair of the noncompact type. Then we have a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let now $\{X_1, \cdots, X_r, X_{r+1}, \cdots, X_n\}$ a basis in $\mathfrak{g}$ such that $\{X_1, \cdots, X_r\}$ is an orthonormal basis for $\mathfrak{p}$ with respect to the Killing form B and $\{X_{r+1}, \cdots, X_n\}$ is an orthonormal basis for $\mathfrak{k}$ with respect to $-B$. Then the Casimir operator $\Gamma = \sum_{i=1}^r X_i^2 - \sum_{j=r+1}^n X_j^2$ is not only right translation invariant, but also left translation invariant. Hence the Laplace operator $\Delta = \sum_{i=1}^n X_i^2$ is translation invariant under K. This operator Δ leads to semigroups which we have written as T^t and P^t before. If f is any smooth zonal function, that is, $f(k_1 x k_2) = f(x)$ for all $x \in G$ and all $k_1, k_2 \in K$, then $X_i f = 0$ for $r < i \leqq n$. Therefore $\Delta f = \Gamma f$ for all smooth zonal functions f. Hence $XT^t f = T^t X f$ and $XP^t f = P^t X f$ for all $X \in \mathfrak{g}$ and all $t > 0$ whenever f is a smooth zonal function. We consider a function f_0 of the form $f_0 = (\partial P^t/\partial t)(f)|_{t=t_0}$ where $t_0 > 0$. For these f_0 we define the Riesz transforms by

$$R_i(f_0) = X_i P^{t_0}(f) , \qquad i = 1, \cdots, n .$$

These R_i are well-defined on a dense subset of $L^p(G)$ for $1 < p < \infty$. (See [7], p. 132-p. 133.) Furthermore, $\sum_{i=1}^r \|R_i(f)\|_p \approx \|f\|_p$ for a zonal $f \in L^p(G)$, $1 < p < \infty$ (see [7], p. 134).

The Killing form is denoted by $(\cdot, \cdot)$ from now on. This form is nondegenerate and has properties:

$$(X, Y) = (Y, X) , \qquad X, Y \in \mathfrak{g} .$$
$$(\sigma X, \sigma Y) = (X, Y) , \qquad \sigma \text{ automorphism of } \mathfrak{g}.$$
$$(X, [Y, Z]) = (Y, [Z, X]) = (Z, [X, Y]) , \qquad X, Y, Z \in \mathfrak{g} .$$

We shall say that an $(r+1)$ tuple of (real valued) functions $F = (u_0, u_1, \cdots, u_r)$ on $G \times (0, \infty)$ is a (generalized) Cauchy-Riemann system if F is smooth on $G \times (0, \infty)$ and zonal on G for each $t > 0$, and satisfies the equations:

(a) $\frac{\partial}{\partial t} u_j = X_j u_0, \ j = 1, \cdots, r.$

(b) $X_i u_j = X_j u_i, \ i, j = 1, \cdots, r.$

(c) $\frac{\partial}{\partial t} u_0 + \sum_{j=1}^{r} X_j u_j = 0.$

LEMMA 13 (cf. [1]). *If* $F = (u_0, u_1, \cdots, u_r)$ *is a Cauchy-Riemann system, then*

$$L(u_j) = \frac{\partial^2}{\partial t^2} u_j + \sum_{j=1}^{r} X_i^2 u_j = 0 , \qquad j = 0, 1, \cdots, r .$$

PROOF. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$,

$$\sum_{i=1}^{r} [X_j, X_i] u_i = 0 , \qquad j = 1, \cdots, r .$$

By (a), (b) and (c),

$$\begin{aligned} L(u_j) &= \frac{\partial^2}{\partial t^2} u_j + \sum_{i=1}^{r} X_i^2 u_j \\ &= \frac{\partial}{\partial t} X_j u_0 + \sum_{i=1}^{r} X_i X_j u_i \\ &= X_j \Big(\frac{\partial}{\partial t} u_0 + \sum_{i=1}^{r} X_i u_i \Big) - \sum_{i=1}^{r} [X_j, X_i] u_i \\ &= - \sum_{i=1}^{r} [X_j, X_i] u_i = 0 . \end{aligned}$$

