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SOME PROPERTIES OF MAR AND MANR

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Abstract. In this paper, we extend the concepts of movability, strong movability, AWR and AWNR for arbitrary metrizable spaces and we show that MAR and AWR are the same concept and that MANR, strong movability and movable AWNR are the same concept. And we prove that the projection of the product $X \times Y$ of a locally compact metric space X and an MAR Y onto X induces the shape equivalence and that the inclusion of a metric space X into a union $X \cup Y$ of X and an MAR Y induces a shape equivalence if X and Y are closed in $X \cup Y$ and if $X \cap Y$ is an MAR.

0. Introduction. The notion of shape for compacta introduced by K. Borsuk [5] was generalized to arbitrary metrizable spaces by K. Borsuk [8] and by R. H. Fox [12]. The concepts of FAR and FANR were induced by K. Borsuk [6] and these are important and interesting in his shape theory as well as AR and ANR in homotopy theory (refer to the book of K. Borsuk [4]). Generalizing these concepts for compacta, S. Godlewski [14] introduced the concepts of MAR and MANR in Fox's shape theory. Some basic properties of MAR and MANR were investigated in [14], [15] and [16].

In Sect. 2 of this paper, we shall give a characterization of MAR which is similar to one of FAR. Movability and strong movability introduced by K. Borsuk [7], [9] can be defined for metrizable spaces. In Sect. 3, we shall show that strong movability coincides with the concept of MANR, as in the case of FANR. And the concepts of AWR and AWNR introduced by S. A. Bogatyi [2] can be also defined for metrizable spaces. In Sect. 4, we shall extend Bogatyi's results in [2], that is, MAR and AWR are equivalent concepts and movable AWNR's are MANR's.

Y. Kodama [18] recently showed that the product of an FAR (resp. a pointed FANR) and an MAR (resp. an MANR) is also an MAR (resp. an MANR) and that the projection of the product $X \times Y$ of a metrizable space X and an FAR Y onto X induces a shape equivalence. In Sect. 5, we shall prove that the projection of the product $X \times Y$ of a locally compact metrizable space X and an MAR Y onto X induces the shape equivalence.

In Sect. 6, we shall show that the inclusion map of a metrizable space X into its union $X \cup Y$ with an MAR Y induces a shape equivalence if X and Y are closed in $X \cup Y$ and if $X \cap Y$ is an MAR. As a corollary, a sum theorem for MANR's or movable metrizable spaces or AWNR's is obtained.

In this paper, AR and ANR mean those for metrizable spaces. As concerns AR and ANR, refer to the books of K. Borsuk [3] or S-T. Hu [17].

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1. The shape in the sence of Fox. Now, we extend the basic notions introduced by R. H. Fox in [12]. Let A be a subset of B. We denote the inclusion map of A into B by $i_{B,A}$. Let X be a subspace of a space X'. The family $\mathrm{Nbd}(X, X')$ of all open neighbourhoods of X in X' is called the *complete neighbourhood system* of X in X'. Consider two arbitrary complete neighbourhood systems $\mathrm{Nbd}(X, X')$ and $\mathrm{Nbd}(Y, Y')$. A *mutation* F: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ from $\mathrm{Nbd}(X, X')$ to $\mathrm{Nbd}(Y, Y')$ is defined as a collection of continuous maps $f: U \to V$, where $U \in \mathrm{Nbd}(X, X')$, $V \in \mathrm{Nbd}(Y, Y')$ such that

(M-1) For each $f: U \to V$ in F and for each $U' \in \mathrm{Nbd}(X, X')$ and $V' \in \mathrm{Nbd}(Y, Y')$ such that $U' \subset U$ and $V \subset V'$, $i_{V',V} f i_{U,U'} \in F$.

(M-2) For each $V \in \mathrm{Nbd}(Y, Y')$, there exists $f: U \to V$ in F.

(M-3) For each $f, f': U \to V$ in F, there exists $U' \in \mathrm{Nbd}(X, X')$ such that $U' \subset U$ and $f i_{U,U'} \sim f' i_{U,U'}$.

