Tôhoku Math. Journ.
30 (1978), 367-371.

A NOTE ON EXPONENTIAL MARTINGALES

NORIHIKO KAZAMAKI

(Received December 6, 1976)

1. Introduction. Let (Ω, F, P) be a probability space, given a non-decreasing right continuous family $(F_t)_{0 \leqq t < +\infty}$ of sub σ-fields of F such that F_0 contains all null sets. Let M be a local martingle adapted to (F_t) such that $M_0 = 0$ and $\Delta M_t = M_t - M_{t-} \geqq -1$ for every $t \geqq 0$. Throughout the paper, Z denotes the process defined by the formula

$$Z_t = \exp(M_t - \langle M^c \rangle_t / 2) \prod_{s \leqq t} (1 + \Delta M_s) \exp(-\Delta M_s)$$

where M^c is the continuous part of M and $\langle M^c \rangle$ is the continuous increasing process such that $(M^c)^2 - \langle M^c \rangle$ is a local martingale. Then the process Z is a non-negative local martingale with $Z_0 = 1$ (see C. Doléans-Dade [1]).

Our aim is to give a sufficient condition for Z to be a martingale. Originally, this problem was raised by I. V. Girsanov in [4] to study the transformation of the measure of a Brownian motion.

The reader is assumed to be familiar with the martingale theory as given in [2].

2. On the L^p-integrability of the exponential martingale. In a previous paper [5] we dealt only with continuous local martingales M, and proved that if $\exp(M_t/2)$ is a submartingale, then the process Z is a martingale. We start with such an example of a continuous local martingale M that Z is a uniformly integrable martingale but $\exp(M_t/2) \notin L^1$ for some t. For that, let $B = (B_t, F_t)$ be a one dimensional Brownian motion with $B_0 = 0$, and introduce an F_t-stopping time:

$$\tau = \inf\{t > 0; |B_t| \geqq (t+1)^{1/2}\} .$$

It is clear that $\tau < \infty$ and $|B_\tau| = (\tau + 1)^{1/2}$ with probability 1. If $\tau \in L^1$, then $E[B_\tau^2] = E[\tau]$, so that it is absurd to claim that τ is integrable. Thus $\exp(B_\tau/2)$ is not integrable. On the other hand, the process $\{\exp(B_{t \wedge \tau} - (t \wedge \tau)/2), F_t\}$ being a martingale, we get

$$\begin{aligned} 1 &= E[\exp(B_{n \wedge \tau} - (n \wedge \tau)/2)] \\ &\leqq E[\exp(B_\tau - \tau/2)] + E[\exp(B_n - n/2); n < \tau] \end{aligned}$$

for every $n \geqq 1$. As $|B_n| < (n+1)^{1/2}$ on $\{n < \tau\}$, the second term on the right hand side is dominated by $\exp((n+1)^{1/2} - n/2)$, which converges to 0 as $n \to \infty$. Therefore we find

$$E[\exp(B_\tau - \tau/2)] = 1\ . \tag{1}$$

Now let $\alpha\colon [0, 1[\to [0, \infty[$ be an increasing homeomorphic function, and set

$$\theta_t = \begin{cases} \alpha(t) \wedge \tau & \text{if} \quad 0 \leqq t < 1 \\ \tau & \text{if} \quad t \geqq 1\ . \end{cases}$$

Then each θ_t is an F_t-stopping time. Almost all sample functions of (θ_t) are non-decreasing and continuous, so that the process M defined by $M_t = B_{\theta_t}$ is a continuous local martingale. From (1) it follows that the process $Z = \exp(M - \langle M \rangle/2)$ is a uniformly integrable martingale over (F_{θ_t}), but $\exp(M_1/2)$ is not integrable because $\theta_1 = \tau$.