Therefore $L(u_j) = 0, \ j = 1, \cdots, r$. When $j = 0$, by (a) and (c),

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \Big(\frac{\partial}{\partial t} u_0 + \sum_{j=1}^{r} X_j u_j \Big) = \frac{\partial^2}{\partial t^2} u_0 + \sum_{j=1}^{r} X_j \frac{\partial}{\partial t} u_j \\ &= \frac{\partial^2}{\partial t^2} u_0 + \sum_{j=1}^{r} X_j^2 u_0 = L(u_0) . \end{aligned}$$

We say that a Cauchy-Riemann system $F = (u_0, u_1, \cdots, u_r)$ belongs to the space H^p, $p > 0$, if

$$\begin{aligned} \| F \|_p &= \sup_{t>0} \Big(\int_G | F |^p dx \Big)^{1/p} \\ &= \sup_{t>0} \Big(\int_G (u_0^2 + u_1^2 + \cdots + u_r^2)^{p/2} dx \Big)^{1/p} < \infty . \end{aligned}$$

LEMMA 14 ([8], p. 234). *Let* $F=(u_0, u_1, \cdots, u_r)$ *be a Cauchy-Riemann system; then* $s=|F|^p=(u_0^2+u_1^2+\cdots+u_r^2)^{p/2}$ *satisfies the inequality* $L(s)\geqq 0$ *for* $p\geqq (r-1/r)$.

We shall now show the fact that the space H^p may be identified with the space of all zonal functions in $L^p(G)$ when $1<p<\infty$.

THEOREM 15. *Suppose that* $F=(u_0, u_1, \cdots, u_r)\in H^p$, $1<p<\infty$. *Then there exists a zonal function* $f_0\in L^p(G)$ *such that*

$$u_0(x, t)=P^t f_0(x)\ ,\quad u_j(x, t)=P^t f_j(x)\ ,\quad j=1, \cdots, r$$

where $f_j=R_j(f_0)$ *Riesz transforms of* f_0, $j=1, \cdots, r$. *Conversely, if* $f_0\in L^p(G)$, $1<p<\infty$, *is zonal and* $f_j=R_j(f_0)$, $u_0(x, t)=P^t f_0(x)$ *and* $u_j(x, t)=P^t f_j(x)$, $j=1, \cdots, r$, *then* $F=(u_0, u_1, \cdots, u_r)\in H^p$. *Furthermore* $\|f_0\|_p\approx \|F\|_p$. *Hence for* $1<p<\infty$ *the space* H^p *is identified with the space consisting of all zonal functions in* $L^p(G)$.

PROOF. Let f_0 be of the form $f_0=(\partial P^t/\partial t)(f)|_{t=t_0}$, $t_0>0$ and a zonal function in $L^p(G)$, $1<p<\infty$. We put $u_0=P^t f_0$, $f_j=R_j f_0$ and $u_j=P^t f_j$, $j=1, \cdots, r$. Then we have, since we may assume f is zonal,

$$u_j=P^t f_j=P^t R_j f_0=P^t X_j P^{t_0} f=X_j P^{t+t_0} f_0$$

and so

$$\frac{\partial}{\partial t}u_j=X_j P^t\Big(\frac{\partial}{\partial t}P^t f|_{t=t_0}\Big)=X_j P^t f_0=X_j u_0\ ,$$

$$\begin{aligned} X_i u_j &= X_i X_j P^{t+t_0} f = X_j X_i P^{t+t_0} f+[X_i, X_j]P^{t+t_0} f \\ &= X_j u_i+[X_i, X_j]P^{t+t_0} f \\ &= X_j u_i\ ,\qquad i, j=1, \cdots, r\ , \end{aligned}$$

because $[\mathfrak{p}, \mathfrak{p}]\subset \mathfrak{k}$. We also have

$$\begin{aligned} \frac{\partial}{\partial t}u_0+\sum_{i=1}^r X_i u_i &= \frac{\partial}{\partial t}P^t\Big(\frac{\partial}{\partial t}P^t f|_{t=t_0}\Big)+\sum_{i=1}^r X_i^2 P^{t+t_0} f \\ &= \Big(\frac{\partial^2}{\partial t^2}+\sum_{i=1}^r X_i^2\Big)P^{t+t_0} f=0\ . \end{aligned}$$

Since $\sum_{i=1}^r \|R_i f\|_p\approx \|f\|_p$ $(1<p<\infty)$, it follows that

$$\sup_{t>0}\|(u_0^2+u_1^2+\cdots+u_r^2)^{1/2}\|_p<\infty\ ,$$

that is, $F=(u_0, u_1, \cdots, u_r)\in H^p$ and $\|F\|_p\approx \|f_0\|_p$.

