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ON DERIVATIONS OF AW^*-ALGEBRAS

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Abstract. Some elementary results on derivations of continuous fields of C^*-algebras are used to prove that every derivation of an AW^*-algebra of type III (or of type I) is inner, and also that if a given quotient of an AW^*-algebra is known to have only inner derivations, its tensor product with a separable commutative C^*-algebra with unit also has this property.

1. Introduction. It was proved by Sakai and Kadison ([20], [12]) that every derivation of a W^*-algebra is inner. Using this, Sakai proved that every derivation of a simple C^*-algebra with unit is inner ([21]).

In the present paper it is shown that the W^*-algebra theorem can be deduced from the simple C^*-algebra theorem. More precisely, a proof is given, using Sakai's theorem on simple C^*-algebras with unit, that every derivation of an AW^*-algebra of type III is inner. For W^*-algebras, the general theorem follows by Sakai's method of passing to the tensor product with an algebra of type III ([20]). Possibly this method could also be applied to AW^*-algebras, and this might be of interest to investigate. Soon after I obtained the theorem for AW^*-algebras of type III (and, independently, Deel obtained the theorem for AW^*-algebras of type II_1 with centre-valued trace—[3]), the method of spectral analysis was introduced by Arveson and Borchers ([1], [2]), yielding the theorem in any AW^*-algebra (see [17])—indeed, more generally—in [18] it was shown how to prove the simple C^*-algebra theorem by this method.

In section 2, some easy lemmas on derivations of continuous fields of C^*-algebras are presented, giving sufficient conditions for every derivation to be inner. In section 4 it is shown that a derivation of an AW^*-algebra is inner if it is inner on a reduced subalgebra of central support one. By results of Feldman and Glimm ([8], [9]), an AW^*-algebra of type III has a reduced subalgebra of central support one which is a continuous field of simple C^*-algebras; to this the lemmas of section 2 can be applied.

Another application of section 2 is as follows. Let B be a C^*-algebra and C a separable commutative C^*-algebra with unit. Suppose that B has the property that if a sequence of derivations converges simply (with respect to the norm of B) then it converges in norm. Then every derivation of $B \otimes C$ is inner, if B has only inner derivations.

In section 3 it is shown that any quotient of an AW^*-algebra has the property required of B above. For automorphisms instead of derivations, this was proved in [6], but there are two gaps in the proof in [6], so a complete proof is given here (and the gaps in [6] filled). It should be remarked that while for W^*-algebras this theorem is essentially due to Kallman (see [6] for reference), for quotients of W^*-algebras the same methods seem to be needed as for AW^*-algebras.

Besides the fact that every derivation of a simple C^*-algebra with unit is inner ([21]), we shall also use the fact that a derivation of any C^*-algebra is continuous ([19]).

2. Derivations of continuous fields of C^*-algebras.

NOTATION 2.1. In this paragraph, T will denote a compact Hausdorff space, and $((A(t))_{t\in T}, A)$ a continuous field of C^*-algebras on T (see 10.3 of [5]). Thus, A is a C^*-algebra, and for each $t \in T$ there is a morphism $a \mapsto a(t)$ of A onto $A(t)$ such that $a \mapsto ||a(t)||$ is continuous for any $a \in A$, and $||a|| = \sup ||a(t)||$.

If δ is a derivation of A, we shall denote by $\delta(t)$ the derivation of $A(t)$ induced by δ.

LEMMA 2.2. *Let δ be a derivation of A. Then δ is in the norm closure of the space of inner derivations of A if (and only if) for each $t \in T$ and $\varepsilon > 0$ there exists $a \in A$ such that $||\delta(s) - \mathrm{ad}\, a(s)|| < \varepsilon$ for all s in a neighbourhood of t.*

PROOF. By compactness, there exist a covering $(G_1, \cdots, G_n)$ of T by open sets and a finite family $(a_1, \cdots, a_n)$ of elements of A such that for each $i = 1, \cdots, n$,

$$||\delta(t) - \mathrm{ad}\, a_i(t)|| \leqq \varepsilon\,, \qquad t \in G_i\,.$$

Choose a partition of unity $(f_1, \cdots, f_n)$ subordinate to the covering $(G_1, \cdots, G_n)$, and set

$$\sum f_i a_i = a\,.$$

Then

$$||\delta(t) - \mathrm{ad}\, a(t)|| \leqq \varepsilon\,, \qquad t \in T\,.$$

LEMMA 2.3. *The space of inner derivations of A is norm-closed if there exists $0 < K \leqq 2$ such that for each $t \in T$ and $a \in A$,*

$$||\mathrm{ad}\, a(t)|| \geqq K \inf_{z \in \mathrm{centre}\, A} ||a(t) - z(t)||\,.$$

