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DEFORMATIONS OF SASAKIAN STRUCTURES AND ITS APPLICATION TO THE BRIESKORN MANIFOLDS

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1. Introduction. Let (M, ϕ, ξ, η, g) be a Sasakian manifold which admits an infinitesimal automorphism μ. Under further conditions of μ we will show that we can construct a deformation of a Sasakian structure with respect to μ. When $\mu = \alpha \cdot \xi$ for some real number α such that $1 + \alpha > 0$, then the deformation with respect to μ is called D-deformation where D is a distribution defined by $\eta = 0$. Denote by $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the Sasakian manifold deformed with respect to μ, we will find that D is also defined by $\tilde{\eta} = 0$. From this fact our deformation is a generalization of D-deformation. But we can find many deformations which are not D-deformations. In those cases the trajectories of $\tilde{\xi}$ may be different from those of ξ and in many cases there exist trajectories with infinite length.

Recently a contact structure on a Brieskorn manifold is studied by K. Abe, C. J. Hsu and S. Sasaki [1], [4]. And in [1] K. Abe proved that there exist Sasakian structures on Brieskorn manifolds. For the application of the deformation of Sasakian manifolds we will give another proof to his result. For this, we apply these deformations on the standard Sasakian spheres and show that Brieskorn manifolds have almost contact metric structures so that Brieskorn manifolds are invariant submanifolds of these deformed spheres. Because it is known that an invariant submanifold of a Sasakian manifold is also a Sasakian manifold [2], we conclude that every Brieskorn manifold has a Sasakian structure.

2. The deformations of Sasakian manifolds. (M, ϕ, ξ, η, g) is called a Sasakian manifold when the following relations hold for the structure tensor fields. η is a 1-form, ξ is a vector field, ϕ is a $(1, 1)$-tensor field and g is a Riemannian metric on M such that

$$\eta(\xi) = 1 \tag{1}$$

$$\phi^2(X) = -X + \eta(X)\xi \tag{2}$$

$$d\eta(X, Y) = g(\phi(X), Y) \tag{3}$$

$$g(\xi, \xi) = 1 \tag{4}$$

$$N(X, Y) = 0 \tag{5}$$

where X and Y are arbitrary vector fields over M and $N(X, Y)$ is a vector field defined by

$$N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] - (X\cdot\eta(Y) - Y\cdot\eta(X))\xi\ . \tag{6}$$

This tensor field N is called the torsion tensor field of the almost contact structure (ϕ, ξ, η).

THEOREM A. *Let (M, ϕ, ξ, η, g) be a Sasakian manifold and μ be a vector field over M which satisfies the next three conditions*

$$\mathscr{L}_\mu(g) = 0 \tag{7}$$

$$[\mu, \xi] = 0 \tag{8}$$

$$1 + \eta(\mu) > 0 \tag{9}$$

where $\mathscr{L}_\mu$ is the Lie differentiation with respect to μ.

New structure tensor fields denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are defined by the next equations

$$\tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{\xi})\ , \qquad \tilde{\eta} = (1 + \eta(\mu))^{-1}\cdot\eta\ , \qquad \tilde{\xi} = \xi + \mu$$

and

$$\tilde{g}(X, Y) = (1 + \eta(\mu))^{-1}\cdot g(X - \tilde{\eta}(X)\tilde{\xi}, Y - \tilde{\eta}(Y)\tilde{\xi}) + \tilde{\eta}(X)\cdot\tilde{\eta}(Y)$$

where X and Y are vector fields over M.

Then $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also a Sasakian manifold.

PROOF. Let D be the distribution defined by $\eta = 0$. By the definition of $\tilde{\eta}$ we see D is also defined by $\tilde{\eta} = 0$. We prove that the structure tensor fields $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ satisfy the relations similar to (1), (2), (3), (4) and (5). The relations (1) and (4) hold obviously by the definitions of $\tilde{\eta}$ and $\tilde{g}$. For the vector field X which belongs to D we have $\tilde{\eta}(X) = 0$, $\tilde{\eta}\cdot\phi(X) = 0$ and $\phi^2(X) = -X$, hence

$$\begin{aligned}\tilde{\phi}^2(X) &= \tilde{\phi}(\phi(X - \tilde{\eta}(X)\tilde{\xi})) \\ &= \tilde{\phi}(\phi(X)) \\ &= \phi(\phi(X) - \tilde{\eta}(\phi(X))\tilde{\xi}) \\ &= \phi^2(X) \\ &= -X\ .\end{aligned}$$

Because $\tilde{\phi}(\tilde{\xi}) = 0$ and the above equation holds for any vector field which belongs to D, we can conclude

$$\tilde{\phi}^2(X) = -X + \tilde{\eta}(X)\tilde{\xi}$$

for any vector field X on M. Next we will prove that the torsion tensor field $\tilde{N}$ with respect to the almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ vanishes on M. As (ϕ, ξ, η) are structure tensor fields of a Sasakian manifold, the tensor field N with respect to the almost contact structure (ϕ, ξ, η) vanishes and from this it is easily seen that the next three tensor fields vanish.