Since the set of f_0 of the above form is dense in the subspace of all zonal functions of $L^p(G)$, $1<p<\infty$, the above result is also valid for every zonal $f_0\in L^p(G)$ by a limiting argument and the fact that

$$\sum_{i=1}^{r} \| X_i P^t f \|_p \leqq C \left\| \frac{\partial P^t}{\partial t}(f) \right\|_p \leqq \frac{C'}{t} \| f \|_p$$

(see [7], p. 60 and p. 131).

Conversely if $F = (u_0, u_1, \cdots, u_r) \in H^p$, then $\sup_{t>0} \| u_j(\cdot, t) \|_p < \infty$ and $L(u_j) = 0$, $j = 0, 1, \cdots, r$ by Lemma 13. By Theorem 11 there exist zonal functions $f_j \in L^p(G)$, $j = 0, 1, \cdots, r$ such that $u_j = P^t f_j$, $j = 0, 1, \cdots, r$. We set $v_j = P^t R_j f_0$, $j = 1, \cdots, r$. Then we have

$$\frac{\partial}{\partial t} u_j = X_j u_0 = \frac{\partial}{\partial t} v_j , \qquad j = 1, \cdots, r ,$$

that is,

$$\frac{\partial}{\partial t} P^t(R_j f_0) = \frac{\partial}{\partial t} P^t f_j , \qquad j = 1, \cdots, r$$

since $(u_0, v_1, \cdots, v_r)$ is a Cauchy-Riemann system. Hence $P^t(R_j f_0) = P^t f_j$ (see [7], p. 133). By Theorem 2′, $R_j f_0 = f_j$, $j = 1, \cdots, r$.

In order to prove F. and M. Riesz's theorem, we need the idea of nontangential arguments.

Denote by $\mathscr{B}$ a family of all balls $B = B(r)$ with center e and suitably small radius r contained in a local coordinate neighborhood of e in G.

LEMMA 16 (The covering lemma). *Let $\mathscr{S}$ be a family of forms Bx ($x \in G$, $B \in \mathscr{B}$) whose union covers a measurable subset E of G. Then we can select a disjoint subsequence $\{B_j x_j\}$ in φ such that $\sum_j |B_j| \geqq C|E|$, where C is a positive constant depending only on the dimension of G.*

PROOF. See [6], p. 9.

We define the maximal functions for a locally integrable function f and $\mu \in M(G)$ by

$$mf(x) = \sup_{B \in \mathscr{B}} \frac{1}{|B|} \int_{Bx} |f(y)| dy$$

$$m\mu(x) = \sup_{B \in \mathscr{B}} \frac{|\mu|(Bx)}{|B|} .$$

LEMMA 17. *The following inequalities hold:*

(i) *If $f \in L^1(G)$ and $\alpha > 0$, then*

$$|\{x \in G: mf(x) > \alpha\}| \leqq \frac{C}{\alpha} \| f \|_1$$

where C is a constant independent on f and α.

(ii) *If* $\mu \in M(G)$ *and* $\alpha > 0$, *then*

$$|\{x \in G: m\mu(x) > \alpha\}| \leqq \frac{C}{\alpha}||\mu||$$

where C *is a constant independent on* μ *and* α.

(iii) *If* $f \in L^p(G)$, $1 < p \leqq \infty$, *then*

$$||mf||_p \leqq C||f||_p$$

where C *is a constant independent on* f.

Hence the maximal function mf *and* $m\mu$ *is finite almost everywhere.*

PROOF. See [6], pp. 19-22.

PROPOSITION 18. *Let* f *be a function in* $L^p(G)$, $1 \leqq p \leqq \infty$. *Then we have*

(i) $\lim_{r \to 0} \frac{1}{|B(r)|} \int_{B(r)x} f(y)dy = f(x)$

at almost all x *in* G.

