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QUASI-CONFORMAL STABILITY OF FINITELY GENERATED FUNCTION GROUPS

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1. Introduction. Let G be the group of all Möbius transformations. Each $g \in G$ is a conformal self-mapping of the Riemann sphere $\hat{C} = C \cup \{\infty\}$, where C is the complex plane, and g is of the form

$$g(t) = (at + b)/(ct + d) ,$$

where $a, b, c, d \in C$ and $ad - bc = 1$. Hence the group G is a 3-dimensional Lie group isomorphic to $SL(2, C)$ modulo its center. An element $g \in G$, $g(t) = (at + b)/(ct + d)$, not being identity, is called parabolic if $\mathrm{tr}^2 g = (a + d)^2 = 4$; g is called elliptic if $\mathrm{tr}^2 g = (a + d)^2 \in [0, 4)$; in all other cases g is called loxodromic.

Let Γ be any subgroup of G. We denote by $\mathrm{Hom}\,(\Gamma, G)$ the set of all homomorphisms of Γ into G. A homomorphism $\theta\colon \Gamma \to G$ is called parabolic, if $\mathrm{tr}^2\, \theta(\gamma) = 4$ whenever $\gamma \in \Gamma$ is parabolic. We denote by $\mathrm{Hom}_p\,(\Gamma, G)$ the set of all parabolic homomorphisms of Γ into G.

Let Γ be a Kleinian group and let $w\colon \hat{C} \to \hat{C}$ be a quasi-conformal self-mapping of the Riemann sphere. We say that w is compatible with a Kleinian group Γ if $w \circ \Gamma \circ w^{-1} \subset G$. If w is compatible with Γ, then the mapping

$$\Gamma \ni \gamma \mapsto \theta(\gamma) = w \circ \gamma \circ w^{-1} \in G$$

is an isomorphism of Γ onto $w \circ \Gamma \circ w^{-1}$ and we call the isomorphism θ a quasi-conformal deformation of Γ. We denote by $\mathrm{Hom}_{qc}\,(\Gamma, G)$ the set of all quasi-conformal deformations of the Kleinian group Γ into G. We have $\mathrm{Hom}_{qc}\,(\Gamma, G) \subset \mathrm{Hom}_p\,(\Gamma, G) \subset \mathrm{Hom}\,(\Gamma, G)$.

Let Γ be a finitely generated Kleinian group with a system of generators $\{\gamma_1, \cdots, \gamma_N\}$. Then an element $\theta \in \mathrm{Hom}\,(\Gamma, G)$ is uniquely determined by $(\theta(\gamma_1), \cdots, \theta(\gamma_N)) \in G^N$, where G^N is the N times product space of G. We define the set $X(\Gamma; \gamma_1, \cdots, \gamma_N)$ in G^N such as

$$X(\Gamma; \gamma_1, \cdots, \gamma_N) = \{(\theta(\gamma_1), \cdots, \theta(\gamma_N)) \in G^N \,|\, \theta \in \mathrm{Hom}\,(\Gamma, G)\} .$$

Now we identify an element $\theta \in \mathrm{Hom}\,(\Gamma, G)$ with $(\theta(\gamma_1), \cdots, \theta(\gamma_N)) \in X(\Gamma; \gamma_1, \cdots, \gamma_N)$ and we regard $X(\Gamma; \gamma_1, \cdots, \gamma_N)$ as $\mathrm{Hom}\,(\Gamma, G)$. The corresponding spaces for $\mathrm{Hom}_p(\Gamma, G)$ and $\mathrm{Hom}_{qc}(\Gamma, G)$ in G^N are denoted by $X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$ and $X_{qc}(\Gamma; \gamma_1, \cdots, \gamma_N)$, respectively. We define stability of Γ as follows: A finitely generated Kleinian group Γ is said to be quasi-conformally stable (or simply stable) if there exists a system of generators $\{\gamma_1, \cdots, \gamma_N\}$ of Γ and a neighborhood $U(\gamma_1, \cdots, \gamma_N) \subset G^N$ of $(\gamma_1, \cdots, \gamma_N) \in G^N$ in G^N such that

$$X_p(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N) = X_{qc}(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N) .$$

Although it is well-known that stability of Γ in the above definition is independent of choice of generators of Γ, we shall give a proof of this fact for completeness. Namely we have the following proposition.

PROPOSITION. *Let Γ be a quasi-conformally stable group and let $\{\delta_1, \cdots, \delta_M\}$ be an arbitrary system of generators of Γ. Then*

$$X_p(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M) = X_{qc}(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M)$$

for some neighborhood $V(\delta_1, \cdots, \delta_M) \subset G^M$ of $(\delta_1, \cdots, \delta_M) \in G^M$ in G^M.

PROOF. Since Γ is quasi-conformally stable, there exists a system of generators $\{\gamma_1, \cdots, \gamma_N\}$ of Γ and a neighborhood $U(\gamma_1, \cdots, \gamma_N)$ of $(\gamma_1, \cdots, \gamma_N) \in G^N$ in G^N such that

$$X_p(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N) = X_{qc}(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N) .$$

Let $\omega_i(g_1, \cdots, g_N)$ be a word in N letters $g_1, \cdots, g_N$ satisfying $\omega_i(\gamma_1, \cdots, \gamma_N) = \delta_i$ for $i = 1, \cdots, M$ and let $\tilde{\omega}_j(h_1, \cdots, h_M)$ be a word in M letters $h_1, \cdots, h_M$ satisfying $\tilde{\omega}_j(\delta_1, \cdots, \delta_M) = \gamma_j$ for $j = 1, \cdots, N$. We define mappings $\omega: G^N \to G^M$ and $\tilde{\omega}: G^M \to G^N$ as follows;

$$\omega(g_1, \cdots, g_N) = (\omega_1(g_1, \cdots, g_N), \cdots, \omega_M(g_1, \cdots, g_N))$$
$$\tilde{\omega}(h_1, \cdots, h_M) = (\tilde{\omega}_1(h_1, \cdots, h_M), \cdots, \tilde{\omega}_N(h_1, \cdots, h_M)) .$$