$$N_1(X) = [\xi, \phi X] - \phi[\xi, X] = 0 \tag{10}$$

$$N_2(X, Y) = -\eta([\phi X, Y]) - \eta([X, \phi Y]) + (\phi X)\cdot\eta(Y) - (\phi Y)\eta(X) = 0 \tag{11}$$

$$N_3(X) = \xi\cdot\eta(X) - \eta([\xi, X]) = 0 \tag{12}$$

for any vector fields X and Y on M. For the vector fields X and Y which belong to D, we get from (11)

$$N_2(X, Y) = -\eta([\phi X, Y]) - \eta([X, \phi Y]) = 0 .$$

Because $\phi(X) = \tilde{\phi}(X)$ and $\tilde{\eta}(X) = 0$ for the vector field X which belongs to D, using the above equation we conclude

$$\begin{aligned} \tilde{N}(X, Y) &= [X, Y] + \tilde{\phi}([\phi X, Y]) + \tilde{\phi}([X, \phi Y]) - [\phi X, \phi Y] \\ &= N(X, Y) - \phi(\tilde{\eta}([\phi X, Y])\tilde{\xi} + \tilde{\eta}([X, \phi Y])\tilde{\xi}) \\ &= N(X, Y) + (1 + \eta(\mu))^{-1}\cdot N_2(X, Y)\phi(\tilde{\xi}) = 0 \end{aligned} \tag{13}$$

where X and Y belong to D.

Because of (10) and (12), we get $\mathscr{L}_\xi\phi = 0$ and $\mathscr{L}_\xi\eta = 0$ and from these facts and (7), (8) and (3), we get $\mathscr{L}_{\tilde{\xi}}\phi = 0$ and $\mathscr{L}_{\tilde{\xi}}\eta = 0$. Hence we get

$$\mathscr{L}_{\tilde{\xi}}\tilde{\eta} = (1 + \eta(\mu))^{-1}\mathscr{L}_{\tilde{\xi}}\eta = 0$$

and

$$\begin{aligned} (\mathscr{L}_{\tilde{\xi}}\tilde{\phi})(X) &= \mathscr{L}_{\tilde{\xi}}(\phi(X - \tilde{\eta}(X)\tilde{\xi})) - \tilde{\phi}[\tilde{\xi}, X] . \\ &= \phi[\tilde{\xi}, X - \tilde{\eta}(X)\tilde{\xi}] - \phi([\tilde{\xi}, X] - \tilde{\eta}([\tilde{\xi}, X])\tilde{\xi}) = 0 \end{aligned}$$

From these two equations we find for a vector field X on M, we have

$$\tilde{N}(\tilde{\xi}, X) = [\tilde{\xi}, X] + \tilde{\phi}[\tilde{\xi}, \phi X] = 0 . \tag{14}$$

The equations (13) and (14) show that the torsion tensor field $\tilde{N}$ with respect to $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ vanishes on M.

Next we will prove that the relation (3) with respect to $(\tilde{\phi}, \tilde{\eta}, \tilde{g})$ holds. Since (ϕ, ξ, η, g) is structure tensor fields of a Sasakian manifold, the next equation holds for any vector fields X and Y.

$$2g(\phi X, Y) = 2d\eta(X, Y) = X\cdot\eta(Y) - Y\cdot\eta(X) - \eta([X, Y]) .$$

Hence for the vector fields X and Y belonging to D, we have

$$
\begin{aligned}
(15)\qquad 2d\tilde{\eta}(X, Y) &= X\cdot\tilde{\eta}(Y) - Y\cdot\tilde{\eta}(X) - \tilde{\eta}([X, Y]) \\
&= -\tilde{\eta}([X, Y]) \\
&= -(1 + \eta(\mu))^{-1}\eta([X, Y]) \\
&= 2(1 + \eta(\mu))^{-1}g(\phi X, Y) \\
&= 2(1 + \eta(\mu))^{-1}g(\tilde{\phi} X, Y) \\
&= 2\tilde{g}(\tilde{\phi} X, Y)
\end{aligned}
$$

because of the definitions of $\tilde{\eta}$ and $\tilde{g}$, $\phi(X) = \tilde{\phi}(X)$ and $\eta(X) = \tilde{\eta}(X) = 0$. Furthermore since $\mathscr{L}_{\tilde{\xi}}\tilde{\eta} = 0$, the equation

$$
(16)\qquad 2d\tilde{\eta}(\tilde{\xi}, X) = -\tilde{\eta}([\tilde{\xi}, X]) = 0
$$

hold for any vector field X belonging to D. By (15) and (16), it was proved that the relation (3) holds. Thus we have proved that the relations (1), (2), (3), (4) and (5) for $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ hold good. So we can introduce the new Sasakian structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on M. $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\phi}, \tilde{g})$ is called a Sasakian manifold deformed with respect to μ.