(ii) $\lim_{r \to 0} \frac{1}{|B(r)|} \int_{B(r)x} |f(y) - f(x)|dy = 0$

at almost all x *in* G.

(iii) *For any measurable set* E *of* G,

$$\lim_{r \to 0} \frac{|E \cap B(r)x|}{|B(r)|} = 1$$

at almost all x *in* E.

Let μ *be a measure in* $M(G)$, *and* $d\mu = fdx + d\nu$ *the Lebesgue decomposition. Then we have*

(iv) $\lim_{r \to 0} \frac{\mu(B(r)x)}{|B(r)|} = f(x)$

at almost all x *in* G.

PROOF. These follow from the routine argument by Lemma 17.

Let $\Gamma_\alpha^h(x_0) = \{(x, t) \in G \times (0, h): d(x_0, x) < \alpha t\}$, and call it a truncated cone at x_0. We say that a function $u(x, t)$ on $G \times (0, \infty)$ is nontangentially bounded at x_0 provided that for some α and h,

$$\sup\{|u(x, t)|: (x, t) \in \Gamma_\alpha^h(x_0)\} < \infty .$$

LEMMA 19 (cf. [6], p. 201). *Suppose* u *is a smooth function on* $G \times (0, \infty)$ *satisfying the equation* $L(u) = 0$. *Let* E *be a measurable subset of* G *and suppose that* u *is nontangentially bounded at every* $x_0 \in E$.

Then u is convergent as $t \to 0$ at almost every $x_0 \in E$.

PROOF. We may assume E is compact and is contained in some local coordinate neighborhood U of G. We define the open region $\tilde{\mathscr{R}}$ by $\tilde{\mathscr{R}} = \bigcup_{x_0 \in E} \Gamma_\beta^k(x_0)$. We can assume that $|u| < 1$ in $\tilde{\mathscr{R}}$ and $\{x \in G: d(x, E) < 2\beta k\} \subset U$. For fixed α and h with $\beta > \alpha$ and $k > h$, we also define the open region $\mathscr{R}$ by $\mathscr{R} = \bigcup_{x_0 \in E} \Gamma_\alpha^h(x_0)$. For sufficiently small $\varepsilon > 0$ let $D_\varepsilon = \{x \in G: (x, \varepsilon) \in \mathscr{R}\}$ and φ_ε a continuous function on G and less than 1 in absolute value, equal to $u(x, \varepsilon)$ in E and vanishing outside D_ε. Let $\phi_\varepsilon(x, t) = P^t \varphi_\varepsilon(x)$. Define $\psi_\varepsilon(x, t)$ by $u(x, t + \varepsilon) = \phi_\varepsilon(x, t) + \psi_\varepsilon(x, t)$. Since $\{\varphi_\varepsilon(x)\}$ are uniformly bounded in the $L^\infty(G)$ norm, we can find a $\varphi(x) \in L^\infty(G)$ and a subsequence $\{\varphi_{\varepsilon'}\}$ such that $\varphi_{\varepsilon'} \to \varphi$ weakly as $\varepsilon' \to 0$. Hence

$$\phi_{\varepsilon'}(x, t) = P^t \varphi_{\varepsilon'}(x) \to P^t \varphi(x) = \phi(x, t)$$

for each (x, t). Since it is obviously true that $\lim_{\varepsilon \to 0} u(x, t + \varepsilon) = u(x, t)$, we have the existence of the limit $\psi(x, t) = \lim_{\varepsilon' \to 0} \psi_{\varepsilon'}(x, t) = u(x, t) - \phi(x, t)$. Since almost everywhere convergence holds for the function ϕ, it remains to show that $\lim_{t \to 0} \psi(x, t) = 0$ a.e x in E. To this end we consider an auxiliary function $H(x, t)$ on $U \times (0, \infty)$ with the following properties. We divide the boundary $\partial\mathscr{R}$ of $\mathscr{R}$ into three parts: $\partial\mathscr{R} = \mathscr{B}_0 \cup \mathscr{B}_+ \cup \mathscr{B}_h$, where $\mathscr{B}_0 = \{(x, 0) \in \partial\mathscr{R}\}$, $\mathscr{B}_+ = \{(x, t) \in \mathscr{R}: \alpha t = d(x, E)\}$, and $\mathscr{B}_h = \{(x, h) \in \partial\mathscr{R}\}$. H will have the following properties:

(a) H satisfies the equality $L(H) = 0$ in $U \times (0, \infty)$.

