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FAVARD'S SEPARATION THEOREM IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE RETARDATIONS

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1. Introduction. Favard [1] has shown that if a linear almost periodic system

$$\dot{x}(t) = A(t)x + f(t) \tag{1}$$

has a bounded solution and if for every $B(t)$ in the hull $H(A)$, every nontrivial solution $x(t)$ of $\dot{x}(t) = B(t)x$ which is defined and bounded on R (shortly, R-bounded) satisfies the condition

$$\inf_{t \in R} \|x(t)\| > 0 , \tag{2}$$

then system (1) has an almost periodic solution.

Recently, Kato [7] has pointed out that for functional differential equations the replacement of condition (2) in Favard's theorem by the condition

$$\inf_{t \in R} \Big(\sup_{\theta \in [-h,0]} \|x(t+\theta)\| \Big) > 0 \tag{3}$$

is not obvious. However, Kato has shown that condition (2) can be replaced by condition (3) by considering a minimal solution with respect to a new norm $\|\cdot\|$ in $C([-h, 0], R^n)$ defined by

$$\|\phi\| = \left(\int_{-h}^{0} \|\phi(s)\|^2 ds \right)^{1/2} .$$

In this paper, more generally, we shall show that for functional differential equations with infinite retardations, we can replace condition (2) by the conditions

$$\inf_{t \in R} \Big(\sup_{\theta \in (-\infty,0]} \|x(t+\theta)\| e^{\gamma\theta} \Big) > 0 , \qquad \gamma > 0 ,$$

and

$$\inf_{t \in R} \left\{ \sup_{\theta \in [-r,0]} \|x(t+\theta)\|^p + \int_{-\infty}^{0} \|x(t+\theta)\|^p g(\theta) d\theta \right\}^{1/p} > 0$$

by introducing semi-norm

$$\|\phi\|_* = \left\{\int_{-\infty}^{0} \|\phi(\theta)\|^2 e^{2\gamma\theta} d\theta\right\}^{1/2}$$

and

$$\|\phi\|_* = \begin{cases} \left\{\|\phi(0)\|^2 + \int_{-\infty}^{0} \|\phi(\theta)\|^2 g(\theta) d\theta\right\}^{1/2}, & \text{if} \quad r = 0 , \\ \left\{\int_{-r}^{0} \|\phi(\theta)\|^2 d\theta + \int_{-\infty}^{0} \|\phi(\theta)\|^2 g(\theta) d\theta\right\}^{1/2}, & \text{if} \quad r > 0 , \end{cases}$$

for continuous and bounded functions ϕ mapping $(-\infty, 0]$ into R^n, respectively, where $g(\theta)$ is a nondecreasing positive function defined on $(-\infty, 0]$ such that $\int_{-\infty}^{0} g(\theta) d\theta < \infty$.

2. Hale's space and some lemmas. First we shall give a class of Banach spaces considered by Hale [2]. Let x be any vector in R^n and $\|x\|$ be the Euclidean norm of x. Let $B = B((-\infty, 0], R^n)$ be a space of functions mapping $(-\infty, 0]$ into R^n with norm $\|\cdot\|_B$. For any ϕ in B and any σ in $[0, \infty)$, let ϕ^σ be the restriction of ϕ to the interval $(-\infty, -\sigma]$. This is a function mapping $(-\infty, -\sigma]$ into R^n. We shall denote by B^σ the space of such functions ϕ^σ. For any $\eta \in B^\sigma$, we define the semi-norm $\|\eta\|_{B^\sigma}$ of η by

$$\|\eta\|_{B^\sigma} = \inf\{\|\phi\|_B : \phi^\sigma = \eta\} .$$

If x is a function defined on $(-\infty, a)$, $a > 0$, then for each t in $[0, a)$ we define the function x_t by the relation $x_t(s) = x(t+s)$, $-\infty < s \leqq 0$. For numbers a and τ, $a > \tau$, we denote by A_τ^a the class of function x mapping $(-\infty, a)$ into R^n such that x is a continuous function on $[\tau, a)$ and $x_\tau \in B$. The space B is assumed to have the following properties:

(I) B is a Banach space.

(II) If x is in A_τ^a, then x_t is in B for all t in $[\tau, a)$ and x_t is a continuous function of t, where a and τ are constants such that $\tau < a \leqq \infty$.