Now let M be any local martingale such that $M_0 = 0$. As is well-known, it can be split into the continuous part M^c, and the purely discontinuous part M^d, orthogonal to all continuous local martingales. For simplicity, we use the following notations:

$$Y_t = \exp(M_t^c - \langle M^c \rangle_t/2)$$
$$W_t = \exp(M_t^d) \prod_{s \leqq t} (1 + \Delta M_s) \exp(-\Delta M_s)\ .$$

Y is a continuous local martingale, and W is a purely discontinuous local martingale. It is clear that $Z = YW$. By applying the differentiation formulas, C. Doléans-Dade showed in [1] that the process Z must satisfy the stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_{s-} dM_s\ .$$

We now give a sufficient condition for Z to be a martingale.

THEOREM 1. *Let* ε, δ *be two numbers* >0, *and set* $\gamma = (1 + 1/(2\delta))^2(1 + \varepsilon)/\{(1 + 1/(2\delta))^2(1 + \varepsilon) - \varepsilon\}$. *Then we have*

$$\|Z_t\|_\gamma \leqq \|\exp((\delta + 1/2)M_t^c)\|_1^{4\delta/(1+2\delta)^2} \|\exp(M_t^d)\|_{(1+1/(2\delta))^2(1+\varepsilon)}\ . \tag{2}$$

PROOF. Firstly, we show that the inequality

$$\|Y_t\|_{p_\delta} \leqq \|\exp((\delta + 1/2)M_t^c)\|_1^{4\delta/(1+2\delta)^2}\ , \qquad p_\delta = (1 + 2\delta)^2/(1 + 4\delta) > 1 \tag{3}$$

is valid for every $\delta > 0$. For that, set $p = 1 + 4\delta$. Then the exponent conjugate to p is $q = (1 + 4\delta)/(4\delta)$, and so by the Hölder inequality we get

$$\begin{aligned} E[Y_t^{p_\delta}] &= E[\exp(\sqrt{p_\delta/p}M_t^c - p_\delta\langle M^c \rangle_t/2)\exp((p_\delta - \sqrt{p_\delta/p})M_t^c)] \\ &\leqq \{E[\exp(\sqrt{pp_\delta}M_t^c - pp_\delta\langle M^c \rangle_t/2)]\}^{1/p}\{E[\exp((p_\delta - \sqrt{p_\delta/p})qM_t^c)]\}^{1/q}\ . \end{aligned}$$

As the process $\{\exp(\sqrt{pp_\delta}M_t^c - pp_\delta\langle M^c\rangle_t/2), F_t\}$ is a non-negative local martingale, the first term on the right hand side is bounded by 1. Moreover, by a simple calculation, $(p_\delta - \sqrt{p_\delta/p})q = \delta + 1/2$ and $qp_\delta = (1 + 2\delta)^2/(4\delta)$. Thus the inequality (3) is proved.

Secondly, let $1 < p < \infty$. Noticing the inequality $W_t \leqq \exp(M_t^d)$ and applying the Hölder inequality with the exponents p and $q = p/(p-1)$, we have

$$E[Z_t^\gamma] \leqq \{E[Y_t^{p\gamma}]\}^{1/p}\{E[\exp(q\gamma M_t^d)]\}^{1/q} .$$

It is easy to see that $p_\delta > \gamma > 1$. Then, by setting $p = p_\delta/\gamma$, we find

$$\begin{aligned} \|Z_t\|_\gamma &\leqq \|Y_t\|_{p_\delta}\|\exp(M_t^d)\|_\mu \\ &\leqq \|\exp((\delta + 1/2)M_t^c)\|_1^{4\delta/(1+2\delta)^2}\|\exp(M_t^d)\|_\mu \end{aligned}$$

where $\mu = q\gamma = (1 + 1/(2\delta))^2(1 + \varepsilon)$. Thus the theorem is established.