(b) $H \geqq 0$ on $U \times (0, \infty)$.

(c) $H \geqq 2$ on $\mathscr{B}_+ \cup \mathscr{B}_h$.

(d) $\lim_{t \to 0} H(x, t) = 0$ for almost all x in E.

We shall now construct a function H satisfying above properties. Let χ denote the characteristic function of the complement of E. For a constant C to be determined later we set

$$H(x, t) = C\Big(t + \int_U \frac{t}{(t^2 + d(x, y)^2)^{n+1/2}} \chi(y) dy\Big) .$$

The properties (a) and (b) are obvious. For $\mathscr{B}_h$ we can assure (c) by taking C large enough. For any $(x, t) \in \mathscr{B}_+$, the ball $B(x, \alpha t)$ in G whose center is x and which has radius αt lies inside $E^c \cap U$. Hence

$$\begin{aligned} &\int_U \frac{t}{(t^2 + d(x, y)^2)^{n+1/2}} \chi(y) dy \\ &\geqq \int_{B(x, \alpha t)} \frac{t}{(t^2 + d(x, y)^2)^{n+1/?}} dy \\ &= \int_{B(e, \alpha t)} \frac{t}{(t^2 + d(y)^2)^{n+1/2}} dy = \text{constant} . \end{aligned}$$

Taking C large enough we see that the property (c) has been verified. For any $x \in E$ with $d = d(x, E^c) > 0$, we have

$$\begin{aligned}\int_U \frac{t}{(t^2 + d(x, y)^2)^{n+1/2}} \chi(y) dy \\ \leqq \int_U \frac{t}{(t^2 + d^2)^{n+1/2}} \chi(y) dy \\ = \frac{t}{(t^2 + d^2)^{n+1/2}} |U - E| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0 .\end{aligned}$$

Since the set $\{x \in E: d(x, E^c) = 0\}$ is of measure zero, this verifies the property (d). We shall now prove that $|\psi_\varepsilon(x, t)| \leqq H(x, t)$ when $(x, t) \in \mathscr{R}$. If this were not so, Lemma 5(v) implies the existence of a sequence of points (x_k, t_k) converging to a point on the boundary of $\mathscr{R}$ such that $\liminf (H(x_k, t_k) \pm \psi_\varepsilon(x_k, t_k)) < 0$. The functions u and ϕ_ε are both bounded by 1 in absolute value, and therefore $|\psi_\varepsilon| \leqq 2$ and so by property (c) the limit of $\{(x_k, t_k)\}$ must be on $\mathscr{R}_0$. But, since $\psi_\varepsilon(x_k, t_k) \longrightarrow 0$ there, we obtain a contradiction by (b). Consequently we obtain $|\psi_\varepsilon(x, t)| \leqq H(x, t)$ when $(x, t) \in \mathscr{R}$. Hence $|\psi(x, t)| \leqq H(x, t)$ for all $(x, t) \in \mathscr{R}$. Property (d) gives us the desired result, $\lim_{t \to 0} \psi(x, t) = 0$ for almost all x in E. This concludes the proof of Lemma 19.

PROPOSITION 20. *Suppose s is a smooth nonnegative function satisfying the inequality $L(s) \geqq 0$. Let $\Gamma(x)$ be a suitably small truncated cone of x in G contained in a product of a local coordinate neighborhood of x and $(0, \infty)$. We put*

$$s^*(x) = \sup_{(y,t) \in \Gamma(x)} |s(y, t)|$$

and

$$s^+(x) = \sup_{t>0} |s(x, t)| .$$

Then there exists a positive constant C not depending on x and s such that

$$s^*(x) \leqq Cms^+(x) .$$

PROOF. $B_t(x, t)$ will denote the ball of radius t centered at (x, t) and $B(t)$ the ball of radius t centered at e in G. Then we have by Lemma 5,

$$\begin{aligned}s(x, t) \leqq \frac{1}{|B_t(x, t)|} \int_{B_t(x,t)} s(y, t') dy dt' \\ \leqq \frac{C}{t^{n+1}} \int_0^{2t} \int_{B(t)x_0} s(y, t') dy dt'\end{aligned}$$

$$= \frac{C|B(t)|}{t^n} \cdot \frac{1}{|B(t)|} \int_{B(t)x_0} s^+(y)dy$$
$$\leqq Cms^+(x_0)$$

for any $(x, t) \in \Gamma(x_0)$.