(III) All bounded continuous functions mapping $(-\infty, 0]$ into R^n are in B.

(IV) If a sequence $\{\phi_k\}$, $\phi_k \in B$, is uniformly bounded on $(-\infty, 0]$ with respect to the Euclidean norm $\|\cdot\|$ and converges to ϕ uniformly on any compact subset of $(-\infty, 0]$, then $\phi \in B$ and $\|\phi_k - \phi\|_B \to 0$ as $k \to \infty$.

REMARK. Property (IV) is equivalent to the following property: For any $b > 0$ and $\varepsilon > 0$, there exist an $N > 0$ and a $\delta > 0$ such that

$$\{\phi \in B; \|\phi\|_B < \varepsilon\} \supset \{\phi \in B; \sup_{\theta \in [-N, 0]} \|\phi(\theta)\| < \delta\} \cap \{\phi \in B; \sup_{\theta \in (-\infty, 0]} \|\phi(\theta)\| < b\} .$$

(V) There are continuous, increasing and nonnegative functions $b(r)$, $c(r)$ defined on $[0, \infty)$, $b(0) = c(0) = 0$, such that

$$\|\phi\|_B \leqq b(\sup_{\theta \in [-\sigma, 0]} \|\phi(\theta)\|) + c(\|\phi^\sigma\|_{B^\sigma})$$

for any ϕ in B and any $\sigma \geqq 0$.

(VI) If σ is a nonnegative number and ϕ is an element in B, then $T_\sigma\phi$ defined by $T_\sigma\phi(s) = \phi(s + \sigma)$, $s \in (-\infty, -\sigma]$, is an element in B^σ and $\|T_\sigma\phi\|_{B^\sigma} \to 0$ as $\sigma \to \infty$.

In addition, we shall assume that the space B has the following properties;

(VII) B is separable.

(VIII) $\|\phi(0)\| \leqq M_1\|\phi\|_B$ for $M_1 > 0$.

In the following four lemmas, we assume that $f(t, \phi)$ is continuous in $(t, \phi) \in R \times B$ and almost periodic in t uniformly for $\phi \in B$.

LEMMA 1 (cf. Lemma 3 in [5]). *Suppose that $f(t, \phi)$ satisfies the condition*

$$\sup\{\|f(t, \phi)\|; t \in R, \|\phi\|_B \leqq \alpha\} \leqq F(\alpha) < \infty \tag{4}$$

for every $\alpha > 0$.

If the system

$$\dot{x}(t) = f(t, x_t) \tag{5}$$

has a solution $x(t)$ which is bounded on $[0, \infty)$, then for any $g(t, \phi)$ in $H(f)$ the system

$$\dot{x}(t) = g(t, x_t) \tag{6}$$

has an R-bounded solution. More exactly, if $\{x(t + t_k), f(t + t_k, \phi)\}$ converges to $(y(t), g(t, \phi))$, then $y(t)$ is a bounded solution of (6) on $(-\overline{\lim}_{k\to\infty} t_k, \infty)$.

The following lemma can be proved by slightly modifying the proof of Lemma 1 in [6].

LEMMA 2. *If $f(t, \phi)$ is linear in ϕ, then it satisfies condition* (4) *with $F(\alpha) = L\alpha$ for a constant $L > 0$.*

For continuous and bounded function ϕ mapping $(-\infty, 0]$ into R^n, let $\|\phi\|_*$ be a semi-norm which has the following properties:

(a) For any $d > 0$, there exists an $M(d) > 0$ such that if $\|\phi(t)\| \leqq d$ for all $t \in (-\infty, 0]$, then $\|\phi\|_* \leqq M(d)$.

(b) If a sequence $\{\phi_k\}$ is continuous and uniformly bounded on $(-\infty, 0]$ with respect to the Euclidean norm $\|\cdot\|$ and converges to ϕ uniformly on any compact subset of $(-\infty, 0]$, then $\|\phi_k - \phi\|_* \to 0$ as $k \to \infty$.

(c) There exists a $\beta(\alpha)$ such that if $x(t)$ is an R-bounded solution of (5) and satisfies $\|x_t\|_* \leqq \alpha$, $\alpha > 0$, where $f(t, \phi)$ satisfies condition (4) with $F(\alpha) = o(\alpha^3)$ as $\alpha \to \infty$, then $\|x(t)\| \leqq \beta(\alpha)$.