PROOF. Let $a \in A$. Then $t \mapsto \|\operatorname{ad} a(t)\|$ is lower semicontinuous. Hence by compactness, for each $\varepsilon > 0$ there exists a covering $(G_1, \cdots, G_n)$ of T by open sets and a family $(z_1, \cdots, z_n)$ in centre A such that for each $i = 1, \cdots, n$,

$$\varepsilon + \|\operatorname{ad} a(t)\| > K\|a(t) - z_i(t)\| , \qquad t \in G_i .$$

Choose a partition of unity $(f_1, \cdots, f_n)$ subordinate to the covering $(G_1, \cdots, G_n)$, and set

$$\sum f_i z_i = z .$$

Then

$$\varepsilon + \|\operatorname{ad} a(t)\| > K\|a(t) - z(t)\| , \qquad t \in T ,$$

whence

$$\varepsilon + \|\operatorname{ad} a\| > K\|a - z\| .$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\|\operatorname{ad} a\| \geqq K \inf_{z \in \text{centre } A} \|a - z\| , \qquad a \in A .$$

This implies (and, by the closed graph theorem, is equivalent to) closure of the range of the canonical map $A \ni a \mapsto \operatorname{ad} a$, which factorizes through $A/\text{centre } A$—in other words, closure of the space of inner derivations (in the norm topology).

LEMMA 2.4. *The function $t \mapsto \|\delta(t)\|$ is continuous for any derivation δ of A if either of the following conditions holds.*

(i) *$((A(t))_{t \in T}, A)$ is locally trivial, T is first countable, and for each $t \in T$, each simply convergent sequence of derivations of $A(t)$ converges in norm.*

(ii) *T is totally disconnected, and for every family (e_i) of mutually disjoint open and closed subsets of T and every bounded family (a_i) in A, there exists $a \in A$ with $a|e_i = a_i|e_i$, all i.*

PROOF. In case (i), we may suppose that $((A(t))_{t \in T}, A)$ is trivial. Then even $t \mapsto \delta(t)$ is continuous—first in the topology of simple convergence, and then, by the hypothesis, in the norm topology.

In case (ii), to prove that $t \mapsto \|\delta(t)\|$ is continuous, we must prove that it is upper semicontinuous (as, being a supremum of continuous functions, it is automatically lower semicontinuous). Suppose that, on the contrary, for some $t \in T$ and some $\varepsilon > 0$, every neighbourhood of t contains an s such that

$$\|\delta(s)\| \geqq \|\delta(t)\| + \varepsilon .$$

Then, by lower semicontinuity,

$$\|\delta(s)\| > \|\delta(t)\| + \varepsilon/2$$

for all s in a nonempty open subset of each neighbourhood of t.

Choose a maximal familly (e_i) of mutually disjoint open and closed subsets of T such that

$$\|\delta(s)\| > \|\delta(t)\| + \varepsilon/2\,, \qquad s \in e_i\,.$$

Then, by maximality, and the hypothesis that T is totally disconnected, $t \in (\bigcup e_i)^-$. Since each e_i is compact, there exists for each i an element a_i of A of norm one such that

$$\|(\delta a_i)(s)\| > (\|\delta(t)\| + \varepsilon/2)\|a_i(s)\|\,, \qquad s \in e_i\,.$$

Hence by hypothesis there exists $a \in A$ such that

$$\|(\delta a)(s)\| > (\|\delta(t)\| + \varepsilon/2)\|a(s)\|\,, \qquad s \in \bigcup e_i\,.$$

By continuity,

$$\|(\delta a)(t)\| \geqq (\|\delta(t)\| + \varepsilon/2)\|a(t)\|\,,$$

a contradiction. This shows that $t \mapsto \|\delta(t)\|$ must be upper semicontinuous.

COROLLARY 2.5. *Suppose that A satisfies the hypothesis of* 2.3 *and one of the conditions* 2.4 (i), 2.4 (ii). *Let δ be a derivation of A such that each $\delta(t)$ is inner. Then δ is inner.*

PROOF. By 2.4 and 2.2, δ is in the norm closure of the space of inner derivations. By 2.3, this is equal to the space of inner derivations.

LEMMA 2.6. *Let $a \in A$. Then the function $t \mapsto \|\mathrm{ad}\, a(t)\|$ is continuous if either of the following conditions holds.*

(i) *$((A(t))_{t\in T}, A)$ is locally trivial.*

(ii) *For each $t \in T$,*

$$\|\mathrm{ad}\, a(t)\| = 2 \inf_{z \in \mathrm{centre}\, A} \|a(t) - z(t)\|\,.$$

PROOF. In case (i), we may suppose that $((A(t))_{t\in T}, A)$ is trivial, and then even $t \mapsto \mathrm{ad}\, a(t)$ is continuous (as $t \mapsto a(t)$ is).