We shall show $\tilde{\omega} \circ \omega = \mathrm{id}$ on $X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$. Let $(g_1, \cdots, g_N)$ be an arbitrary element of $X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$. Then $g_i = \theta(\gamma_i)$ for an element $\theta \in \mathrm{Hom}_p(\Gamma, G)$ determined uniquely. Therefore we have

$$\begin{aligned}
\tilde{\omega} \circ \omega(g_1, \cdots, g_N) &= \tilde{\omega}(\omega_1(g_1, \cdots, g_N), \cdots, \omega_M(g_1, \cdots, g_N)) \\
&= \tilde{\omega}(\omega_1(\theta(\gamma_1), \cdots, \theta(\gamma_N)), \cdots, \omega_M(\theta(\gamma_1), \cdots, \theta(\gamma_N))) \\
&= \tilde{\omega}(\theta(\omega_1(\gamma_1, \cdots, \gamma_N)), \cdots, \theta(\omega_M(\gamma_1, \cdots, \gamma_N))) \\
&= \tilde{\omega}(\theta(\delta_1), \cdots, \theta(\delta_M)) \\
&= (\tilde{\omega}_1(\theta(\delta_1), \cdots, \theta(\delta_M)), \cdots, \tilde{\omega}_N(\theta(\delta_1), \cdots, \theta(\delta_M))) \\
&= (\theta(\tilde{\omega}_1(\delta_1, \cdots, \delta_M)), \cdots, \theta(\tilde{\omega}_N(\delta_1, \cdots, \delta_M))) \\
&= (\theta(\gamma_1), \cdots, \theta(\gamma_N)) = (g_1, \cdots, g_N) .
\end{aligned}$$

Hence $\tilde{\omega} \circ \omega = \mathrm{id}$ on $X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$. By the same manner we have $\omega \circ \tilde{\omega} = \mathrm{id}$ on $X_p(\Gamma; \delta_1, \cdots, \delta_M)$. Since ω and $\tilde{\omega}$ are continuous, we see that ω is a homeomorphism between $X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$ and $X_p(\Gamma; \delta_1, \cdots, \delta_M)$. Therefore it holds that

$$\omega(X_p(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N)) = X_p(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M)$$

for some neighborhood $V(\delta_1, \cdots, \delta_M) \subset G^M$ of $(\delta_1, \cdots, \delta_M) \in G^M$ in G^M. Hence, for an arbitrary element $(h_1, \cdots, h_M)$ of $X_p(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M)$, there exists a unique element $(g_1, \cdots, g_N) \in X_p(\Gamma; \gamma_1, \cdots, \gamma_N) \cap U(\gamma_1, \cdots, \gamma_N)$ such that $\omega(g_1, \cdots, g_N) = (h_1, \cdots, h_M)$. Since $g_j = w \circ \gamma_j \circ w^{-1}$ for some quasi-conformal mapping w, we have

$$\begin{aligned}
(h_1, \cdots, h_M) &= \omega(g_1, \cdots, g_N) \\
&= (\omega_1(g_1, \cdots, g_N), \cdots, \omega_M(g_1, \cdots, g_N)) \\
&= (\omega_1(w \circ \gamma_1 \circ w^{-1}, \cdots, w \circ \gamma_N \circ w^{-1}), \cdots, \omega_M(w \circ \gamma_1 \circ w^{-1}, \cdots, w \circ \gamma_N \circ w^{-1})) \\
&= (w \circ \omega_1(\gamma_1, \cdots, \gamma_N) \circ w^{-1}, \cdots, w \circ \omega_M(\gamma_1, \cdots, \gamma_N) \circ w^{-1}) \\
&= (w \circ \delta_1 \circ w^{-1}, \cdots, w \circ \delta_M \circ w^{-1}) .
\end{aligned}$$

Hence we see $(h_1, \cdots, h_M) \in X_{qc}(\Gamma; \delta_1, \cdots, \delta_M)$, that is,

$$X_p(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M) \subset X_{qc}(\Gamma; \delta_1, \cdots, \delta_M) \cap V(\delta_1, \cdots, \delta_M) .$$

The converse inclusion relation is evident and we have our proposition.

A Kleinian group Γ is called a function group if Γ has an invariant component. In this paper we shall give a necessary and sufficient condition for finitely generated function groups to be quasi-conformally stable.

2. Notations and definitions. In our following discussions we need the cohomology theory and, in this section, we recall some notations and definitions of cohomology of Kleinian groups following Gardiner and Kra [3].

A representation ρ of a Kleinian group Γ on a finite dimensional vector space E is an anti-homomorphism of Γ into the automorphism group $\mathrm{Aut}\, E$ of E. For a representation ρ of a Kleinian group Γ on E, we define an action of Γ on E by

$$E \times \Gamma \ni (x, \gamma) \mapsto x \cdot \gamma = \rho(\gamma)(x) \in E .$$

A cocycle is a mapping $z: \Gamma \to E$ satisfying $z(\gamma_1 \circ \gamma_2) = z(\gamma_1) \cdot \gamma_2 + z(\gamma_2)$ for $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$. We denote by $Z^1(\Gamma, E)$ the space of all cocycles. A coboundary is a cocycle such that $z(\gamma) = x \cdot \gamma - x$ for some $x \in E$. We denote by $B^1(\Gamma, E)$ the space of all coboundaries. The cohomology space $H^1(\Gamma, E)$ is the space of cocycles factored by the space of coboundaries.

Throughout this paper we denote by e the identity of the group G. Let $\mathfrak{G}$ be the Lie algebra of G. Then the Lie algebra $\mathfrak{G}$ is identified with the tangent space $T_e(G)$ of G at e. The adjoint representation $\mathrm{Ad}(A)$: $\mathfrak{G} \longrightarrow \mathfrak{G}$ of G in $\mathfrak{G}$ is the differential at e of the map

$$G \ni B \mapsto A^{-1} \circ B \circ A \in G .$$

Since $\mathrm{Ad}\colon \Gamma \longrightarrow \mathrm{Aut}\,\mathfrak{G}$ yields a representation of Γ on $\mathfrak{G}$, we have a well defined cohomology space $H^1(\Gamma, \mathfrak{G})$. A cocycle $z \in Z^1(\Gamma, \mathfrak{G})$ is called parabolic if $z|_{\Gamma_0} \in B^1(\Gamma_0, \mathfrak{G})$ for any parabolic cyclic subgroup Γ_0 of Γ. We denote by $PZ^1(\Gamma, \mathfrak{G})$ the space of all parabolic cocycles. The space of parabolic cohomology $PH^1(\Gamma, \mathfrak{G})$ is the space of parabolic cocycles factored by the space of coboundaries.