Existence of such a semi-norm $\|\cdot\|_*$ will be discussed in Sections 3 and 4.

For an R-bounded and continuous function $x(t)$, put

$$\lambda(x) = \sup\{\|x_t\|_*; t \in R\} .$$

LEMMA 3. *Suppose that $f(t, \phi)$ satisfies condition (4) and that system (5) has an R-bounded solution. Let $\Lambda(f)$ be defined by*

$$\Lambda(f) = \inf\{\lambda(x); x(t) \text{ is an } R\text{-bounded solution of } (5)\} .$$

Then for every $g(t, \phi) \in H(f)$, we have $\Lambda(g) = \Lambda(f)$.

PROOF. First of all, we note that $\lambda(x) < \infty$ if $x(t)$ is R-bounded by property (a). For every $\varepsilon > 0$, there exists an R-bounded solution of (5) such that $\lambda(x) \leqq \Lambda(f) + \varepsilon$. Since $x(t)$ is an R-bounded solution of (5), for every $g(t, \phi) \in H(f)$, system (6) has a solution $y(t)$ to which $\{x(t + t_k)\}$ converges uniformly on any compact interval in R for some sequence $\{t_k\}$ by Lemma 1. Then

$$\|y_t\|_* - \|x_{t+t_k}\|_* \leqq \|x_{t+t_k} - y_t\|_* \to 0 \qquad \text{as} \quad k \to \infty$$

by property (b). This implies

$$\Lambda(g) \leqq \lambda(y) \leqq \lambda(x) \leqq \Lambda(f) + \varepsilon ,$$

and hence $\Lambda(g) \leqq \Lambda(f)$. On the other hand, $g(t, \phi) \in H(f)$ is almost periodic uniformly for $\phi \in B$ and $f(t, \phi) \in H(g)$, and hence $\Lambda(f) \leqq \Lambda(g)$. Thus we have $\Lambda(g) = \Lambda(f)$ for every $g(t, \phi) \in H(f)$.

LEMMA 4. *Suppose that $f(t, \phi)$ satisfies condition (4) with $F(\alpha) = o(\alpha^3)$ as $\alpha \to \infty$ and that system (5) has an R-bounded solution. Then there exists an R-bounded solution $x(t)$ of (5) with the property $\lambda(x) = \Lambda(f)$.*

PROOF. By the definition of $\Lambda(f)$, there exists a sequence $\{x^k(t)\}$ of R-bounded solution of (5) such that $\lambda(x^k) \leqq \Lambda(f) + 1/k \leqq \Lambda(f) + 1$. Since $\|x_t^k\|_* \leqq \Lambda(f) + 1$, there exists a $\beta > 0$ such that $\|x^k(t)\| < \beta$ for all k and all $t \in R$ by property (c). Let K be such that

$$K = \{\phi \in B; \|\phi(\theta)\| \leqq \beta \text{ on } \theta \in (-\infty, 0], \|\phi(\theta_1) - \phi(\theta_2)\| \leqq F(b(\beta))|\theta_1 - \theta_2|, \\ \theta_1, \theta_2 \in (-\infty, 0]\} ,$$

where $b(\cdot)$ is the one given in property (V) of the space B. Clearly, K is a compact subset of B. Since $\|\dot{x}^k(t)\| \leqq F(b(\beta))$ for all k and all

t, $x^k{}_t \in K$ for all k and all $t \in R$. Thus $\{x^k(t)\}$ has a subsequence $\{x^{k_j}(t)\}$ which converges to an R-bounded solution $x(t)$ of (5). On the other hand, by using the same arguments as in the proof of Lemma 3, we have $\lambda(x) \leqq \Lambda(f)$. That is, $\lambda(x) = \Lambda(f)$.

3. The space $\mathscr{C}$ with norm $\sup_{\theta \in (-\infty, 0]} ||\phi(\theta)|| e^{\gamma\theta}$. The following class of Banach spaces has been discussed by Hino in [4] as one of Hale's spaces.

DEFINITION 1. The space $\mathscr{C}$ consists of all continuous functions mapping $(-\infty, 0]$ into R^n such that $\phi(\theta)e^{\gamma\theta} \to 0$ as $\theta \to -\infty$ with norm $||\phi||_{\mathscr{C}} = \sup_{\theta \in (-\infty, 0]} ||\phi(\theta)|| e^{\gamma\theta}$, $\gamma > 0$.