In case (ii), the left side of the equation is lower semicontinuous and the right side upper semicontinuous.

COROLLARY 2.7. *Let $a \in A$. Suppose that A has a unit, and that for each $t \in T$,*

$$\|\mathrm{ad}\, a(t)\| = 2 \inf_{\lambda \in \boldsymbol{C}} \|a(t) - \lambda\|\,,$$

$$\|a(t) - z(t)\|^2 + |z(t) - \lambda|^2 \leqq \|a(t) - \lambda\|^2\,, \qquad \lambda \in \boldsymbol{C}\,,$$

where $z(t) \in \boldsymbol{C}$ is such that

$$\|a(t) - z(t)\| = \inf_{\lambda \in C} \|a(t) - \lambda\| .$$

Then $z \in$ centre A.

PROOF. By hypothesis, 2.6 (ii) holds. Hence by 2.6, $t \mapsto \|a(t) - z(t)\|$ is continuous, and, in particular, since T is compact, is bounded, say by M.

Fix $t \in T$, $\varepsilon > 0$. Then there exists a neighbourhood V of t such that

$$\|a(s) - z(t)\| \leqq \|a(s) - z(s)\| + \varepsilon , \qquad s \in V ,$$

as both sides are continuous in s. By hypothesis, for any s,

$$\|a(s) - z(s)\|^2 + |z(s) - z(t)|^2 \leqq \|a(s) - z(t)\|^2 .$$

Hence, for $s \in V$,

$$\|a(s) - z(s)\|^2 + |z(s) - z(t)|^2 \leqq (\|a(s) - z(s)\| + \varepsilon)^2 ;$$
$$|z(s) - z(t)|^2 \leqq \varepsilon(2M + \varepsilon) .$$

This shows that z is continuous, that is, that $z \in$ centre A.

3. Convergence of derivations in certain C^*-algebras.

LEMMA 3.1. *Let A be a C^*-algebra, and let δ be a derivation of A. Denote by M_n the C^*-algebra of $n \times n$ complex matrices, $n = 1, 2, \cdots$. Then for each n the derivation $\delta \otimes 1$ of $A \otimes M_n$ satisfies*

$$\|\delta \otimes 1\| = \|\delta\| .$$

PROOF. This is 4.1 of [7].

COROLLARY 3.2. *Let A be a C^*-algebra and let α be an (adjoint-preserving) automorphism of A. Then for each $n = 1, 2, \cdots$ the automorphism $\alpha \otimes 1$ of $A \otimes M_n$ satisfies*

$$\|\alpha \otimes 1 - 1\| = \|\alpha - 1\| .$$

PROOF. Clearly

$$\|\alpha - 1\| \leqq \|(\alpha - 1) \otimes 1\| = \|\alpha \otimes 1 - 1\| \leqq 2 .$$

Therefore equality holds trivially if $\|\alpha - 1\| = 2$.

If $\|\alpha - 1\| < 2$, then by Theorem 7 of [13] α is universally weakly inner. Representing A as a C^*-algebra of operators, then, we have a unitary operator $u \in A''$ such that

$$\alpha = \mathrm{Ad}\, u | A .$$

Since u is unitary, we have $\|uau^* - a\| = \|ua - au\|$ for any $a \in A$, whence

$$\|\alpha - 1\| = \|\delta\|$$

where $\delta = \operatorname{ad} u | A''$. By 3.1,

$$\|\delta\| = \|\delta \otimes 1\| ,$$

and since $\delta \otimes 1 = \operatorname{ad} (u \otimes 1) | A'' \otimes M_n$,

$$\|\delta \otimes 1\| = \|\alpha \otimes 1 - 1\| .$$

The conclusion follows from these three norm equalities.

COROLLARY 3.3. *Let A be a C^*-algebra, and let B be a sub-C^*-algebra of A which is simple and finite-dimensional. Let e be a minimal projection of B, and denote by p the unit of B. Suppose that there is a partial isometry $v \in pA(1-p)$ such that $p + v^*v$ is central in A. Let $(\delta_1, \delta_2, \cdots)$ be a sequence of derivations of A such that $\|\delta_n a\|$ converges to zero for each $a \in A$, and $\|\delta_n | A(1 - p - v^*v)\|$ converges to zero. Then $\|\delta_n\| - \|\delta_n | eAe\|$ converges to zero.*