THEOREM 21 (F. and M. Riesz's theorem). *Let* $F = (u_0, u_1, \cdots, u_r)$ *be a Cauchy-Riemann system. If* $F \in H^1$, *then there exist functions* $f_j \in L^1(G)$, $j = 0, 1, \cdots, r$ *such that* $u_j(x, t) = P^t f_j(x)$, $j = 0, 1, \cdots, r$.

PROOF. Since $F \in H^1$, there exist measures $\mu_j \in M(G)$, $j = 0, 1, \cdots, r$ such that $u_j(x, t) = P^t \mu_j(x)$, $j = 0, 1, \cdots, r$ by Theorem 11. By Lemma 14, $s = |F|^{p_0}$, $1 > p_0 > ((r-1)/r)$, satisfies the inequality $L(s) \geqq 0$. From Proposition 12, there exists a majorant $h(x, t)$ of $s(x, t)$ such that $L(h) = 0$ and

$$\sup_{t>0} ||s(\cdot, t)||_p = \sup_{t>0} ||h(\cdot, t)||_p = ||f||_p$$

where $h(x, t) = P^t f(x)$, a nonnegative $f \in L^p(G)$, $p = (1/p_0)$. From Lemma 4,

$$s^+(x) = \sup_{t>0} |s(x, t)| \leqq \sup_{t>0} |h(x, t)| \leqq CMf(x) .$$

By Lemma 3, Lemma 17 and Proposition 20,

$$||s^*||_p^p = \int s^{*p}(x)dx \leqq C \int (ms^+)^p(x)dx$$
$$\leqq C \int s^{+p}(x)dx \leqq C \int Mf^p(x) \leqq C \int f^p(x)dx$$
$$= C||f||_p^p$$

where $s^*(x) = \sup_{(y,t) \in \Gamma(x)} |s(y, t)|$ for a truncated cone $\Gamma(x)$ of x. Hence s is nontangentially bounded at almost every $x \in G$. By Lemma 19, we get the existence of the almost everywhere limit $\lim_{t \to 0} u_j(x, t) = f_j(x)$, $j = 0, 1, \cdots, r$. Since $\sup_{t>0} h(x, t) = \sup_{t>0} P^t f(x) \in L^p(G)$, we obtain $f_j \in L^1(G)$ by the dominated convergence theorem, and the fact that $d\mu_j(x) = f_j(x)dx$, $j = 0, 1, \cdots, r$, follows immediately from Theorem 2′.

COROLLARY 22. *Let* $F = (u_0, u_1, \cdots, u_r)$ *be a Cauchy-Riemann system; then* $F \in H^p$, $1 \leqq p < \infty$ *if and only if*

$$\sup_{t>0} |F(x, t)| = F^+(x) \in L^p(G) .$$

Moreover, $||F||_p \approx ||F^+||_p$.

PROOF. We will use notations used in the proof of the above theorem. In the above theorem, we see that $F^+(x) \leqq CMf(x)^{1/p_0}$. Hence, by Lemma 4,

$$\begin{aligned}\|F^+\|_p^p &= \int F^+(x)^p dx \leqq C\int Mf(x)^{p/p_0}dx \\ &\leqq C\|f\|_{p/p_0}^{p/p_0} = C\|F\|_p^p .\end{aligned}$$

This proves the "only if" part.

Conversely,

$$\begin{aligned}\|F\|_p^p &= \sup_{t>0}\int |F(x, t)|^p dx \\ &\leqq \int \sup_{t>0} |F(x, t)|^p dx = \|F^+\|_p^p < \infty .\end{aligned}$$

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