It is easily seen that the space $\mathscr{C}$ has properties (I)~(VIII).

For bounded functions ϕ in $\mathscr{C}$, if we define $||\phi||_*$ by

$$||\phi||_* = \left\{\int_{-\infty}^0 ||\phi(\theta)||^2 e^{2\gamma\theta} d\theta\right\}^{1/2},$$

then it has properties (a), (b), and (c). It is clear that it has properties (a) and (b). We shall show that it has property (c). Assume that $x(t)$ is an R-bounded solution of (5) and $||x_t||_* \leqq \alpha$, $\alpha > 0$. Clearly, for any $T > 0$

$$e^{-\gamma T}\left(\int_{-T}^0 ||x(t+\theta)||^2 d\theta\right)^{1/2} \leqq \left(\int_{-T}^0 ||x(t+\theta)||^2 e^{2\gamma\theta} d\theta\right)^{1/2} \leqq ||x_t||_* \leqq \alpha,$$

and hence property (c) follows from Lemma 2 and the following lemma.

LEMMA 5 (cf. Lemma 4 in [7]). *Suppose that $f(t, \phi)$ satisfies condition* (4) *with $F(\alpha) = o(\alpha^3)$ as $\alpha \to \infty$. Then for any $\alpha > 0$, there exists a constant $\beta > 0$ such that if $x(t)$ is an R-bounded solution of system* (5) *and satisfies $\sup_{t \in R}\left(\int_{-T}^0 ||x(t+\theta)||^2 d\theta\right)^{1/2} \leqq \alpha$ for some $T > 0$, we have $||x(t)|| \leqq \beta$ for all $t \in R$.*

Here we should note that this $||\cdot||_*$ has the following property;

(d) If $x^1(t)$ and $x^2(t)$ are R-bounded continuous functions, then

$$\{||x_t^1||_*^2 + ||x_t^2||_*^2\}/2 = ||y_t||_*^2 + ||z_t||_*^2,$$

where $y(t) = \{x^1(t) + x^2(t)\}/2$ and $z(t) = \{x^1(t) - x^2(t)\}/2$.

4. The space $\mathscr{B}$ with norm $\left\{(\sup_{\theta \in [-r, 0]} ||\phi(\theta)||)^p + \int_{-\infty}^0 ||\phi(\theta)||^p g(\theta) d\theta\right\}^{1/p}$. We shall discuss a class of Banach spaces considered by Naito in [8] as one of Hale's spaces.

DEFINITION 2. Let $r \geqq 0$, $p \geqq 1$, and let $g(\theta)$ be a nondecreasing positive function defined on $(-\infty, 0]$ such that $\int_{-\infty}^{0} g(\theta)d\theta < \infty$. The space $\mathscr{B}$ consists of all functions ϕ mapping $(-\infty, 0]$ into R^n, which are Lebesgue measurable on $(-\infty, 0]$ and are continuous on $[-r, 0]$ with norm $||\phi||_{\mathscr{B}} = \left\{(\sup_{\theta\in[-r,0]} ||\phi(\theta)||)^p + \int_{-\infty}^{0} ||\phi(\theta)||^p g(\theta)d\theta\right\}^{1/p}$. When $r = 0$, we do not assume the continuity of ϕ at $\theta = 0$.

It is easily shown that the space $\mathscr{B}$ also has properties (I)~(VIII).

For continuous and bounded function ϕ mapping $(-\infty, 0]$ into R^n, we can consider

$$||\phi||_* = \begin{cases} \left\{||\phi(0)||^2 + \int_{-\infty}^{0} ||\phi(\theta)||^2 g(\theta)d\theta\right\}^{1/2} & \text{if} \quad r = 0\,, \\ \left\{\int_{-r}^{0} ||\phi(\theta)||^2 d\theta + \int_{-\infty}^{0} ||\phi(\theta)||^2 g(\theta)d\theta\right\}^{1/2} & \text{if} \quad r > 0 \end{cases}$$

which has properties (a), (b), (c), and (d). It is clear that it has properties (a), (b), and (d). Assume that $x(t)$ is an R-bounded solution of (5) and $||x_t||_* \leqq \alpha$, $\alpha > 0$. If $r = 0$, then it satisfies

$$||x(t)|| \leqq ||x_t||_* \leqq \alpha\,.$$

Since

$$\left(\int_{-r}^{0} ||x(t+\theta)||^2 d\theta\right)^{1/2} \leqq ||x_t||_* \leqq \alpha\,, \qquad \text{if} \quad r > 0\,,$$

property (c) follows from Lemma 5.