PROOF. First, by hypothesis,

$$\|\delta_n\| - \|\delta_n | A(p + v^*v)\| \to 0 .$$

Next, denote the C^*-algebra $pAp \otimes M_2$ by C. If (e_{ij}) is a system of matrix units for M_2, then we may identify $A(p + v^*v)$ with fCf where $f = p \otimes e_{11} + vv^* \otimes e_{22}$, in such a way that p corresponds to $p \otimes e_{11}$ and v to $vv^* \otimes e_{12}$. By 3.1,

$$\|\delta_n | pAp\| = \|(\delta_n | pAp) \otimes 1\| .$$

Denote $(\delta_n | pAp) \otimes 1$ by D_n. Then

$$\|D_n | fCf - \delta_n | A(p + v^*v)\| \to 0 .$$

Hence

$$\|\delta_n | A(p + v^*v)\| - \|\delta_n | pAp\| \to 0 .$$

Finally, by 3.1,

$$\|\delta_n | pAp\| - \|\delta_n | eAe\| \to 0 .$$

THEOREM 3.4*. *Let A be an AW^*-algebra, and let $(\delta_1, \delta_2, \cdots)$ be a sequence of derivations of A. Suppose that $\|\delta_n a\|$ converges to zero for each $a \in A$. Then $\|\delta_n\|$ converges to zero.*

PROOF. By Theorem 2 of [12], all δ_n are zero on the centre of A.

Suppose first that A has no finite type I direct summand. Then there exist mutually orthogonal projections $e_1, e_2, \cdots$ such that for each $i = 1, 2, \cdots$, e_i is equivalent to 2^{-i}; i.e., there exists a subfactor A_i of

* See Appendix.

A of type I_{2^i} containing e_i as a minimal projection. (Then $A = A_i \otimes A_i'$ where A_i' is the relative commutant of A_i, $i = 1, 2, \cdots$.) Suppose that for some $\varepsilon > 0$, $\|\delta_n\| > \varepsilon$ for infinitely many n. Fix $i = 1, 2, \cdots$. Since A_i is finite-dimensional, $\|\delta_n | A_i\| \to 0$. Hence by 3.1,

$$\|\delta_n\| - \|\delta_n | e_i A e_i\| \to 0 .$$

Then for each i there exist $n_i \geqq i$ and $a_i \in e_i A e_i$ of norm one such that $\|\delta_{n_i} a_i\| > \varepsilon/2$. Thus, passing to a subsequence of $(\delta_1, \delta_2, \cdots)$, we have a sequence $(a_1, a_2, \cdots)$ of elements of norm one of mutually orthogonal reduced subalgebras $e_1Ae_1, e_2Ae_2, \cdots$ of A, such that

$$\|\delta_n a_n\| > \varepsilon/2, \quad \text{all } n .$$

This is in contradiction with Phillips' Lemma (use 2 of [6] with $\theta_n(\sum_{k \in K} e_k) = \delta_n(\sum_{k \in K} a_k)$).

Consider now the case that A is finite and of type I. If A has only finitely many homogeneous direct summands then A is a finitely generated module over its centre, whence $\|\delta_n\| \to 0$. If the number of (maximal) homogeneous direct summands of A is infinite, then a modified form of the argument above is applicable. It is enough to find a sequence $(e_1, e_2, \cdots)$ of mutually orthogonal projections in A such that for each $i = 1, 2, \cdots$,

$$\|\delta_n\| - \|\delta_n | e_i A e_i\| \to 0 .$$

Choose $(e_1, e_2, \cdots)$ as follows. First, for each $k = 1, 2, \cdots$ define $r(k) = 1, 2, \cdots$ by

$$2^{r(k)} \leqq k < 2^{r(k)+1} .$$

Choose e_1 so that the component of e_1 in the homogeneous direct summand of A of type I_k is zero if $r(k) < 1$ and is equivalent to $k^{-1}2^{r(k)-1}$ if $r(k) \geqq 1$. Choose e_2 orthogonal to e_1 and such that the component of e_2 in the homogeneous direct summand of A of type I_k is zero if $r(k) < 2$ and is equivalent to $k^{-1}2^{r(k)-2}$ if $r(k) \geqq 2$. Continue in this way. It is easily verified that for each i the hypotheses of 3.3 are verified with $e = e_i$ and B a suitable algebra of order 2^i. This shows that each e_i has the desired property.

THEOREM 3.5. *Let A be an AW^*-algebra, let J be a closed two-sided ideal of A, and let $(\delta_1, \delta_2, \cdots)$ be a sequence of derivations of A/J. If $\|\delta_n a\|$ converges to zero for each $a \in A/J$, then $\|\delta_n\|$ converges to zero.*

PROOF. Let π denote the canonical surjection from A to A/J. It is enough to show that $\|\delta_n \cdot \pi\|$ converges to zero. The proof of this is identical to that of 3.4 with δ_n replaced by $\delta_n \cdot \pi$.