Let Π be the vector space of quadratic polynomials with complex coefficients. We define a representation ρ of a Kleinian group Γ on Π by

$$(\rho(\gamma)(v))(t) = v(\gamma(t))/\gamma'(t)$$

for $v \in \Pi$, $\gamma \in \Gamma$, and $t \in \boldsymbol{C}$. We can thus define the cohomology space $H^1(\Gamma, \Pi)$. A cocycle $z \in Z^1(\Gamma, \Pi)$ is called parabolic if $z|_{\Gamma_0} \in B^1(\Gamma_0, \Pi)$ for any parabolic cyclic subgroup Γ_0 of Γ. We denote by $PZ^1(\Gamma, \Pi)$ the space of all parabolic cocycles. The space of the parabolic cohomology is denoted by $PH^1(\Gamma, \Pi)$. The following is due to Gardiner and Kra [3] and will be used later.

ISOMORPHISM THEOREM (Gardiner and Kra [3]). *Let Γ be a Kleinian group. Then $PH^1(\Gamma, \mathfrak{G})$ is isomorphic to $PH^1(\Gamma, \Pi)$.*

3. For a non-elementary Kleinian group Γ we denote by $\Omega(\Gamma)$ the region of discontinuity of Γ and by $\Lambda(\Gamma)$ the limit set of Γ. Let $A(\Omega(\Gamma), \Gamma)$ be the Banach space of bounded quadratic holomorphic forms on $\Omega(\Gamma)$ with respect to Γ and let $A_1(\Omega(\Gamma), \Gamma)$ be the open unit ball in $A(\Omega(\Gamma), \Gamma)$. If the Kleinian group Γ is finitely generated, then we have the anti-linear injective mapping

$$\beta^*\colon A(\Omega(\Gamma), \Gamma) \mapsto PH^1(\Gamma, \Pi) ,$$

which is the so-called Bers map.

Now we shall prove the following theorem. The essential part of our proof of the theorem is due to [3]. However, for the sake of completeness, we shall restate the proof of it.

THEOREM 1. *Let Γ be a non-elementary finitely generated Kleinian group. If $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$, then Γ is quasi-conformally stable.*

PROOF. Let λ be the Poincaré density on $\Omega(\Gamma)$. For $(g, \phi) \in G \times A_1(\Omega(\Gamma), \Gamma)$, we set $\mu = \lambda^{-2}\bar{\phi}$ on $\Omega(\Gamma)$ and $\mu = 0$ on $\Lambda(\Gamma)$ and observe that μ is a Beltrami coefficient for Γ. Let $w = g \circ w^{\mu}$, where w^{μ} is the μ-conformal self-mapping of $\hat{C}$ that fixes $0, 1, \infty$. For each element $\gamma \in \Gamma$, there is an element $\gamma^{(g,\phi)} \in G$ such that

$$w \circ \gamma = \gamma^{(g,\phi)} \circ w .$$

We set

$$f((g, \phi)) = (\gamma_1^{(g,\phi)}, \cdots, \gamma_N^{(g,\phi)}) \in G^N ,$$

where $\{\gamma_1, \cdots, \gamma_N\}$ is a system of generators of Γ. The mapping f is analytic and $f(G \times A_1(\Omega(\Gamma), \Gamma))$ is contained in $X_{qc}(\Gamma; \gamma_1, \cdots, \gamma_N)$. Now, by the isomorphism theorem stated in Section 2, the image of the tangent space $\mathfrak{G} \times A(\Omega(\Gamma), \Gamma)$ of $G \times A_1(\Omega(\Gamma), \Gamma)$ at $(e, 0)$ under the tangent linear map $(df)(e, 0)$ at $(e, 0)$ is canonically isomorphic to the linear space of cocycles that correspond to the Bers cohomology space $\beta^*(A(\Omega(\Gamma), \Gamma))$.

Let A be a set of suffices α such that $\omega_\alpha(g_1, \cdots, g_N)$ is the word in N letters $g_1, \cdots, g_N$ satisfying $\omega_\alpha(\gamma_1, \cdots, \gamma_N) = e$ and let B be a set of suffices β such that $\omega_\beta(g_1, \cdots, g_N)$ is the word in N letters $g_1, \cdots, g_N$ satisfying $\mathrm{tr}^2\, \omega_\beta(\gamma_1, \cdots, \gamma_N) = 4$ and $\omega_\beta(\gamma_1, \cdots, \gamma_N) \neq e$. Consider two mappings $F_\alpha: G^N \to G$ and $F_\beta: G^N \to C$ such as

$$F_\alpha(g_1, \cdots, g_N) = \omega_\alpha(g_1, \cdots, g_N) , \quad F_\beta(g_1, \cdots, g_N) = \mathrm{tr}^2\, \omega_\beta(g_1, \cdots, g_N) - 4 .$$

Then

$$\{\bigcap_{\alpha \in A} F_\alpha^{-1}(e)\} \cap \{\bigcap_{\beta \in B} F_\beta^{-1}(0)\} = X_p(\Gamma; \gamma_1, \cdots, \gamma_N)$$

and we see that

$$\text{(a)} \qquad f(G \times A_1(\Omega(\Gamma), \Gamma)) \subset \{\bigcap_{\alpha \in A} F_\alpha^{-1}(e)\} \cap \{\bigcap_{\beta \in B} F_\beta^{-1}(0)\} .$$

For the tangent linear mappings $(dF_\alpha)(\gamma_1, \cdots, \gamma_N)$ and $(dF_\beta)(\gamma_1, \cdots, \gamma_N)$ we have the isomorphism

$$\{\bigcap_{\alpha \in A} \ker (dF_\alpha)(\gamma_1, \cdots, \gamma_N)\} \cap \{\bigcap_{\beta \in B} \ker (dF_\beta)(\gamma_1, \cdots, \gamma_N)\} \cong PZ^1(\Gamma, \mathfrak{G}) ,$$

where $\ker (dF_\alpha)(\gamma_1, \cdots, \gamma_N)$ and $\ker (dF_\beta)(\gamma_1, \cdots, \gamma_N)$ are the kernels of the linear mappings $(dF_\alpha)(\gamma_1, \cdots, \gamma_N)$ and $(dF_\beta)(\gamma_1, \cdots, \gamma_N)$, respectively (see [3] and [7]).