For example, by setting $\varepsilon = 1$ and $\delta = 1/2$, we get

$$\|Z_t\|_{8/7} \leqq \|\exp(M_t^c)\|_1^{1/2}\|\exp(M_t^d)\|_8 .$$

COROLLARY. *If there exist two numbers $\varepsilon, \delta > 0$ such that the processes $\{\exp((\delta+1/2)M_t^c)\}$ and $\{\exp((1+1/(2\delta))^2(1+\varepsilon)M_t^d)\}$ are submartingales, then Z is a martingale.*

PROOF. By (3) there exists a constant $\gamma > 1$ such that $\sup\{E[Z_s^\gamma]; 0 \leqq s \leqq t\} < \infty$ for each t, and so the family $(Z_s)_{0\leqq s\leqq t}$ is uniformly integrable. This completes the proof.

In particular, if M_t^d is bounded from above, then for every ε and $\delta > 0$ $\exp((1 + 1/(2\delta))^2(1 + \varepsilon)M_t^d)$ is a submartingale. So we get:

THEOREM 2. *Suppose that there exists a positive constant K such that $\sup\{M_s^d; 0 \leqq s \leqq t\} \leqq K$.*

If $\exp(M_t^c/2)$ is a submartingale, then Z is a martingale. Here the constant K may depend on t.

PROOF. As $\Delta M_t \geqq -1$ for every t, Z is a non-negative supermartingale, so that $E[Z_t] \leqq 1$ for every t. Therefore, it is a martingale if and only if $E[Z_t] = 1$ for every t. By the definition of a local martingale there exists a non-decreasing sequence (T_n) of F_t-stopping times with $\lim_n T_n = \infty$ such that for every n the process $(Z_{t\wedge T_n}, F_t)$ is a uniformly integrable martingale. Namely, for each n, $E[Z_{T_n}] = 1$. As Z is non-negative, we have

$$1 = E[Z_{t\wedge T_n}] \leqq E[Z_t] + E[Z_{t\wedge T_n}; t > T_n] .$$

Therefore, to prove $E[Z_t] = 1$, it suffices to show that the second term on the right hand side converges to 0 as $n \to \infty$. From the assumption it follows that $\sup\{W_s; 0 \leqq s \leqq t\}$ is dominated by some constant C which may depend on t. On the other hand, it is proved in [5] that the process Y is a martingale if $\exp(M_t^c/2)$ is a submartingale. That is, $E[Y_t|F_{t\wedge T_n}] = Y_{t\wedge T_n}$ for every n. As $\{t > T_n\}$ belongs to $F_{t\wedge T_n}$, we get

$$E[Z_{t\wedge T_n}; t > T_n] \leqq CE[Y_t; t > T_n] .$$

and the right hand side converges to 0 as $n \to \infty$. This completes the proof.

Let now M be a locally square integrable martingale and $\langle M \rangle$ be the predictable increasing process such that $M^2 - \langle M \rangle$ is a local martingale. It should be noted that if $\exp(\langle M \rangle_t/2) \in L^1$, then $\exp(M_t^c/2) \in L^1$. Indeed, as $\langle M^c \rangle \leqq \langle M \rangle$, the Schwarz inequality implies that

$$\begin{aligned} E[\exp(M_t^c/2)] &\leqq E[\exp(M_t^c/2 - \langle M^c \rangle_t/4) \exp(\langle M \rangle_t/4)] \\ &\leqq \{E[Y_t]\}^{1/2}\{E[\exp(\langle M \rangle_t/2)]\}^{1/2} \\ &\leqq \{E[\exp(\langle M \rangle_t/2)]\}^{1/2} . \end{aligned}$$

However, the converse is not true. For such an example, see [5].