REMARK 3.6. The analogues of 3.4 and 3.5 for (adjoint-preserving) automorphisms are 4 and 5 of [6]. I should like to take this opportunity to give an explanation of the second-last sentence in the proof of 4 of [6], and also to give an alternative proof of 4 of [6] in the finite type I case, which must be used in order for the proof of 5 of [6] to be valid in this case. Both these aims can be achieved by remarking that the proof of 3.3 above, with δ_n replaced by $\varphi_n - 1$ and 3.1 replaced by 3.2, and with the first sentence omitted, gives a proof of 4 of [6] starting from the point where all $\varphi_n - 1$ are assumed to be zero on the centre.

4. Some C^*-algebras with only inner derivations.

LEMMA 4.1. *Let B be an AW^*-algebra, let $b \in B$, and let I be a closed two-sided ideal of B. Then there exists $z \in \text{centre}\, B$ such that*

$$\|\mathrm{ad}\,(b + I)\| \geqq \|b - z + I\| .$$

PROOF. By the Dixmier approximation theorem (the proof of which—see Chapitre III, § 5 of [4]—is valid in an AW^*-algebra), there exists $z \in \text{centre}\, B$ such that z is in the norm closure of the convex hull of the set $\{ubu^{-1} | u \text{ unitary in } B\}$. Then

$$\begin{aligned} \|b - z + I\| &\leqq \sup_u \|b - ubu^{-1} + I\| \\ &= \sup_u \|bu - ub + I\| \\ &\leqq \|\mathrm{ad}\,(b + I)\| . \end{aligned}$$

THEOREM 4.2. *Let B be an AW^*-algebra, let I be a closed two-sided ideal of B, and let C be a separable commutative C^*-algebra with unit. Denote the C^*-algebra tensor product of B/I and C by A. Assume that every derivation of B/I is inner (this holds, in particular, if $I = 0$ or if I is maximal). Then every derivation of A is inner.*

PROOF. Consider the continuous field of C^*-algebras $((A(t))_{t \in T}, A)$ where $T = \text{Prim}\, C$ and $A \to A(t)$ is the quotient map of A corresponding to the closed two-sided ideal $(B/I) \otimes t$.

By 4.1 the hypothesis of 2.3 is verified with $K = 1$. By 3.5, condition 2.4 (i) is satisfied. Hence the conclusion follows by 2.5.

REMARK 4.3. In the case that B is a factor of type I and $I = 0$, 4.2 is due to Lance (it is Theorem 2.3 of [14]).

LEMMA 4.4. *Let A be an AW^*-algebra, and let δ be a derivation of A. Suppose that there exists a projection $e \in A$ of central support 1 such that $\delta(eAe) \subset eAe$ and the derivation $\delta | eAe$ is inner. Then δ is inner.*

PROOF. Replacing δ by $\delta - \mathrm{ad}\, x_0$, where $x_0 \in eAe$ satisfies $\delta | eAe = \mathrm{ad}\, x_0 | eAe$, we may suppose that $\delta | eAe = 0$.

Since $\delta = \delta_1 + i\delta_2$ for unique derivations δ_1, δ_2 preserving adjoints, we may suppose that δ itself preserves adjoints. Then $\exp \delta$ is an automorphism of A.

By 4.1, if $\delta = \mathrm{ad}\, y$ for some $y \in A$ then y may be chosen such that $\|y\| \leqq \|\delta\|$. This shows that it is enough to prove the conclusion of the lemma with A replaced by a nonzero direct summand. Passing, then, to a suitable direct summand, and replacing e by a subprojection if necessary, we may suppose that there is a family (e_{ij}) of matrix units in A (possibly infinite) such that $e_{11} = e$.

By Theorem 64 of [16], there exists a unique unitary $u \in A$ such that

$$ue_{ii} = (\exp \delta)(e_{i1})e_{1i} \ .$$

By simple calculations we deduce that $uxu^{-1} = (\exp \delta)(x)$ whenever either $x \in eAe$ or $x = e_{ij}$. If x is arbitrary in A then, for all i and j,

$$\begin{aligned} ue_{ii}xe_{jj}u^{-1} &= (ue_{i1}u^{-1})(ue_{1i}xe_{j1}u^{-1})(ue_{1j}u^{-1}) \\ &= (\exp \delta)(e_{i1}e_{1i}xe_{j1}e_{1j}) \\ &= (\exp \delta)(e_{ii}xe_{jj}) \ ; \end{aligned}$$

hence, with f_i denoting $ue_{ii}u^{-1}$,

$$\begin{aligned} f_iuxu^{-1}f_j &= f_i(\exp \delta)(x)f_j \ , \quad \text{all} \quad i,\ j \ ; \\ uxu^{-1} &= (\exp \delta)(x) \ . \end{aligned}$$

Replacing δ by $t\delta$, with t real, let us denote the unitary obtained as above by u_t. We shall now show that $t \mapsto u_t$ is a norm-continuous one-parameter group.