We assume that $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$. Since $(df)(e, 0)(\mathfrak{G} \times A(\Omega(\Gamma), \Gamma))$ is isomorphic to the linear space of cocycles that correspond to the Bers cohomology, we see from our assumption that $(df)(e, 0)(\mathfrak{G} \times A(\Omega(\Gamma), \Gamma))$ is isomorphic to $PZ^1(\Gamma, \Pi)$. Hence we see that $(df)(e, 0)(\mathfrak{G} \times$

$A(\Omega(\Gamma), \Gamma))$ is isomorphic to $PZ^1(\Gamma, \mathfrak{G})$. Therefore we have

$$\text{(b)} \qquad (df)(e, 0)(\mathfrak{G} \times A(\Omega(\Gamma), \Gamma)) = \{\bigcap_{\alpha \in A} \ker (dF_\alpha)(\gamma_1, \cdots, \gamma_N)\} \cap \{\bigcap_{\beta \in B} \ker (dF_\beta)(\gamma_1, \cdots, \gamma_N)\} .$$

Thus using (a) and (b), we see, as in the same manner as in the proof of Theorem 8.4 in [3] that Γ is quasi-conformally stable (see also the key lemma, in proving Theorem 8.4, in [7]).

4. In the following two sections we shall show the converse of Theorem 1 for non-elementary finitely generated function groups.

We call the group consisting of only the identity e to be trivial. For our purpose, we use the Maskit's Combination Theorems I and II in [5], where the amalgamated subgroups and the conjugated subgroups are cyclic or trivial.

First we shall prove

LEMMA 1. *Let Γ be the non-elementary Kleinian group which is constructed from Γ_1 and Γ_2 by application of Combination Theorem I, where the amalgamated subgroup $H = \Gamma_1 \cap \Gamma_2$ be parabolic cyclic or elliptic cyclic or trivial. Let θ_i be an element of* $\mathrm{Hom}_p(\Gamma_i, G)$ *for $i = 1, 2$. Then there exists an element $\theta \in \mathrm{Hom}_p(\Gamma, G)$ such that $\theta = \theta_i$ on Γ_i for $i = 1, 2$ if and only if $\theta_1 = \theta_2$ on H.*

PROOF. It is sufficient to prove the if part. Let $\Gamma_1 = H + \sum_\sigma Ha_\sigma$ and $\Gamma_2 = H + \sum_\tau Hb_\tau$ be the right coset representations of Γ_1 and Γ_2, respectively. By Combination Theorem I we see that Γ is the free product of Γ_1 and Γ_2 with the amalgamated subgroup H. Therefore, for any $\gamma \in \Gamma$, we have a unique representation

$$\gamma = h \circ \gamma_1 \circ \cdots \circ \gamma_n ,$$

where $h \in H$ and γ_j is some of a_σ or b_τ, and γ_j and γ_{j+1} are not contained simultaneously in the same $\Gamma_i (i = 1$ or $2)$. We set $\tilde{\theta} = \theta_1|_H = \theta_2|_H$ and define the mapping $\theta: \Gamma \to G$ by

$$\theta(\gamma) = \tilde{\theta}(h) \circ \theta_{i_1}(\gamma_1) \circ \cdots \circ \theta_{i_n}(\gamma_n) ,$$

where $\gamma = h \circ \gamma_1 \circ \cdots \circ \gamma_n$ is a unique representation of γ and $i_k = 1$ if $\gamma_k \in \Gamma_1$ and $i_k = 2$ if $\gamma_k \in \Gamma_2$. It is easily shown that $\theta = \theta_i$ on Γ_i by the definition of θ. We can prove by using induction on n as in the proof of Theorem 1 in [6] that θ is a homomorphism of Γ into G. Moreover, for any parabolic element $\gamma \in \Gamma$, there exists a parabolic element $\tilde{\gamma} \in \Gamma_i$ $(i = 1$ or $2)$ and an element $\delta \in \Gamma$ such that $\gamma = \delta \circ \tilde{\gamma} \circ \delta^{-1}$ (see [5]). Therefore we have $\theta(\gamma) = \theta(\delta) \circ \theta(\tilde{\gamma}) \circ \theta(\delta)^{-1}$. Since $\theta = \theta_i \in \mathrm{Hom}_p(\Gamma_i, G)$ on Γ_i, we have $\mathrm{tr}^2 \theta(\gamma) = \mathrm{tr}^2 \theta(\tilde{\gamma}) = \mathrm{tr}^2 \theta_i(\tilde{\gamma}) = 4$. Hence θ is an element of $\mathrm{Hom}_p(\Gamma, G)$ and we have proved our lemma.

Next we shall show that, for a parabolic cyclic or an elliptic cyclic group H, the parabolic homomorphism $\theta \in \mathrm{Hom}_p(H, G)$ sufficiently close to the identity is the conjugation by an element $x \in G$ which is sufficiently close to the identity element e, that is, we can prove

LEMMA 2. *Let H be a parabolic or an elliptic cyclic group with a generator h. Then there exists a neighborhood $U(h)$ of h in G and a neighborhood $V(e)$ of e in G such that $g = x \circ h \circ x^{-1}$ for an arbitrary element g of $X_p(H; h) \cap U(h)$ and for some $x \in V(e)$.*