The mutation $I_{(X,X')}$: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(X, X')$ is the collection of all inclusions $i_{U,U'}$ where $U', U \in \mathrm{Nbd}(X, X')$, $U' \subset U$. Consider two mutations F: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ and G: $\mathrm{Nbd}(Y, Y') \to \mathrm{Nbd}(Z, Z')$. The *composition* GF: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Z, Z')$ of the mutations F and G is the mutation being the collection of all compositions gf such that $f \in F$, $g \in G$ and gf can be defined. Then $FI_{(X,X')} = I_{(Y,Y')}F = F$.

Two mutations F, G: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ are *homotopic* (or F *is homotopic to* G, notation: $F \sim G$) if

(HM) For each $f \in F$ and $g \in G$ such that $f, g: U \to V$, there exists $U' \in \mathrm{Nbd}(X, U)$ such that $f i_{U,U'} \sim g i_{U,U'}$.

Any metric space X can be embedded in an ANR as a closed subset by the well-known Kuratowski-Wojdysławski theorem. We say that X and Y are *shape-equivalent* (or *have the same shape type*, notation: $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$) if there are ANR's P and Q which contain X and Y as closed subsets, respectively, and mutations F: $\mathrm{Nbd}(X, P) \to \mathrm{Nbd}(Y, Q)$ and G: $\mathrm{Nbd}(Y, Q) \to \mathrm{Nbd}(X, P)$ such that $GF \sim I_{(X,P)}$ and $FG \sim I_{(Y,Q)}$.

If the mutations F and G satisfy only the condition $FG \sim I_{(Y,Q)}$, then we say that X *shape-dominates* Y (notation: $\mathrm{Sh}\, X \geqq \mathrm{Sh}\, Y$). By Theorem 3.2 in [12], the choice of ANR's P and Q and the manner of embeddings of X and Y in P and Q, respectively, as a closed subsets, are immaterial.

Let $X \subset X' \subset X''$ and $Y \subset Y' \subset Y''$. A mutation F': $\mathrm{Nbd}(X, X'') \to \mathrm{Nbd}(Y, Y'')$ is an *extension* of a mutation F: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ if for each $V' \in \mathrm{Nbd}(Y, Y'')$ and each $f: U \to V' \cap Y'$ in F, there is an $f': U' \to V'$ in F' such that $f' i_{U',X} = i_{V',V'\cap Y'} f i_{U,X}$.

LEMMA 1-1. *Let X and Y be closed subsets of X' and Y' respectively, X'' a metric space containing X' as a closed set and Q an* ANR *containing Y' as a closed set.* (i) *Each mutation F*: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ *has an extension $\bar{F}$*: $\mathrm{Nbd}(X, X'') \to \mathrm{Nbd}(Y, Q)$. (ii) *If F is homotopic to a mutation G*: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$, *then $\bar{F}$ is homotopic to each extension $\bar{G}$*: $\mathrm{Nbd}(X, X'') \to \mathrm{Nbd}(Y, Q)$ *of G.*

PROOF. (i) Consider the collection of continuous maps

$$\bar{F} = \left\{ \bar{f}: U \to V \,\middle|\, \begin{array}{l} U \in \mathrm{Nbd}(X, X''),\ V \in \mathrm{Nbd}(Y, Q), \\ \exists f: U \cap X' \to V \cap Y' \text{ in } F \text{ s.t. } \bar{f} i_{U,U\cap X'} = i_{V,V\cap Y'} f \end{array} \right\}.$$