5. Existence theorem for almost periodic solutions of linear systems.

LEMMA 6. *Let $r > 0$ and $\phi(\theta)$ be defined on $[-r, 0]$. If $\phi(\theta)$ satisfies a Lipschitz condition*

$$||\phi(\theta_1) - \phi(\theta_2)|| \leqq L|\theta_1 - \theta_2|\,, \qquad \theta_1, \theta_2 \in [-r, 0]\,,$$

then

$$\left(\int_{-r}^{0} ||\phi(\theta)||^2 d\theta\right)^{1/2} \geqq \{\min(r/3, (\sup_{\theta\in[-r,0]} ||\phi(\theta)||)/3L)\}^{1/2} \times (\sup_{\theta\in[-r,0]} ||\phi(\theta)||)\,.$$

For the proof, see ([7], pp 87–88).

THEOREM. *Suppose that $A(t, \phi)$ is continuous in $(t, \phi) \in R \times \mathscr{C}$ $(R \times \mathscr{B})$, linear in ϕ and almost periodic in t uniformly for $\phi \in \mathscr{C}(\mathscr{B})$, and that*

(*) *for every $B(t, \phi) \in H(A)$, every nontrivial R-bounded solution of the system*

(7) $$\dot{x}(t) = B(t, x_t)$$

satisfies the condition

(8) $$\inf_{t \in R} \|x_t\|_{\mathscr{C}} > 0 \quad (\inf_{t \in R} \|x_t\|_{\mathscr{B}} > 0) .$$

Then for any almost periodic function $f(t)$, the system

(9) $$\dot{x}(t) = A(t, x_t) + f(t)$$

has an almost periodic solution, whenever it has a bounded solution on $[0, \infty)$.

PROOF. There exists an R-bounded solution $x(t)$ of (9) with the minimal semi-norm $\lambda(x)$ by Lemmas 2 and 4.

Now we shall show that for each $B(t, \phi) + g(t) \in H(A + f)$, the system

(10) $$\dot{x}(t) = B(t, x_t) + g(t)$$

has a unique R-bounded solution with the minimal semi-norm.

Let $x^1(t)$ and $x^2(t)$ be R-bounded solutions of (10) with the minimal semi-norm. Clearly, $z(t) = \{x^1(t) - x^2(t)\}/2$ is a solution of the homogeneous system (7) and $y(t) = \{x^1(t) + x^2(t)\}/2$ is a solution of system (10). By property (d), we have

$$\{\|x_t^1\|_*^2 + \|x_t^2\|_*^2\}/2 = \|y_t\|_*^2 + \|z_t\|_*^2 ,$$

which implies

(11) $$\inf_{t \in R} \|z_t\|_* = 0 .$$

Assume that $\sup_{t \in R} \|z(t)\| = \delta > 0$. Clearly, $\delta < \infty$. Then there exists an $L_1 > 0$ such that $\sup_{t \in R} \|B(t, z_t)\| \leqq L_1$ by property (V) and Lemma 2.

(i) The case where the space is $\mathscr{C}$. The relation (11) implies that for any $\varepsilon > 0$, there exists a $t_0 \in R$ such that

(12) $$\|z_{t_0}\|_* = \left\{\int_{-\infty}^0 \|z(t_0 + \theta)\|^2 e^{2\gamma\theta} d\theta\right\}^{1/2} < \varepsilon .$$

There exists a $T > 1$ such that

(13) $$\sup_{\theta \in (-\infty, -T]} \|z(t_0 + \theta)\| e^{\gamma\theta} \leqq \delta e^{-\gamma T} < \varepsilon .$$