First, let us note that $u_t^{-1}e = u_te = e$, and that therefore, for any $x \in A$, $u_txe = (\exp t\delta)(xe)$.

To demonstrate the group property, we have for s and t real and x arbitrary in A,

$$\begin{aligned} u_su_txe &= (\exp s\delta)(u_txe) = (\exp s\delta)(\exp t\delta)(xe) \\ &= (\exp (s + t)\delta)(xe) = u_{s+t}xe \ . \end{aligned}$$

In order to show continuity, it is enough to show continuity at 0. We have

$$\begin{aligned} \|u_t - 1\| &= \sup_{\|x\| \leqq 1} \|(u_t - 1)xe\| \\ &= \sup_{\|x\| \leqq 1} \|(\exp t\delta)(xe) - xe\| \\ &\leqq \|\exp t\delta - 1\| \ . \end{aligned}$$

Using Stone's theorem we deduce that there exists $h = h^* \in A$ such that for all real t,

$$u_t = \exp ith .$$

Then, for all $x \in A$,

$$(\exp t\delta)(x) = (\exp ith)x(\exp - ith) = (\exp t(\text{ad } ih))(x) .$$

Differentiation yields

$$\delta = \text{ad } ih .$$

LEMMA 4.5. (Feldman, [8]). *Let A be an AW^*-algebra of type III (or of type I). Then there exists a projection $e \in A$ of central support 1 such that in the subalgebra eAe each projection is equivalent to its central support.*

PROOF. This is Theorem 1 of [8], if A is of type III. If A is of type I, e may be chosen such that eAe is commutative.

COROLLARY 4.6. *Let A be an AW^*-factor. Then every derivation of A is inner.*

PROOF. There exists a nonzero projection $e \in A$ such that eAe is simple. (If A is of type I, a minimal projection will do for e; if of type II, a finite projection will do; if of type III, a projection with the property of 4.5 will do.) Let δ be a derivation of A. Replacing δ by $\delta - \text{ad}\,[\delta(e), e]$, we may suppose that $\delta(e) = 0$. Then $\delta(eAe) \subset eAe$, whence by the theorem of Sakai $\delta | eAe$ is inner. Hence by 4.4, δ is inner.

LEMMA 4.7. (Glimm, [9]). *Let A be an AW^*-algebra. Then for each maximal ideal t of centre A, tA is a closed two-sided ideal of A. Denote by T the set of all such t, i.e., the spectrum of centre A, with the Jacobson topology. Then for each $a \in A$ the map $T \ni t \mapsto \|x + tA\|$ is continuous.*

PROOF. Although by tA we mean $\{za | z \in t, a \in A\}$, in fact, by the Cohen factorization theorem (page 268 of [11]), tA is equal to its linear span, and is closed, and is therefore a closed two-sided ideal (the smallest containing t).

Upper semicontinuity of $t \mapsto \|a + tA\|$ is a special case of Lemma 9 of [9]. (It can be seen somewhat more directly in the present case.)

Lower semicontinuity in the case that A is a W^*-algebra is Lemma 10 of [9]. In the AW^*-algebra case, suppose that $\|a + tA\| > 1$ and $\|a + sA\| < 1$ for some s in every neighbourhood of t. Then by upper semicontinuity there exists a family (e_i) of closed open subsets of T

such that $||a + sA|| < 1$ for all $s \in \bigcup e_i$, and $t \in (\bigcup e_i)^-$. Then $||ae_i|| \leqq 1$; hence $||a(\bigcup e_i)^-|| \leqq 1$, $||a + tA|| \leqq 1$, a contradiction.

COROLLARY 4.8. (Halpern, [10]). *Let A be an AW^*-algebra and let $a \in A$. Then there exists $z \in$ centre A such that*

$$||\mathrm{ad}\, a|| = 2||a - z|| .$$

PROOF. In the case that A is a factor of type I, or indeed any primitive C^*-algebra, this result is due to Stampfli ([22]). Stampfli showed, moreover, that the inequality in 2.7 holds, with A in place of $A(t)$.

Let t be a maximal ideal of centre A. Then Halpern showed in [10] that Stampfli's results also hold for A/tA, even though this quotient is not known to be primitive (in case A is of type II). (Halpern first noted that local comparability of projections in A implies comparability in A/tA, so that the closed two-sided ideals of A/tA form a chain, and then applied Stampfli's result to each primitive quotient of A/tA, deducing it for A/tA by passing to the limit.)