EXAMPLE. Let $(S^{2n+1}, \phi, \xi, \eta, g)$ be the unit sphere with the standard Sasakian structure and be imbedded in the Euclidean space E^{2n+2} with coordinates $(x^1, y^1, \cdots, x^{n+1}, y^{n+1})$. At the point P on S^{2n+1} we put

$$
\xi_P = \sum_{j=1}^{n+1}(x^j\partial y^j - y^j\partial x^j) \quad\text{and}\quad \eta_P = \sum_{j=1}^{n+1}(x^j dy^j - y^j dx^j)
$$

where ∂x^j and ∂y^j are vector fields over E^{2n+2} usually denoted by $\partial/\partial x^j$ and $\partial/\partial y^j$ respectively and $(x^1, y^1, \cdots, x^{n+1}, y^{n+1})$ is the coordinates of the point P. When we introduce a complex structure J on E^{2n+2} as $z^j = x^j + iy^j$ for $j = 1, 2, \cdots, n+1$, then ϕ is defined as the restriction of J on D which is the orthogonal complement of $R\cdot\xi$ in the tangent space at each point on S^{2n+1} and 0 on $R\cdot\xi$. The Riemannian metric g on S^{2n+1} is induced by that of E^{2n+2}.

When we put

$$
\mu = \sum_{j=1}^{n+1} r_j(x^j\partial y^j - y^j\partial x^j)
$$

where $(r_1, r_2, \cdots, r_{n+1})$ is a $(n+1)$-tuple of real numbers such that

$$
1 + \sum_{j=1}^{n+1} r_j((x^j)^2 + (y^j)^2) > 0
$$

on S^{2n+1}, then μ satisfies the conditions of Theorem A. And the new trajectory of $\tilde{\xi}$ with the initial condition P is given by

$$x^j(t) = x^j \cos(1 + r_j)t - y^j \sin(1 + r_j)t$$
$$y^j(t) = x^j \sin(1 + r_j)t + y^j \cos(1 + r_j)t$$

for $j = 1, 2, \cdots, n + 1$.

3. Sasakian structures for the Brieskorn manifolds. Let C^{n+1} be the complex vector space of $(n + 1)$-tuples of complex numbers $Z = (z_1, z_2, \cdots, z_{n+1})$ and $(a_1, a_2, \cdots, a_{n+1})$ be a $(n + 1)$-tuple of positive integers. Then the Brieskorn manifold $B^{2n-1}(a_1, a_2, \cdots, a_{n+1})$ is defined as a submanifold of C^{n+1} by the following two equations.

$$(z_1)^{a_1} + \cdots + (z_{n+1})^{a_{n+1}} = 0 . \tag{17}$$

$$z_1 \cdot \bar{z}_1 + \cdots + z_{n+1} \cdot \bar{z}_{n+1} = 0 . \tag{18}$$

We define a mapping F of C^{n+1} onto C by

$$F(z_1, z_2, \cdots, z_{n+1}) = (z_1)^{a_1} + \cdots + (z_{n+1})^{a_{n+1}} .$$

Then an analytic subvariety in C^{n+1} defined by $F = 0$ may have singularity only at the origin 0 of C^{n+1}. If we consider S^{2n+1} as the unit sphere in C^{n+1} defined by (18), $B^{2n-1}(a_1, \cdots, a_{n+1}) = S^{2n+1} \cap F^{-1}(0)$ and $B^{2n-1}(a_1, \cdots, a_{n+1})$ is a $(2n - 1)$ dimensional submanifold in S^{2n+1}. Denoting by $x_1, y_1, \cdots, x_{n+1}, y_{n+1}$ the real coordinates of C^{n+1} such that $z_j = x_j + iy_j (j = 1, \cdots, n + 1)$, we define a real vector field $\tilde{\xi}$ on C^{n+1} by

$$\tilde{\xi} = \sum_{j=1}^{n+1} A_j(x_j \partial y_j - y_j \partial x_j)$$

where $A_j = (a_j)^{-1}A$ for a positive constant A, $(j = 1, \cdots, n + 1)$. This vector field is tangent to S^{2n+1}. From the example of §2, we know that $\tilde{\xi}$ satisfies the conditions of the theorem A and furthermore $\mu = \tilde{\xi} - \xi$ also satisfies the same conditions because of $1 + \eta(\mu) = \eta(\tilde{\xi}) > 0$. We define $\mu = \tilde{\xi} - \xi$ and apply the theorem A to $(S^{2n+1}, \phi, \xi, \eta, g)$ with respect to μ and denote by $(S^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the deformed Sasakian sphere with respect to μ. The explicit formulas of $\tilde{\eta}$ and $\tilde{\xi}$ are given by