Then we will show that $\bar{F}$: $\mathrm{Nbd}(X, X'') \to \mathrm{Nbd}(Y, Q)$ is a mutation. (M-1): trivial. (M-2): For each $V \in \mathrm{Nbd}(Y, Q)$, $V \cap Y' \in \mathrm{Nbd}(Y, Y')$, so there is an $f: U_0 \to V \cap Y'$ in F. There is a $U' \in \mathrm{Nbd}(X, X'')$ such that $U' \cap X' = U_0$. Since V is an ANR and U_0 is closed in U', there is an extension $\bar{f}: U \to V$ of $i_{V,V\cap Y'} f'$ to some $U \in \mathrm{Nbd}(U_0, U') \subset \mathrm{Nbd}(X, X'')$. Obviously $U \cap X = U_0$. (M-3): Let $\bar{f}, \bar{f}': U \to V$ be in $\bar{F}$. There are $f, f': U \cap X' \to V \cap Y'$ in F such that $\bar{f} i_{U,U\cap X'} = i_{V,V\cap Y'} f$ and $\bar{f}' i_{U,U\cap X'} = i_{V,V\cap Y'} f'$. There exists a $U'' \in \mathrm{Nbd}(X, U)$ such that $f i_{U\cap X', U''\cap X'} \sim f' i_{U\cap X', U''\cap X'}$, that is, $\bar{f} i_{U,U''\cap X'} \sim \bar{f}' i_{U,U''\cap X'}$. Since V is an ANR and since $U'' \cap X'$ is closed in U'', there is some $U' \in \mathrm{Nbd}(U'' \cap X', U'') \subset \mathrm{Nbd}(X, U)$ such that $\bar{f} i_{U,U'} \sim \bar{f} i_{U,U'}$. Thus $\bar{F}$ is a mutation. Obviously, $\bar{F}$ is an extension of F.

(ii) Let $\bar{f}, \bar{g}: U \to V$ be in F and in G, respectively. Since $V \cap Y' \in \mathrm{Nbd}(Y, Y')$, there are $f: U_1 \to V \cap Y'$ in F and $g: U_2 \to V \cap Y'$ in G. Obviously, $F \sim G$ implies $f i_{U_1,X} \sim g i_{U_2,X}$. Since $\bar{F}$ and $\bar{G}$ are extensions of F and G, respectively, there are $f': U_1' \to V$ in $\bar{F}$ and $g': U_2' \to V$ in $\bar{G}$ such that $f' i_{U_1',X} = i_{V,V\cap Y'} f i_{U_1,X}$ and $g' i_{U_2',X} = i_{V,V\cap Y'} g i_{U_2,X}$. Note that $f' i_{U_1',X} \sim \bar{f} i_{U,X}$ and $g' i_{U_2',X} \sim \bar{g} i_{U,X}$, so $\bar{f} i_{U,X} \sim \bar{g} i_{U,X}$. Since V is an ANR, there is some $U' \in \mathrm{Nbd}(X, U)$ such that $\bar{f} i_{U,U'} \sim \bar{g} i_{U,U'}$. □

Let $\bar{F}$: $\mathrm{Nbd}(X, X'') \to \mathrm{Nbd}(Y, Y'')$ and $\bar{G}$: $\mathrm{Nbd}(Y, Y'') \to \mathrm{Nbd}(Z, Z'')$ be extensions of F: $\mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ and G: $\mathrm{Nbd}(Y, Y') \to \mathrm{Nbd}(Z, Z')$, respectively, where $X \subset X' \subset X''$, $Y \subset Y' \subset Y''$ and $Z \subset Z' \subset$

Z''. Then $\bar{G}\bar{F}$ is obviously an extension of GF. The following corollary is obvious.

COROLLARY 1-2. *Let X and Y be closed subsets of X' and Y', respectively, and let* $F\colon \mathrm{Nbd}(X, X') \to \mathrm{Nbd}(Y, Y')$ *and* $G\colon \mathrm{Nbd}(Y, Y') \to \mathrm{Nbd}(X, X')$ *be mutations. If* $FG \sim I_{(Y,Y')}$ *then* $\mathrm{Sh}\, X \geqq \mathrm{Sh}\, Y$ *and if, in addition,* $GF \sim I_{(X,X')}$ *then* $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$.