3. Application. In this section, for simplicity, we deal only with continuous local martingales. The extension to the general case is not difficult. Let M be a continuous local martingale with $M_0 = 0$, and assume that the process Z defined as before is a uniformly integrable martingale. Then we can consider a change of the underlying probability measure dP by the formula $d\hat{P} = Z_\infty dP$. As is proved in [6], for any P-continuous local martingale X, $\hat{X} = X - \langle X, M \rangle$ is a $\hat{P}$-continuous local martingale such that $\langle \hat{X} \rangle = \langle X \rangle$ under either probability measure. Here $\langle X, M \rangle = (\langle X + M \rangle - \langle X \rangle - \langle M \rangle)/2$. We now apply Theorem 1 to give a sufficient condition for the process $\hat{X}$ to be a $\hat{P}$-martingale.

THEOREM 3. *Let M be a continuous local martingale, and assume that the exponential local martingale Z is uniformly integrable. Let δ be a number > 0. Then the inequality*

$$(4) \qquad \hat{E}[\hat{X}_t^*] \leqq C_\delta \|\exp((\delta + 1/2)M_t)\|_1^{4\delta/(1+2\delta)^2} \|X_t\|_{(1+1/(2\delta))^2} , \quad 0 \leqq t < \infty$$

is valid for every continuous local martingale X. Here $\hat{X}_t^* = \sup\{|\hat{X}_s|; 0 \leqq s \leqq t\}$ and C_δ is a positive constant depending only on δ.

PROOF. By the Davis theorem (see [3]) we have

$$\hat{E}[\hat{X}_t^*] \leqq 4\sqrt{2}\,\hat{E}[\langle \hat{X} \rangle_t^{1/2}] .$$

From the definition of $d\hat{P}$ it follows that the expectation on the right hand side is $E[Z_t\langle X\rangle_t^{1/2}]$. Set now $p=(1+2\delta)^2/(1+4\delta)$. Then the exponent conjugate is $q=(1+1/(2\delta))^2$. We apply the Hölder inequality with the exponents p and q to this term:

$$\hat{E}[\langle\hat{X}\rangle_t^{1/2}]\leqq\|Z_t\|_p\|\langle X\rangle_t^{1/2}\|_q . \tag{5}$$

According to Theorem 1, the first term on the right hand side of (5) is smaller than $\|\exp((\delta+1/2)M_t)\|_1^{4\delta/(1+2\delta)^2}$. Furthermore, by a result of D. L. Burkholder and R. F. Gundy (see [3]), the second term is also smaller than $C_q\|X_t\|_q$, where C_q is a positive constant depending only on q. Thus the theorem is proved.

Consequently, if for some $\delta>0$ the process $\exp((\delta+1/2)M_t)$ is a submartingale, then for every $L^{(1+1/(2\delta))^2}$-integrable continuous martingale X relative to dP, $\hat{X}$ is a martingale relative to $d\hat{P}$.

More generally, we can show that the inequality

$$\hat{E}[(\hat{X}_t^*)^p]\leqq C_{p,\delta}\|\exp((\delta+1/2)M_t)\|_1^{4\delta/(1+2\delta)^2}\|X_t^*\|_{(1+1/(2\delta))^2p}^p , \quad 0<p<\infty$$

is valid for every P-continuous local martingale X.

References

[1] C. Doléans-Dade, Quelques applications de la formule de changement de variables pour les semi-martingales, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 16 (1970), 181–194.

[2] C. Doléans-Dade and P. A. Meyer, Intégrales stochastiques par rapport aux martingales locales, Séminaire de Probabilités IV, Université de Strasbourg, Lecture Notes in Math. 124, Springer-Verlag, (1970), 77-107.

[3] R. K. Getoor and M. J. Sharpe, Conformal martingales, Inventiones Math. 16 (1972), 271–308.

[4] I. V. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Teor. Veroyatnost i. Primen 5 (1960), 314–330.

[5] N. Kazamaki, On a problem of Girsanov, Tôhoku Math. J. 29 (1977), 597–600.

[6] J. H. van Schuppen and E. Wong, Transformation of local martingales under a change of law, Ann. Prob. 2 (1974), 879–888.

Department of Mathematics
Toyama University
Toyama, 930 Japan