$$\tilde{\eta} = \sum_{j=1}^{n+1} (K)^{-1}(x_j dy_j - y_j dx_j)$$

and

$$\tilde{\xi} = \sum_{j=1}^{n+1} A_j(x_j \partial y_j - y_j \partial x_j)$$

where $K(x_1, y_1, \cdots, x_{n+1}, y_{n+1}) = \sum_{j=1}^{n+1} A_j((x_j)^2 + (y_j)^2)$.
Next we construct an contact metric structure on $B^{2n-1}(a_1, \cdots, a_{n+1})$. We will show that the above vector field $\tilde{\xi}$ is tangent to $B^{2n-1}(a_1, \cdots, a_{n+1})$. Let P and Q be real valued functions on C^{n+1} defined by

$$P(x_1, y_1, \cdots, x_{n+1}, y_{n+1}) = \text{the real part of } F(z_1, \cdots, z_{n+1})$$

and

$$Q(x_1, y_1, \cdots, x_{n+1}, y_{n+1}) = \text{the imaginary part of } F(z_1, \cdots, z_{n+1}) ,$$

then it is easy to see that $\tilde{\xi}P = -AQ$ and $\tilde{\xi}Q = AP$. These prove that the restriction of the vector field $\tilde{\xi}$ to $B^{2n-1}(a_1, \cdots, a_{n+1})$ is tangent to B^{2n-1} where $B^{2n-1}(a_1, \cdots, a_{n+1})$ is denoted simply by B^{2n-1}. We denote by ι the inclusion mapping from B^{2n-1} and by ι^* the induced mapping from the differential forms over S^{2n+1} to those of B^{2n-1}. We define four tensor fields $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ on B^{2n-1} by the next equations

$$\hat{\phi} = \tilde{\phi}_{|B^{2n-1}} , \qquad \hat{\xi} = \tilde{\xi}_{|B^{2n-1}} , \qquad \hat{\eta} = \iota^*\tilde{\eta} \quad \text{and} \quad \hat{g} = \iota^*\tilde{g} . \tag{19}$$

We will prove that these four tensor fields define a contact metric structure on B^{2n-1}. Because of the definitions (19), it is sufficient to show that the restriction $\tilde{\phi}_{|B^{2n-1}}$ is well defined. Since the restrictions of ϕ, $\tilde{\phi}$ and J to D are the same mapping and $F^{-1}(0) - \{0\}$ is a complex submanifold of C^{n+1}, for any tangent vector X of B^{2n-1} which belongs to D on S^{2n+1} we get

$$\tilde{\phi}(X)(F) = (J(X))(F) = i \cdot X(F) = 0 \tag{20}$$

because of the Cauchy-Riemann equations for the complex analytic function F. Since $T_PB^{2n-1} = (D_P + R\cdot\tilde{\xi}_P) \cap T_PB^{2n-1} = (D_P \cap T_PB^{2n-1}) + R\cdot\hat{\xi}_P$ is an orthogonal decomposition with respect to $\hat{g}_P$ where T_PB^{2n-1} is the tangent space at the point P of B^{2n-1} and since $\tilde{\phi}(\hat{\xi}) = \hat{\phi}(\hat{\xi}) = 0$, we find from (20) that $\hat{\phi}$ is a mapping from the tangent space T_PB^{2n-1} to itself. Hence we find that $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ define a contact metric structure on B^{2n-1}. Generally let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold and M^{2n-1} be a submanifold in M^{2n+1}. If it is satisfied that $\phi(T_PM^{2n-1}) \subset T_PM^{2n-1}$ for any point P on M^{2n-1}, the submanifold M^{2n-1} is called an invariant submanifold. In this case a contact metric structure on M^{2n-1} is induced from $(M^{2n+1}, \phi, \xi, \eta, g)$ by the equations similar to (19). M. Okumura proved that if $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold, then the induced contact metric structure on M^{2n-1} is also a Sasakian structure [2]. Since B^{2n-1} is an invariant submanifold of $(S^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, we have

THEOREM B. *Every Brieskorn manifold admits many Sasakian structures.*

REFERENCES

[1] K. ABE, On a generalization of the Hopf fibration, I, Tôhoku Math. J., 29 (1977), 335–374.

[2] M. OKUMURA, On contact metric immersion, Kōdai Math. Sem. Rep. 20 (1968), 389-409.
[3] S. SASAKI, Almost contact manifold, Lecture Notes of Tôhoku University Vol. I, II, III.
[4] S. SASAKI AND C. J. HSU, On a property of Brieskorn manifolds, Tôhoku Math. J., 28 (1976), 67-78.

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