By 4.7, A defines a continuous field of C^*-algebras on T, the spectrum of centre A, which we may write as $((A/tA)_{t \in T}, A)$. Since the continuous fields $a \mapsto a + tA$, $a \in A$, are closed under multiplication by scalar-valued fields and are norm-complete, by 11.5.3 of [5] A is the C^*-algebra defined by the continuous field of C^*-algebras $((A/tA)_{t \in T}, A)$.

Thus it is seen that 2.7 may be applied to the field $((A/tA)_{t \in T}, A)$; this yields the desired conclusion.

QUESTION 4.9. Does 4.8 hold for a quotient of an AW^*-algebra? (Cf. 4.1.)

THEOREM 4.10. *Let A be an AW^*-algebra of type III (or of type I). Then every derivation of A is inner.*

PROOF. The case that A is of type I is of course due to Kaplansky ([15]); an alternate proof is given by the following argument intended primarily for the case that A is of type III.

By 4.4 and 4.5 we may suppose that every projection in A is equivalent to its central support. Then A/tA is simple for every maximal ideal t of centre A. (Every closed two-sided ideal of A is generated by its projections, and any projection not in tA is equivalent to 1 modulo tA.) Hence by the theorem of Sakai every derivation of each A/tA is inner.

As in 4.7, A is the C^*-algebra defined by the continuous field of C^*-algebras $((A/tA)_{t \in T}, A)$.

By 4.1 the hypothesis in 2.3 is fulfilled with $K = 1$. Hence by 2.5 every derivation of A is inner.

ADDED APRIL 1, 1977. The method of this paper can be used to show that certain quotients of AW^*-algebras have the property that every derivation is inner. The quotients to which the method is applicable are precisely those which Ringrose in "Derivations of quotients of von Neumann algebras" (preprint) has shown to have this property by a different method: the quotients by intersections of maximal two-sided ideals, such that orthogonal families of central projections can be lifted. Such a C^*-algebra is a continuous field of simple C^*-algebras, and satisfies the hypotheses of 2.5, and hence has only inner derivations.

REFERENCES

[1] W. ARVESON, On groups of automorphisms of operator algebras, J. Functional Analysis 15 (1974), 217–243.
[2] W. J. BORCHERS, Über Ableitungen von C^*-Algebren, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, Nr. 2 (1973).
[3] J. C. DEEL, Derivations of AW^*-algebras, Proc. Amer. Math. Soc. 42 (1974), 85–95.
[4] J. DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1957.
[5] J. DIXMIER, Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.
[6] G. A. ELLIOTT, Convergence of automorphisms in certain C^*-algebras, J. Functional Analysis 11 (1972), 204–206.
[7] G. A. ELLIOTT, On lifting and extending derivations of approximately finite-dimensional C^*-algebras, J. Functional Analysis 17 (1974), 395–408.
[8] J. FELDMAN, Some connections between topological and algebraic properties in rings of operators, Duke Math. J. 23 (1956), 365–370.
[9] J. G. GLIMM, A Stone-Weierstrass theorem for C^*-algebras, Ann. of Math. 72 (1960), 216–244.
[10] H. HALPERN, The norm of an inner derivation of an AW^*-algebra, preprint.
[11] E. HEWITT AND K. A. ROSS, Abstract harmonic analysis II, Springer, New York, 1970.
[12] R. V. KADISON, Derivations of operator algebras, Ann. of Math. 83 (1966), 280–293.
[13] R. V. KADISON AND J. R. RINGROSE, Derivations and automorphisms of operator algebras, Commun. Math. Phys. 4 (1967), 32–63.
[14] E. C. LANCE, Automorphisms of certain operator algebras, Amer. J. Math. 91 (1969), 160–174.
[15] I. KAPLANSKY, Modules over operator algebras, Amer. J. Math. 75 (1953), 839–858.
[16] I. KAPLANSKY, Rings of operators, Benjamin, New York, 1968.
[17] D. OLESEN, Derivations of AW^*-algebras are inner, Pacific J. Math. 53 (1974), 555–561.
[18] D. OLESEN AND G. K. PEDERSEN, Derivations of C^*-algebras have semi-continuous generators, Pacific J. Math. 53 (1974), 563–572.
[19] S. SAKAI, On a conjecture of Kaplansky, Tôhoku Math. J. 12 (1960), 31–33.
[20] S. SAKAI, Derivations of W^*-algebras, Ann. of Math. 83 (1966), 273–279.
[21] S. SAKAI, Derivations of simple C^*-algebras, J. Functional Analysis 2 (1968), 202–206.
[22] J. G. STAMPFLI, The norm of a derivation, Pacific J. Math. 33 (1970), 737–747.