PROOF. First we assume that H is a parabolic cyclic group with a generator h. Let $F_1: G \to G$ and $F_2: G \to C$ be the mappings such as

$$F_1(g) = g \circ h \circ g^{-1} \quad \text{and} \quad F_2(g) = \mathrm{tr}^2 g - 4 ,$$

respectively. The kernel of the tangent linear map $(dF_2)(h)$ of F_2 at h is identified with the space of coboundaries $B^1(H, \mathfrak{G})$ for H (see Lemma 8.3 in [3]) so that

$$\begin{aligned}\dim (dF_2)(h)(T_h(G)) &= \dim T_h(G) - \dim B^1(H, \mathfrak{G}) \\ &= \dim \mathfrak{G} - \dim B^1(H, \mathfrak{G}) ,\end{aligned}$$

where $T_h(G)$ is the tangent space of G at h. Since $\mathfrak{G} \cong \Pi$ and $B^1(H, \mathfrak{G}) \cong B^1(H, \Pi)$ and since $\dim \Pi > \dim B^1(H, \Pi)$ (for instance, see [6]), we see that $\dim (dF_2)(h)(T_h(G)) \neq 0$. Hence we have $(dF_2)(h) \neq 0$. Therefore, there exists a neighborhood $U'(h)$ of h in G such that $U'(h) \cap F_2^{-1}(0)$ is a complex submanifold of $U'(h)$ with

$$T_h(U'(h) \cap F_2^{-1}(0)) = \ker (dF_2)(h)$$

for the tangent space $T_h(U'(h) \cap F_2^{-1}(0))$ of $U'(h) \cap F_2^{-1}(0)$ at h. The image of $\mathfrak{G}$ under the tangent linear mapping $(dF_1)(e)$ is identified with the space of coboundaries $B^1(H, \mathfrak{G})$ for H (see [7]). Since the kernel of $(dF_2)(h)$ is also the space of coboundaries, we see that the linear mapping

$$(dF_1)(e): \mathfrak{G} \mapsto T_h(U'(h) \cap F_2^{-1}(0))$$

is surjective. Therefore, by the implicit function theorem, we have a neighborhood $V(e)$ of e in G such that $F_1(V(e))$ is the neighborhood of h in $U'(h) \cap F_2^{-1}(0)$, that is,

$$F_1(V(e)) = U(h) \cap F_2^{-1}(0)$$

for some neighborhood $U(h)$ of h in G. Since $F_2^{-1}(0) = X_p(H; h)$, we see that $F_1(V(e)) = X_p(H; h) \cap U(h)$. Hence, for any $g \in X_p(H; h) \cap U(h)$, it holds that $g = F_1(x)$ for some $x \in V(e)$. Therefore, we have our lemma in the case of a parabolic cyclic group H.

Next we assume that H is an elliptic cyclic group of order ν with a generator h. We define mappings $F_1: G \to G$ and $F_2: G \to G$ such as

$$F_1(g) = g \circ h \circ g^{-1} \quad \text{and} \quad F_2(g) = g^{\nu} ,$$

respectively. The kernel of $(dF_2)(h)$ is identified with the space of cocycles $Z^1(H, \mathfrak{G})$ for H, and for an elliptic cyclic group H we have $Z^1(H, \mathfrak{G}) = B^1(H, \mathfrak{G})$ (see [7]). Let V_0 be a neighborhood of h in G and Y_0 be a complex submanifold of V_0 satisfying $T_h(Y_0) = \ker(dF_2)(h)$ and $F_2^{-1}(e) \cap V_0 \subset Y_0$ (see [3]). Since the image of $\mathfrak{G}$ under the tangent linear mapping $(dF_1)(e)$ is identified with the space of coboundaries $B^1(H, \mathfrak{G})$ for H and since the kernel of $(dF_2)(h)$ is also the space of coboundaries, we see that the linear mapping

$$(dF_1)(e): \mathfrak{G} \mapsto T_h(Y_0)$$

is surjective. Hence, we have a neighborhood $V(e)$ of e in G such that

$$F_1(V(e)) = Y_0 \cap U(h)$$

for some neighborhood $U(h)$ of h in V_0. Since $F_1(G) \subset F_2^{-1}(e) = X_p(H; h) = X(H; h)$, we have $X(H; h) \cap U(h) \subset Y_0 \cap U(h) = F_1(V(e)) \subset X(H; h) \cap U(h)$. So it holds that $F_1(V(e)) = X(H; h) \cap U(h)$. Therefore, for any $g \in X(H; h) \cap U(h)$, we have $g = F_1(x)$ for some $x \in V(e)$. Thus we have also proved our lemma for the case of an elliptic cyclic group H.

The contents in general situation of the above lemma is seen in [7].

LEMMA 3. *Let Γ be a non-elementary Kleinian group which is constructed from finitely generated Kleinian groups Γ_1 and Γ_2 by application of Combination Theorem I, where the amalgamated subgroup $H = \Gamma_1 \cap \Gamma_2$ be parabolic cyclic or elliptic cyclic or trivial. Let $\{\gamma_1, \cdots, \gamma_{M-1}, h\}$ and $\{\delta_1, \cdots, \delta_{N-1}, h\}$ be systems of generators for Γ_1 and Γ_2, respectively, where h is a generator for H. Then there exist neighborhoods $U(\gamma_1, \cdots, \gamma_{M-1}, h)$ of $(\gamma_1, \cdots, \gamma_{M-1}, h) \in G^M$ in G^M and $V(\delta_1, \cdots, \delta_{N-1}, h)$ of $(\delta_1, \cdots, \delta_{N-1}, h) \in G^N$ in G^N such that*

$$(f_1, \cdots, f_M, g_1, \cdots g_N) \in X_p(\Gamma; \gamma_1, \cdots, \gamma_{M-1}, h, \delta_1, \cdots, \delta_{N-1}, h) \\ \cap \{U(\gamma_1, \cdots, \gamma_{M-1}, h) \times V(\delta_1, \cdots, \delta_{N-1}, h)\}$$

for an arbitrary element $(f_1, \cdots, f_M)$ of $X_p(\Gamma_1; \gamma_1, \cdots, \gamma_{M-1}, h) \cap U(\gamma_1, \cdots, \gamma_{M-1}, h)$ and for some element $(g_1, \cdots, g_N)$ of $X_p(\Gamma_2; \delta_1, \cdots, \delta_{N-1}, h) \cap V(\delta_1, \cdots, \delta_{N-1}, h)$.