Since

$$\|z(t + \theta_1)e^{\gamma\theta_1} - z(t + \theta_2)e^{\gamma\theta_2}\| \leqq L_2|\theta_1 - \theta_2| \qquad \text{for} \quad \theta_1, \theta_2 \in [-T, 0] ,$$

where $L_2 = L_1 + \gamma\delta$, it follows from (12) and Lemma 6 that

$$\varepsilon^2 > \int_{-T}^{0} \|z(t_0+\theta)e^{\gamma\theta}\|^2 d\theta$$
$$\geqq \min\{T/3, (\sup_{\theta\in[-T,0]} \|z(t_0+\theta)e^{\gamma\theta}\|)/3L_2\} \times (\sup_{\theta\in[-T,0]} \|z(t_0+\theta)e^{\gamma\theta}\|)^2 .$$

Hence we have

$$\sup_{\theta\in[-T,0]} \|z(t_0+\theta)e^{\gamma\theta}\| \leqq \max\{\sqrt{3}\varepsilon, \sqrt[3]{3L_2\varepsilon^2}\}, \tag{14}$$

because $T > 1$. By (13) and (14), we have $\inf_{t\in R}\|z_t\|_{\mathscr{C}} = 0$, which contradicts to condition (8). Thus $z(t) = 0$ on R.

(ii) The case where the space is $\mathscr{B}$. Define $\|z_t\|_{**}$ by

$$\|z_t\|_{**} = \sup_{\theta\in[-r,0]} \|z(t+\theta)\| + \left(\int_{-\infty}^{0} \|z(t+\theta)\|^p g(\theta)d\theta\right)^{1/p} .$$

Then, we have

$$\|z_t\|_{\mathscr{B}} \leqq \|z_t\|_{**} . \tag{15}$$

It follows from (11) that for any $\varepsilon > 0$, there exists a $t_0 \in R$ such that $\|z_{t_0}\|_* < \varepsilon$, that is,

$$\begin{cases} \|z(t_0)\| < \varepsilon & \text{if } r = 0 , \\ \left(\int_{-r}^{0} \|z(t_0+\theta)\|^2 d\theta\right)^{1/2} < \varepsilon & \text{if } r > 0 \end{cases} \tag{16}$$

and

$$\left(\int_{-\infty}^{0} \|z(t_0+\theta)\|^2 g(\theta)d\theta\right)^{1/2} < \varepsilon . \tag{17}$$

(ii.1) The case where $1 \leqq p < 2$. By Hölder's inequality, we have

$$\begin{aligned} \left(\int_{-\infty}^{0} \|z(t+\theta)\|^p g(\theta)d\theta\right)^{1/p} &= \left(\int_{-\infty}^{0} \|z(t+\theta)\|^p g(\theta)^{p/2} g(\theta)^{1-p/2} d\theta\right)^{1/p} \\ &\leqq \left\{\left(\int_{-\infty}^{0} \|z(t+\theta)\|^2 g(\theta)d\theta\right)^{p/2} \times \left(\int_{-\infty}^{0} (g(\theta)^{1-p/2})^{2/(2-p)} d\theta\right)^{(2-p)/2}\right\}^{1/p} \\ &\leqq \left(\int_{-\infty}^{0} \|z(t+\theta)\|^2 g(\theta)d\theta\right)^{1/2} \times \left(\int_{-\infty}^{0} g(\theta)d\theta\right)^{(2-p)/p} , \end{aligned} \tag{18}$$

because $1 < 2/p$. By Lemma 6, it holds that

$$\begin{aligned} \left(\int_{-r}^{0} \|z(t+\theta)\|^2 d\theta\right)^{1/2} &\geqq \{\min(r/3, (\sup_{\theta\in[-r,0]} \|z(t+\theta)\|)/3L_1)\}^{1/2} \\ &\quad \times (\sup_{\theta\in[-r,0]} \|z(t+\theta)\|) , \end{aligned} \tag{19}$$

if $r > 0$. By (16), (17), (18), and (19), we have

$$\|z_{t_0}\|_{**} \leqq \begin{cases} \varepsilon + \left(\int_{-\infty}^{0} g(\theta)d\theta\right)^{(2-p)/p} \times \varepsilon & \text{if} \quad r = 0\,, \\ \max\{\sqrt{3\varepsilon/r}\,,\ \sqrt[3]{3L_1\varepsilon^2}\} + \left(\int_{-\infty}^{0} g(\theta)d\theta\right)^{(2-p)/p} \times \varepsilon & \text{if} \quad r > 0\,, \end{cases}$$

which implies that $\inf_{t\in R}\|z_t\|_{**} = 0$. Therefore $\inf_{t\in R}\|z_t\|_{\mathscr{B}} = 0$ by (15), which contradicts to condition (8). Thus $z(t) = 0$ on R.