2. **MAR and MANR.** Let X be a closed subset of Y. A mutation $R\colon \mathrm{Nbd}(Y, Y') \to \mathrm{Nbd}(X, X')$ is called a *mutational retraction* if $r|X = \mathrm{id}$ for each $r \in R$, where X' and Y' are metric spaces containing X and Y, respectively, as closed subsets. We say that X is a *mutational retract* of Y ([14]) if there exists a mutational retraction $R\colon \mathrm{Nbd}(Y, P) \to \mathrm{Nbd}(X, P)$ for some ANR P containing Y as a closed subset. By Theorem 3.1 in [14], the choice of an ANR P and the manner of an embedding of Y in P as a closed subset are immaterial, and moreover the notion of mutational retract is topological invariant. We say that X is a *mutational neighbourhood retract of* Y ([14]) if X is mutational retract of some closed neighbourhood of X in Y. Obviously, this notion is also topological invariant.

PROPOSITION 2-1. *Let X be a closed subset of Y and let X and Y be closed subsets of X' and Y', respectively. If there is a mutational retraction* $R\colon \mathrm{Nbd}(Y, Y') \to \mathrm{Nbd}(X, X')$, *then X is a mutational retract of Y. If there are a closed neighbourhood W of X in Y and a mutational retraction* $R'\colon \mathrm{Nbd}(W, Y') \to \mathrm{Nbd}(X, X')$, *then X is a mutational neighbourhood retract of Y.*

PROOF. We may assume that $X' \cap Y' = X$. Then $X' \cup Y'$ is a metric space in which X' and Y' are closed (Lemma 4.7 in [14]). Embed $X' \cup Y'$ in some ANR P as a closed subset. By 1-1, there are extensions $\bar{R}\colon \mathrm{Nbd}(Y, P) \to \mathrm{Nbd}(X, P)$ and $\bar{R}'\colon \mathrm{Nbd}(W, P) \to \mathrm{Nbd}(X, P)$ of R and R', respectively. Consider the subcollections of $\bar{R}$ and $\bar{R}'$ consisting of all member r such that $r|X = \mathrm{id}$, these are mutational retractions. □.

We say that X is a *mutational absolute retract* (shortly: MAR) (resp. a *mutational absolute neighbourhood retract* (shortly: MANR)) if X is a mutational retract (resp. a mutational neighbourhood retract) of each metric space Y containing X as a closed subset ([14]). These concepts of MAR and MANR are not only topological invariant but shape invariant in both senses of Fox and of Borsuk ([15] 3.8, 3.11, 4.11 and 4.13) and these are the extensions of the concepts of AR (FAR) and ANR (FANR) respectively ([14] 4.2-5).

The following characterizations are established by S. Godlewski [14], [15].

THEOREM 2-2. *MAR's (resp. MANR's) are the same as mutational retracts of* AR's *(resp.* ANR's*) (Theorem* 4.9 *and* 4.11 *in* [14]).

THEOREM 2-3. *A metric space X is an* MAR *if and only if* $\mathrm{Sh}\,X$ *is trivial, that is, X has the same shape type as a one-point space (Theorem* 3.5 *in* [15]).

As an extension of Theorem 6.1 and Corollary 6.3 in Ch. VIII of [4], the following characterization of MAR holds.

THEOREM 2-4. *Let X be a closed subset of an* AR *P. Then the following conditions are equivalent.*

(i) *X is an* MAR.

(ii) *Every neighbourhood U of X contains a neighbourhood U_0 of X which is contractible in U.*

(iii) *X is contractible in each neighbourhood.*

(iv) *For each neighbourhood U of X, there is a continuous map $r: P \to U$ such that* $r|X = \mathrm{id}$.

PROOF. (i) $\Rightarrow$ (ii): There is a mutational retraction $R: \mathrm{Nbd}(P, P) \to \mathrm{Nbd}(X, P)$. Then for each $U \in \mathrm{Nbd}(X, P)$, there is an $r: P \to U$ in R. Since $r(X) = X$ is contractible in $r(P)$ and since U is an ANR, it is easy to see that there is some $U_0 \in \mathrm{Nbd}(X, P)$ which is contractible in U.

(ii) $\Rightarrow$ (iii): trivial.