PROOF. We assume that H is a parabolic cyclic group. We choose a neighborhood $U(\gamma_1, \cdots, \gamma_{M-1}, h)$ sufficiently small so that $f_M = x \circ h \circ x^{-1}$ for each element $(f_1, \cdots, f_M) \in X_p(\Gamma_1; \gamma_1, \cdots, \gamma_{M-1}, h) \cap U(\gamma_1, \cdots, \gamma_{M-1}, h)$ and for some $x \in G$ which is sufficiently close to the identity element e. This is possible by Lemma 2. Now we define a parabolic homomorphism $(g_1, \cdots, g_N) \in X_p(\Gamma_2; \delta_1, \cdots, \delta_{N-1}, h)$ by

$$g_i = x \circ \delta_i \circ x^{-1}, \qquad (i = 1, \cdots, N-1), \qquad g_N = x \circ h \circ x^{-1}.$$

Since x is sufficiently close to the identity element e, we see

$$(g_1, \cdots, g_N) \in X_p(\Gamma_2; \delta_1, \cdots, \delta_{N-1}, h) \cap V(\delta_1, \cdots, \delta_{N-1}, h)$$

for some small neighborhood $V(\delta_1, \cdots, \delta_{N-1}, h)$. Let θ_1 and θ_2 be the corresponding parabolic homomorphisms to $(f_1, \cdots, f_M)$ and $(g_1, \cdots, g_N)$, respectively. Then θ_1 and θ_2 satisfy $\theta_1|_H = \theta_2|_H$. Therefore, we see by Lemma 1 that $(f_1, \cdots, f_M, g_1, \cdots, g_N)$ is an element of $X_p(\Gamma; \gamma_1, \cdots, \gamma_{M-1}, h, \delta_1, \cdots, \delta_{N-1}, h)$. Clearly $(f_1, \cdots, f_M, g_1, \cdots, g_N)$ is also contained in $U(\gamma_1, \cdots, \gamma_{M-1}, h) \times V(\delta_1, \cdots, \delta_{N-1}, h)$. Thus we have proved our lemma for a parabolic cyclic group H. In the other cases, we can obtain the desired by the same manner as above.

Lemma 3 implies that a parabolic homomorphism $\theta_1 \in \mathrm{Hom}_p(\Gamma_1, G)$ sufficiently close to the identity homomorphism is restriction of some parabolic homomorphism $\theta \in \mathrm{Hom}_p(\Gamma, G)$ on Γ_1 which is sufficiently close to the identity homomorphism. We can also prove that any $\theta_2 \in \mathrm{Hom}_p(\Gamma_2, G)$ sufficiently close to the identity homomorphism is restriction of some $\theta \in \mathrm{Hom}_p(\Gamma, G)$ on Γ_2 which is sufficiently close to the identity homomorphism.

LEMMA 4. *Let Γ be a non-elementary Kleinian group which is constructed from finitely generated Kleinian group Γ_1 and an element $f \in G$ by application of Combination Theorem II, where the conjugated subgroups H_1 and H_2 are parabolic cyclic or elliptic cyclic or trivial. Let $\{\gamma_1, \cdots, \gamma_L, h_1, h_2\}$ be a system of generators of Γ_1, where h_i is a generator of H_i $(i = 1, 2)$ with $f \circ h_1 \circ f^{-1} = h_2$. Let Γ_2 be the group generated by f. Then there exists a neighborhood $U(\gamma_1, \cdots, \gamma_L, h_1, h_2)$ of $(\gamma_1, \cdots, \gamma_L, h_1, h_2) \in G^{L+2}$ in G^{L+2} and a neighborhood $V(f)$ of f in G such that*

$$(g_1, \cdots, g_{L+2}, g) \in X_p(\Gamma; \gamma_1, \cdots, \gamma_L, h_1, h_2, f) \cap \{U(\gamma_1, \cdots, \gamma_L, h_1, h_2) \times V(f)\}$$

for an arbitrary element $(g_1, \cdots, g_{L+2}) \in X_p(\Gamma_1; \gamma_1, \cdots, \gamma_L, h_1, h_2) \cap U(\gamma_1, \cdots, \gamma_L, h_1, h_2)$ and for some $g \in X_p(\Gamma_2; f) \cap V(f)$.

PROOF. We assume that H is a parabolic cyclic group. We choose a sufficiently small neighborhood $U(\gamma_1, \cdots, \gamma_L, h_1, h_2)$ so that $g_{L+1} = x_1 \circ h_1 \circ x_1^{-1}$

and $g_{L+2}=x_2\circ h_2\circ x_2^{-1}$ for each element $(g_1,\cdots,g_{L+2})\in X_p(\Gamma_1;\gamma_1,\cdots,\gamma_L,h_1,h_2)\cap U(\gamma_1,\cdots,\gamma_L,h_1,h_2)$ and for some $x_1\in G$ and $x_2\in G$ which are sufficiently close to the identity element e. Let θ_1 be the parabolic homomorphism corresponding to $(g_1,\cdots,g_{L+2})$. Now we define a mapping $\theta:\{\gamma_1,\cdots,\gamma_L,h_1,h_2,f\}\to G$ by

$$\theta(\gamma_i)=\theta_1(\gamma_i)(i=1,\cdots,L)\,,\quad \theta(h_j)=\theta_1(h_j)(j=1,2)\,,\quad \theta(f)=x_2\circ f\circ x_1^{-1}\,.$$

Then we have

$$\begin{aligned}&\theta(f)\circ\theta(h_1)\circ\theta(f)^{-1}\circ\theta(h_2)^{-1}\\&\quad=(x_2\circ f\circ x_1^{-1})\circ(x_1\circ h_1\circ x_1^{-1})\circ(x_2\circ f\circ x_1^{-1})^{-1}\circ(x_2\circ h_2\circ x_2^{-1})^{-1}\\&\quad=x_2\circ f\circ h_1\circ f^{-1}\circ h_2^{-1}\circ x_2^{-1}\,.\end{aligned}$$