(ii.2) The case which $p \geqq 2$. It is easily seen that

$$(20) \qquad \left(\int_{-\infty}^{0} \|z(t+\theta)\|^2 g(\theta)d\theta\right)^{1/p} \geqq \left\{\left(\int_{-\infty}^{0} \|z(t+\theta)\|^p g(\theta)d\theta\right)\Big/\delta^{p-2}\right\}^{1/p},$$

because $p \geqq 2$. By (16), (17), (19), and (20), we have

$$\|z_{t_0}\|_{**} \leqq \begin{cases} \varepsilon + \delta^{(p-2)/p} \times \varepsilon^{2/p} & \text{if} \quad r = 0\,, \\ \max\{\sqrt{3\varepsilon/r}\,, \sqrt[3]{3L_1\varepsilon^2}\} + \delta^{(p-2)/p} \times \varepsilon^{2/p} & \text{if} \quad r > 0\,, \end{cases}$$

which implies that $\inf_{t\in R}\|z_t\|_{**} = 0$. Therefore $\inf_{t\in R}\|z_t\|_{\mathscr{B}} = 0$ by (15), which contradicts to condition (8). Thus $z(t) = 0$ on R. Thus system (10) has a unique R-bounded solution with the minimal semi-norm.

Let $p(t)$ be the solution of (9) with the minimal semi-norm. It is easy to see that if $(y,\ C(t,\phi) + h) \in H(p, A + f)$, $y(t)$ is the solution of the system

$$\dot{x}(t) = C(t, x_t) + h(t)$$

with the minimal semi-norm by Lemma 3. Let $\{\tau_k\}$ be a sequence such that $A(t+\tau_k, \phi) + f(t+\tau_k) \to B(t,\phi) + g(t)$ uniformly on $R \times S$ as $k \to \infty$, where S is any compact subset of $\mathscr{C}(\mathscr{B})$. Suppose that $p(t+\tau_k)$ is not uniformly convergent on R. Then, by the same idea as in the proof of Theorem 5 in [5], we can find two solutions $\eta(t) \in H(p)$ and $\zeta(t) \in H(p)$ of some system in the hull $H(A + f)$ which satisfies

$$\|\eta_0 - \zeta_0\|_* > \varepsilon$$

for some $\varepsilon > 0$. Thus we can find two minimal solutions of some system in the hull. This contradicts the uniqueness of the minimal solution. Thus we see that $p(t)$ is an almost periodic solution of (9). This completes the proof.

REMARK. If we define a number β by

$$(21) \qquad \beta = \inf\left\{\mathrm{Re}\,\lambda\colon \int_{-\infty}^{0} |e^{\lambda\theta}|^p g(\theta)d\theta < \infty\right\},$$

where $g(\theta)$ is the one given in Definition 2, then β is clearly nonpositive. If $\beta \neq 0$, we can regard our theorem with the space $\mathscr{B}$ as a corollary of our theorem with the space $\mathscr{C}$. Furthermore, we can replace the assumption $(*)$ in our theorem with $\mathscr{B}$ by the following assumption:

$(**)$ there exists a γ, $\beta < -\gamma < 0$, such that for every $B(t, \phi) \in H(A)$, every nontrivial R-bounded solution of system (7) satisfies the condition

$$\inf_{t \in R} \|x_t\|_{\mathscr{C}} > 0 \,, \tag{22}$$

where $\|x_t\|_{\mathscr{C}} = \sup_{\theta \in (-\infty, 0]} \|x(t+\theta)\| e^{\gamma\theta}$.