(iii) $\Rightarrow$ (iv): For each $U \in \mathrm{Nbd}(X, P)$, the inclusion $i_{U,X}$ is homotopic to a constant map. Since U is an ANR, there is an extension $r: P \to U$ of $i_{U,X}$ by the Homotopy Extension Theorem.

(iv) $\Rightarrow$ (i): This is a direct consequence of the following lemma:

LEMMA 2-5. *Let X be a closed subset of Y and suppose that Y is deformable into X. Then X is a mutational retract of Y if for each neighbourhood U of X in Y there is a continuous map $r: Y \to U$ such that* $r|X = \mathrm{id}$.

PROOF. Consider the collection of continuous maps

$$R = \{r: Y \to U \,|\, U \in \mathrm{Nbd}(X, Y),\ r|X = \mathrm{id}\}\,.$$

Let $d: Y \times I \to Y$ be a deformation of Y into X. Since each $r \in R$ is homotopic to $rd_1 = d_1$, it is easy to see that $R: \mathrm{Nbd}(Y, Y) \to \mathrm{Nbd}(X, Y)$ is a mutational retraction. By 2-1, X is a mutational retract of Y. □

3. Movable and strongly movable metric spaces. Let X be a closed subset of Y. We say that X is *movable in* Y if for each neighbourhood U of X in Y, there is a neighbourhood U_0 of X in Y which satisfies the following condition:

m(X, U): For each neighbourhood V of X in Y, there is a continuous map $f: U_0 \to V \cap U$ such that $i_{U,V\cap U}f \sim i_{U,U_0}$,

and we say that X is *strongly movable in* Y if above U_0 satisfies the following condition:

sm (X, U): For each neighbourhood V of X in Y, there is a continuous map $f: U_0 \to V \cap U$ such that $f|X = \mathrm{id}$ and $i_{U,V\cap U}f \sim i_{U,U_0}$.

A metric space X is said to be *movable* (resp. *strongly movable*) if X is movable (resp. strongly movable) in an ANR containing X as a closed subset. (Compare with the definition in Ch. V of [4].) A movable metric space defined above is movable in the sense of Kozlowski-Segal [19], for if X is movable in an ANR P, then $\mathrm{Nbd}(X, P)$ is a movable inverse system (cf. [13]). In the above definitions of movability and strong movability, the choice of ANR P and the manner of an embedding of X in P as a closed subset are immaterial, and moreover these concepts are hereditary shape invariant, that is, the following theorem holds.

THEOREM 3-1. *Let X and Y be closed subsets of* ANR*'s P and Q respectively, and* $\mathrm{Sh}\, X \geqq \mathrm{Sh}\, Y$ (i.e., *X shape-dominates Y*). *If X is movable* (*resp. strongly movable*) *in P, then Y is also movable* (*resp. strongly movable*) *in Q.*

PROOF. There are mutations $F: \mathrm{Nbd}(X, P) \to \mathrm{Nbd}(Y, Q)$ and G: $\mathrm{Nbd}(Y, Q) \to \mathrm{Nbd}(X, P)$ such that $FG \sim I_{(Y,Q)}$. For each $V \in \mathrm{Nbd}(Y, Q)$, there is an $f: U \to V$ in F. Since X is (strongly) movable in P, there is a $U_0 \in \mathrm{Nbd}(X, U)$ satisfying the condition m (X, U) (the condition sm (X, U)). Then there is a $g: V_0' \to U_0$ in G and since $FG \sim I_{(Y,Q)}$, there is a $V_0 \in \mathrm{Nbd}(Y, Q)$ such that $fi_{U,U_0}gi_{V_0',V_0} \sim i_{V,V_0}$. We will show that V_0 satisfies the condition m (Y, V) (the condition sm (Y, V)).