Since $f\circ h_1\circ f^{-1}\circ h_2^{-1}=e$, we see $\theta(f)\circ\theta(h_1)\circ\theta(f)^{-1}\circ\theta(h_2)^{-1}=e$. For any relation $\omega(\gamma_1,\cdots,\gamma_L,h_1,h_2)=e$ in Γ_1, it holds that $\omega(\theta(\gamma_1),\cdots,\theta(\gamma_L),\theta(h_1),\theta(h_2))=\omega(\theta_1(\gamma_1),\cdots,\theta_1(\gamma_L),\theta_1(h_1),\theta_1(h_2))=\theta_1(\omega(\gamma_1,\cdots,\gamma_L,h_1,h_2))=\theta_1(e)=e$. From Combination Theorem II, we see that relations in Γ are consequences of the relations in Γ_1 and the relation $f\circ h_1\circ f^{-1}\circ h_2^{-1}=e$. Therefore, we can extend the mapping θ to Γ such that θ to be an element of $\mathrm{Hom}(\Gamma,G)$. Hence $(g_1,\cdots,g_{L+2},x_2\circ f\circ x_1^{-1})$ corresponding to the homomorphism θ is an element of $X(\Gamma;\gamma_1,\cdots,\gamma_L,h_1,h_2,f)$. Since x_1 and x_2 are sufficiently close to e, we see

$$\begin{aligned}(g_1,\cdots,g_{L+2},x_2\circ f\circ x_1^{-1})\in X(\Gamma;\gamma_1,\cdots,\gamma_L,h_1,h_2,f)\\\cap\{U(\gamma_1,\cdots,\gamma_L,h_1,h_2)\times V(f)\}\end{aligned}$$

for some small neighborhood $V(f)$. Moreover, for any parabolic element $\gamma\in\Gamma$, there exist a parabolic element $\tilde\gamma\in\Gamma_1$ and an element $\delta\in\Gamma$ such that $\gamma=\delta\circ\tilde\gamma\circ\delta^{-1}$ (see [5]). Therefore, we see that $\theta(\gamma)=\theta(\delta)\circ\theta(\tilde\gamma)\circ\theta(\delta)^{-1}$. Since $\theta=\theta_1$ on Γ_1 by the definition of θ and since $\theta_1\in\mathrm{Hom}_p(\Gamma_1,G)$, we have $\mathrm{tr}^2\,\theta(\gamma)=\mathrm{tr}^2\,\theta(\tilde\gamma)=\mathrm{tr}^2\,\theta_1(\tilde\gamma)=4$. Hence θ is an element of $\mathrm{Hom}_p(\Gamma,G)$, which shows $(g_1,\cdots,g_{L+2},x_2\circ f\circ x_1^{-1})\in X_p(\Gamma;\gamma_1,\cdots,\gamma_L,h_1,h_2,f)$. Thus we have our lemma in the case of a parabolic cyclic group H. We can also prove lemma in the remainder cases by the same argument as above.

Lemma 4 implies that a parabolic homomorphism $\theta_1\in\mathrm{Hom}_p(\Gamma_1,G)$ sufficiently close to the identity homomorphism is restriction of some parabolic homomorphism $\theta\in\mathrm{Hom}_p(\Gamma,G)$ on Γ_1 which is sufficiently close to the identity homomorphism.

5. A finitely generated Kleinian group Γ is called a basic group if Γ has a simply connected invariant component and contains no accidental parabolic transformations. Hence a basic group is either elementary or quasi-Fuchsian or degenerate (see [5]). Let Γ be a non-elementary finitely

generated function group. Then Maskit [5] proved that Γ can be constructed from basic groups $\Gamma_1, \cdots, \Gamma_s$ by using Combination Theorems I and II, where in each step of applying Combination Theorems, the amalgamated subgroups and the conjugated subgroups are parabolic cyclic or elliptic cyclic or trivial.

Now the converse of Theorem 1 for non-elementary finitely generated function groups is obtained by using the following two lemmas.

LEMMA 5. *Let Γ be a non-elementary Kleinian group constructed from finitely generated Kleinian groups Γ_1 and Γ_2 by application of Combination Theorem I, where the amalgamated subgroup $H = \Gamma_1 \cap \Gamma_2$ is parabolic cyclic or elliptic cyclic or trivial. If Γ is quasi-conformally stable, then Γ_1 and Γ_2 are also quasi-conformally stable.*

PROOF. Let θ_1 be an arbitrary element of $\mathrm{Hom}_p(\Gamma_1, G)$ which is sufficiently close to the identity homomorphism. Then, by Lemma 3, there exists an element $\theta \in \mathrm{Hom}_p(\Gamma, G)$ with $\theta = \theta_1$ on Γ_1 and θ is sufficiently close to the identity homomorphism. Since Γ is quasi-conformally stable, we see $\theta(\gamma) = w \circ \gamma \circ w^{-1}$ for some quasi-conformal mapping $w: \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$. Since $\theta = \theta_1$ on Γ_1 we have $\theta_1(\gamma) = w \circ \gamma \circ w^{-1}$ for all $\gamma \in \Gamma_1$. This shows the quasi-conformal stability of Γ_1. In the same way we can see the quasi-conformal stability of Γ_2.

LEMMA 6. *Let a non-elementary Kleinian group Γ be constructed from a finitely generated Kleinian group Γ_1 and an element $f \in G$ by application of Combination Theorem II, where the conjugated subgroups H_1 and H_2 be parabolic cyclic or elliptic cyclic or trivial. If Γ is quasi-conformally stable, then Γ_1 is also quasi-conformally stable.*

PROOF. Let θ_1 be an arbitrary element of $\mathrm{Hom}_p(\Gamma_1, G)$ which is sufficiently close to the identity homomorphism. Then, by Lemma 4, there exists an element $\theta \in \mathrm{Hom}_p(\Gamma, G)$ with $\theta = \theta_1$ on Γ_1 and θ is sufficiently close to the identity homomorphism. Since Γ is quasi-conformally stable, we see $\theta(\gamma) = w \circ \gamma \circ w^{-1}$ for some quasi-conformal mapping $w: \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$. Since $\theta = \theta_1$ on Γ_1, we have $\theta_1(\gamma) = w \circ \gamma \circ w^{-1}$ for all $\gamma \in \Gamma_1$. This shows the quasi-conformal stability of Γ_1.