In fact, for the number γ in $(**)$, the space $\mathscr{C}$ is naturally and continuously imbeded into $\mathscr{B}$, that is, there exists a constant $d(\gamma)$ such that

$$\|\phi\|_{\mathscr{B}} \leqq d(\gamma) \|\phi\|_{\mathscr{C}} \qquad \text{for} \quad \phi \in \mathscr{C} \tag{23}$$

(cf. Lemma 3.3 in [9]). Let $A(t, \phi)$ be a function defined on $R \times \mathscr{B}$ satisfying the assumptions in our theorem with $\mathscr{B}$. Conditions (22) and (23) imply that the restriction $\tilde{A}$ of A on $R \times \mathscr{C}$ satisfies the assumptions in our theorem with $\mathscr{C}$. Suppose that $f(t)$ is an almost periodic function for which system (9) has a bounded solution on $[0, \infty)$. By Lemma 1, system (9) has an R-bounded solution, which is obviously an R-bounded solution of the system

$$\dot{x}(t) = \tilde{A}(t, x_t) + f(t) \,. \tag{24}$$

Then, Theorem with $\mathscr{C}$ says that system (24) has an almost periodic solution $p(t)$. Since $\tilde{A}(t, p_t) = A(t, p_t)$ for $t \in R$, $p(t)$ is a solution of system (9).

In the same ways as above, we can replace the condition (3) in [7] by the condition $(**)$, where β is assumed to be $-\infty$.

6. Autonomous linear system. Consider an autonomous linear system

$$\dot{x}(t) = A(x_t) \,, \tag{25}$$

where $A(\phi)$ is a bounded linear operator on $\mathscr{B}$ into R^n. We assume that the function $g(\theta)$ in Definition 2 satisfies the condition

$$g(\theta_1 + \theta_2) \leqq g(\theta_1) g(\theta_2) \qquad \text{for} \quad \theta_1, \theta_2 \in (-\infty, 0] \,. \tag{26}$$

Then it has been proved by Naito (Theorem 4.4 in [8]) that there exist two positively invariant spaces S and U such that

$$\mathscr{B}((-\infty, 0], R^n) = S \oplus U$$

with the properties that

(i) every solution of (25) starting from S tends to zero as $t \to \infty$,

(ii) $\dim U < \infty$, and the solutions of (25) starting from U are governed by an autonomous linear system of ordinary differential equations all of whose eigenvalues have nonnegative real part.

Hence, by the same arguments as in ([7], p. 91), we can show that if $x(t)$ is a nontrivial R-bounded solution of (25), then it satisfies condition (8).

REMARK. In order to show that the above decomposition of the space $\mathscr{B}$ according to Theorem 4.4 in [8], we must see that the condition

$$\beta < 0 \tag{27}$$

holds, where β is the one defined by (21). However, Professor Naito informed me that condition (27) follows from condition (26). I represent here a method due to Professor Naito. If condition (26) holds, then there exists a number α such that

$$\alpha = \sup_{\theta<0} (\log g(\theta))/\theta = \lim_{\theta\to-\infty} (\log g(\theta))/\theta ,$$

(cf. Theorem 7.6.1 in [3]). It is clear that $g(\theta) \geqq e^{\alpha\theta}$ for $\theta \in (-\infty, 0]$ and that for any $\gamma < \alpha$, there exists a constant $N(\gamma)$ such that $g(\theta) \leqq N(\gamma)e^{\gamma\theta}$ for $\theta \in (-\infty, 0]$. Since $g(\theta)$ is nondecreasing and integrable, it holds that $0 < \alpha \leqq \infty$. Hence we have the relation

$$\beta = -\alpha/p$$

where p is the one given in Definition 2, which implies condition (27).

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REFERENCES

[1] J. FAVARD, Leçons sur les fonctions presque-périodiques, Gauthier-Villars, Paris, 1933.
[2] J. K. HALE, Dynamical systems and stability, J. Math. Anal. Appl., 26 (1969), 39-59.
[3] E. HILL AND R. S. PHILLIPS, Functional analysis and semi-groups, Colloq. Publ. Amer. Math. Soc., 1957.
[4] Y. HINO, Asymptotic behavior of solutions of some functional differential equations, Tôhoku Math. J., 22 (1970), 98-108.
[5] Y. HINO, Stability and existence of almost periodic solutions of some functional differential equations, Tôhoku Math. J., 28 (1976), 389-409.
[6] J. KATO, Remarks on linear functional differential equations, Funkcial. Ekvac., 12 (1969), 89-98.
[7] J. KATO, Favard's separation theorem in functional differential equations, Funkcial. Ekvac., 18 (1975), 85-98.
[8] T. NAITO, On autonoumous linear functional differential equations with infinite retardations, J. Differential Equs., 21 (1976), 297-315.

[9] T. NAITO, Adjoint equations of autonomous linear functional differential equations with infinite retardations, Tôhoku Math. J., 28 (1976), 135-143.

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