Let $V' \in \mathrm{Nbd}(Y, Q)$. There is an $f': U' \to V' \cap V$ in F, and then there is a $U'' \in \mathrm{Nbd}(Y, U' \cap U)$ such that $i_{V,V'\cap V}f'i_{U',U''} \sim fi_{U,U''}$. By the condition m ($X$, U) (the condition sm (X, U)), there is a continuous map $h: U_0 \to U''$ such that $i_{U,U''}h \sim i_{U,U_0}$ (and $hi_{U_0,X} = i_{U'',X}$). Put $h' = f'i_{U',U''}hgi_{V_0',V_0}: V_0 \to V' \cap V$. Then

$$\begin{aligned} i_{V,V'\cap V}h' &= i_{V,V'\cap V}f'i_{U',U''}hgi_{V_0',V_0} \\ &\sim fi_{U,U''}hgi_{V_0',V_0} \\ &\sim fi_{U,U_0}gi_{V_0',V_0} \\ &\sim i_{V,V_0} . \end{aligned}$$

Therefore V_0 satisfies the condition $\mathrm{m}(Y, V)$. (Since $h|X$ and U'' is an ANR, there is a $U_1 \in \mathrm{Nbd}(X, U_0 \cap U'')$ such that $hi_{U_0,U_1} \sim i_{U'',U_1}$. There is a $g'\colon V_1 \to U_1$ in G and then $i_{U_0,U_1} g' i_{V_1,Y} \sim g i_{V_0',Y}$, so

$$\begin{aligned} h' i_{V_0,Y} &= f' i_{U',U''} h g i_{V_0',Y} \\ &\sim f' i_{U',U''} h i_{U_0,U_1} g' i_{V_1,Y} \\ &\sim f' i_{U',U_1} g' i_{V_1,Y} . \end{aligned}$$

Since $FG \sim I_{(Y,Q)}$, $f' i_{U',U_1} g' i_{V_1,Y} \sim i_{V' \cap V,Y}$, then $h' i_{V_0,Y} \sim i_{V' \cap V,Y}$. By the Homotopy Extension Theorem, there exists $h''\colon V_0 \to V' \cap V$ such that $h'' i_{V_0,Y} = i_{V' \cap V,Y}$ and $h'' \sim h'$. Then $i_{V,V' \cap V} h'' \sim i_{V,V' \cap V} h' \sim i_{V,V_0}$. Therefore V_0 satisfies the condition $\mathrm{sm}(Y, V)$.) □

By the following theorem, MANR is characterized by strong movability. This is a generalization of Theorem 7.6 in Ch. VIII of [4].

THEOREM 3-2. MANR'*s are the same as strongly movable metric spaces.*

PROOF. Let X be an MANR. By 2-2, there are an ANR P containing X as a closed subset and a mutational retraction $R\colon \mathrm{Nbd}(P, P) \to \mathrm{Nbd}(X, P)$. For each $U \in \mathrm{Nbd}(X, P)$, there is an $r_0\colon P \to U$ in R. Since U is an ANR and $r_0 | X = \mathrm{id}$, there is a $U_0 \in \mathrm{Nbd}(X, U)$ such that $r_0 i_{P,U_0} \sim i_{U,U_0}$. For each $V \in \mathrm{Nbd}(X, P)$, there is an $r\colon P \to V \cap U$ in R. Since R is a mutation, $i_{U,V \cap U} r \sim r_0$. Then $i_{U,V \cap U} r i_{P,U_0} \sim r_0 i_{P,U_0} \sim i_{U,U_0}$. Note that $i_{U,V \cap U} r i_{P,U_0}(x) = r(x) = x$ for each $x \in X$. Therefore X is strongly movable in P.

Conversely, let X be strongly movable. Then there is an ANR P in which X is strongly movable. There are W', $W \in \mathrm{Nbd}(X, P)$ which satisfy the conditions $\mathrm{sm}(X, P)$ and $\mathrm{sm}(X, W')$, respectively. Consider the collection of continuous maps

$$R = \left\{ r\colon W \to U \;\middle|\; \begin{array}{c} U \in \mathrm{Nbd}(X, W),\ r|X = \mathrm{id},\ i_{W',U} r \sim i_{W',W} \\ r(W) \text{ is contained in some } U_0 \in \mathrm{Nbd}(X, W) \\ \text{satisfying } \mathrm{sm}(X, U) \end{array} \right\}.$$