REMARK. Above two lemmas are converse of Abikoff's theorems under the assumption that the amalgamated subgroups and the conjugated subgroups are parabolic cyclic or elliptic cyclic or trivial (see W. Abikoff's papers "Constructability and Bers stability of Kleinian groups" in Discontinuous groups and Riemann surfaces, Ann. of Math. Studies, 79 (1974) and "On the decomposition and deformation of Kleinian groups" in Contributions to analysis, Academic Press, 1974).

Now we can prove the following.

THEOREM 2. *Let Γ be a non-elementary finitely generated function group. If Γ is quasi-conformally stable, then*

$$PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma)) .$$

PROOF. As stated already, the non-elementary finitely generated function group Γ is decomposed into basic groups $\Gamma_1, \cdots, \Gamma_s$, where, in each step of applying Combination Theorems, the amalgamated subgroups and the conjugated subgroups are parabolic cyclic or elliptic cyclic or trivial. Now we assume that Γ is quasi-conformally stable. Then we see by Lemma 5 and Lemma 6 that $\Gamma_1, \cdots, \Gamma_s$ are all quasi-conformally stable. On the other hand, the degenerate basic groups are not quasi-conformally stable (for instance, see corollary of Theorem 11.2 in [3]). Hence $\Gamma_1, \cdots, \Gamma_s$ are elementary or quasi-Fuchsian. Therefore, we have $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$ by Theorem 5 in [6].

The above Theorem 1 and Theorem 2 together with Theorem 5 in [6] yield the following.

THEOREM 3. *Let Γ be a non-elementary finitely generated function group. Then the following three conditions are equivalent to each other*:

(1) $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$,

(2) *Γ is quasi-conformally stable, and*

(3) *Γ is decomposed into elementary or quasi-Fuchsian basic groups.*

REMARK. We denote by $\mathscr{C}$ the class of all non-elementary Kleinian groups which can be built up in a finite number of steps from the basic groups by using Combination Theorems I and II, where, in each step of applying Combination Theorems, the amalgamated subgroups and the conjugated subgroups are parabolic cyclic or elliptic cyclic or trivial. The non-elementary finitely generated function groups are contained in the class $\mathscr{C}$. Further, it is not so difficult to show that the class $\mathscr{C}_1$ introduced by Maskit [5] is a proper subclass of our class $\mathscr{C}$. It is proved in [6] that, for a non-elementary finitely generated function group Γ, the condition $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$ is equivalent to the condition that Γ is decomposed into elementary or quasi-Fuchsian basic groups. However, the equivalency of these two conditions also holds for groups contained in the class $\mathscr{C}$ (the proof of this fact is already given in the proof of Theorem 5 in [6]). Hence we have the following.

THEOREM 3'. *Let Γ be a Kleinian group in the class $\mathscr{C}$. Then the following three conditions are equivalent to each other*:

(1) $PH^1(\Gamma, \Pi) = \beta^*(A(\Omega(\Gamma), \Gamma))$,

(2) *Γ is quasi-conformally stable, and*

(3) *Γ is decomposed into elementary or quasi-Fuchsian basic groups.*

PROOF. It is sufficient to prove only the equivalency of statements (1) and (2) and the proof of this fact is essentially given in the proof of Theorem 2.

6. Finally we state an application of Theorem 3'. Let Γ be a Kleinian group and let w be a quasi-conformal self-mapping of the Riemann sphere which is compatible with Γ. If Γ is constructed from Kleinian groups Γ_1 and Γ_2 by application of Combination Theorem I, where the amalgamated subgroup $H = \Gamma_1 \cap \Gamma_2$ is parabolic cyclic or elliptic cyclic or trivial, then we see that the Kleinian group $w \circ \Gamma \circ w^{-1}$ is constructed from Kleinian groups $w \circ \Gamma_1 \circ w^{-1}$ and $w \circ \Gamma_2 \circ w^{-1}$ by application of Combination Theorem I, where the amalgamated subgroup $w \circ H \circ w^{-1} = (w \circ \Gamma_1 \circ w^{-1}) \cap (w \circ \Gamma_2 \circ w^{-1})$ is parabolic cyclic or elliptic cyclic or trivial (see [5]). On the other hand, if Γ is constucted from Γ_1 and an element $f \in G$ by application of Combination Theorem II, where the conjugated subgroups H_1 and H_2 are parabolic cyclic or elliptic cyclic or trivial, then we see that the Kleinian group $w \circ \Gamma \circ w^{-1}$ is constructed from the Kleinian group $w \circ \Gamma_1 \circ w^{-1}$ and an element $w \circ f \circ w^{-1} \in G$ by application of Combination Theorem II, where the conjugated subgroups $w \circ H_1 \circ w^{-1}$ and $w \circ H_2 \circ w^{-1}$ are parabolic cyclic or elliptic cyclic or trivial (see also [5]). Therefore, if a Kleinian group Γ is in the class $\mathscr{C}$, then the quasi-conformal deformation $w \circ \Gamma \circ w^{-1}$ of Γ is also in the class $\mathscr{C}$. Hence we have the following.

THEOREM 4. *Let Γ be a Kleinian group in the class $\mathscr{C}$ and assume that Γ is quasi-conformally stable. Then the quasi-conformal deformation of Γ is also quasi-conformally stable. In particular, the quasi-conformal deformation of a non-elementary finitely generated quasi-conformally stable function group is also quasi-conformally stable.*

PROOF. Let w be a quasi-conformal self-mapping of $\hat{C}$ which satisfies $w \circ \Gamma \circ w^{-1} \subset G$. Since Γ is in $\mathscr{C}$ and since Γ is quasi-conformally stable, we see by Theorem 3' that Γ is decomposed into elementary or quasi-Fuchsian basic groups $\Gamma_1, \cdots, \Gamma_s$. Hence the groups $w \circ \Gamma_1 \circ w^{-1}, \cdots,$ $w \circ \Gamma_s \circ w^{-1}$ are elementary or quasi-Fuchsian. Since $w \circ \Gamma \circ w^{-1}$ is in $\mathscr{C}$

and since $w \circ \Gamma \circ w^{-1}$ is decomposed into $w \circ \Gamma_1 \circ w^{-1}, \cdots, w \circ \Gamma_s \circ w^{-1}$, we see again by Theorem 3′ that $w \circ \Gamma \circ w^{-1}$ is also quasi-conformally stable.

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