Appendix (ADDED APRIL 14, 1978)

Convergence of derivations in separable C^*-algebras

A.1. THEOREM. *Let A be a separable C^*-algebra. Then the following conditions are equivalent.*

(i) *Each simply convergent sequence of derivations of A is convergent in norm.*

(ii) *A is the direct sum of a commutative C^*-algebra and a unital C^*-algebra with continuous trace.*

PROOF. By [A4], every derivation of A is the simple limit of a sequence of inner derivations. Therefore, condition (i) implies that the set of derivations of A is separable in norm. Hence by 3 of [A3], A is the direct sum of a commutative C^*-algebra, finitely many simple unital C^*-algebras, and a unital C^*-algebra with only trivial central sequences. In a simple unital C^*-algebra, condition (i) implies immediately that every central sequence is trivial. It follows that A has continuous trace. (By 18 of [A3], A has Hausdorff primitive spectrum, and every derivation of A is inner. By 2.4 of [A1] A is postliminary, and therefore liminary. Hence by 3 of [A2], A has continuous trace.)

Conversely, condition (ii) implies by 3 of [A2] that every derivation of A is inner, and hence by 3 of [A3] that every central sequence of A is trivial (see 2.4 of [A1] for a more direct proof of this last property). Combining the last two properties (assuming that A has a unit) yields (i) immediately.

A.2. COROLLARY. *Let $n = 1, 2\cdots$ and $\varepsilon > 0$. Then there exists $d = d(n, \varepsilon) = 1, 2, \cdots$ such that: if H is a separable Hilbert space with dimension at least d and if $x_1, \cdots, x_n$ belong to the unit ball of $B(H)$, then there exists $y \in B(H)$ with $\|y - C\| = \inf_{\lambda\in C}\|y - \lambda\| = 1$ and $\|yx_i - x_iy\| < \varepsilon$, $i = 1, \cdots, n$.*

PROOF. Assume that the conclusion is false (that is, d does not exist). We shall deduce an absurdity, by constructing a separable C^*-algebra A which satisfies A.1(i) but not A.1(ii).

There exists in $N \cup \{\infty\}$ a sequence (k_p) converging to ∞ such that for each $p = 1, 2, \cdots$ there exist $x_{1,p}, \cdots, x_{n,p} \in B(H_p)$, where $\dim H_p = k_p$, such that for $y \in B(H_p)$,

$$\|yx_{i,p} - x_{i,p}y\| < \varepsilon , \qquad i = 1, \cdots, n \Rightarrow \|y - C\| < 1 .$$

Denote by A the sub-C^*-algebra of $B(\bigoplus H_p)$ generated by the compact operators on all the H_p together with the operators $x_i = \bigoplus x_{i,p}$, $i = 1, \cdots, n$. Then for $y \in A$,

$$\|yx_i - x_iy\| < \varepsilon\,, \qquad i = 1, \cdots, n \Rightarrow \|y - \oplus_p C\| < 1\,.$$

Equivalently,

$$\|y - \oplus_p C\| \leqq \varepsilon^{-1} \sup_{i=1,\cdots,n} \|yx_i - x_iy\|\,, \qquad y \in A\,.$$

This implies that A satisfies A.1 (i), since by [A4] every derivation of a separable C^*-algebra is the simple limit of a sequence of inner derivations (see the bottom of page 122 of [A3] for an elementary proof of this). However, A clearly does not satisfy A.1 (ii).

A.3. PROBLEM. A.2 also holds with H a nonseparable Hilbert space, for the trivial reason that in this case for any $x_1, \cdots, x_n \in B(H)$ there exists a projection $e \neq 0, 1$ such that $ex_i - x_ie = 0$, $i = 1, \cdots, n$. It is easy to see that in A.2 y can always be chosen to be selfadjoint. Is it always possible to choose y of the form $2e - 1$ where e is a projection $(\neq 0, 1)$?

ADDITIONAL REFERENCES

[A1] C. A. AKEMANN AND G. K. PEDERSEN, Central sequences and inner derivations of separable C^*-algebras, Amer. J. Math. (to appear).

[A2] G. A. ELLIOTT, Some C^*-algebras with outer derivations, Rocky Mountain J. Math. 2 (1973), 501–506.

[A3] G. A. ELLIOTT, Some C^*-algebras with outer derivations, III, Ann. of Math. 106 (1977), 121–143.

[A4] G. K. PEDERSEN, Lifting derivations from quotients of separable C^*-algebras, Proc. Nat. Acad. Sci. USA 73 (1976), 1414–1415.

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