Then we will show that $R\colon \mathrm{Nbd}(W, W) \to \mathrm{Nbd}(X, W)$ is a mutational retraction. (M-1): trivial. (M-2): This is easily seen from the strongly movability of X and the condition $\mathrm{sm}(X, W')$ of W. (M-3): Let r, $r'\colon W \to U$ be in R. Since there is a continuous map $k\colon W' \to U$ such that $k|X = \mathrm{id}$ by the condition $\mathrm{sm}(X, P)$ and since U is an ANR, there are a $V \in \mathrm{Nbd}(X, W')$ and a continuous map $h\colon W' \to U$ such that $h|V = \mathrm{id}$. Since $r \in R$, there are a $U_0 \in \mathrm{Nbd}(X, W)$ satisfying $\mathrm{sm}(X, U)$ such that $r(W) \subset U_0$. From the condition $\mathrm{sm}(X, U)$ of U_0, there is a continuous map $f: U_0 \to V$ such that $i_{U,V}f \sim i_{U,U_0}$. Define a continuous map $\bar{r}: W \to V$ by $\bar{r}(x) = f(r(x))$ for each $x \in W$. Then $r \sim i_{U,V}\bar{r}$. Since $i_{W',U}r \sim i_{W',W}$, then $r \sim i_{U,V}\bar{r} = hi_{W',U}i_{U,V}\bar{r} \sim hi_{W',U}r \sim hi_{W',W}$. Similarly, $r' \sim hi_{W',W}$. Thus $r \sim r'$. □

4. **AWR and AWNR.** Let X be a closed subset of Y and of X'. We say that X is a *weak retract of* Y *in* X' if for each neighbourhood U of X in X', there is a continuous map $r: Y \to U$ such that $r|X = \mathrm{id}$ and that X is a *weak retract* (resp. a *weak neighbourhood retract*) *of* Y if X is a weak retract of Y (resp. of a closed neighbourhood of X in Y) in an ANR P containig X as a closed subset. (Compare the definition in [2] Sect. 3.) It is easy to see that if X is a weak retract of Y in some metric space X' containing X as a closed subset, then X is a weak retract of Y. Thus the choice of an ANR P and the manner of an embedding of X in P as a closed subset are immaterial in the above definition.

A metric space X is said to be an *absolute weak retract* (shortly: AWR) (resp. an *absolute weak neighbourhood retract* (shortly: AWNR)) if X is a weak retract (resp. a weak neighbourhood retract) of each metric space Y containing X as a closed subset (see [2]). The following characterization of AWR and AWNR is an extension of Theorem 4 in [2] Sect. 3.

THEOREM 4-1. AWR'*s* (*resp.* AWNR'*s*) *are the same as weak retracts of* AR'*s* (*resp.* ANR'*s*).

PROOF. It is obvious that AWR's are weak retracts of AR's containing those as a closed sets. Conversely, let X be a weak retract of an AR P and suppose X is a closed subset of a metric space Y. For each neighbourhood U of X in P, there is a continuous map $r: P \to U$ such that $r|X = \mathrm{id}$. Since P is an AR, there is a continuous map $f: Y \to P$ such that $f|X = \mathrm{id}$. Then $rf: Y \to U$ is a continuous map such that $rf|X = \mathrm{id}$. Thus X is a weak retract of Y, therefore X is an AWR.

Next, let X be an AWNR and let P be an ANR containing X as a closed subset. Then X is a weak retract of some closed neighbourhood W of X in P. Clearly, X is a weak retract of $\mathrm{int}\,W$. Since $\mathrm{int}\,W$ is open in the ANR P, $\mathrm{int}\,W$ is an ANR. Conversely, let X be a weak retract of an ANR P and suppose X is a closed subset of a metric space Y. Then there exist a closed neighbourhood W of X in Y, a continuous map $f: W \to P$ such that $f|X = \mathrm{id}$ and for each neighbourhood U of X in P, there is a continuous map $r: P \to U$ such that $r|X = \mathrm{id}$. This implies that X is a weak neighbourhood retract of Y. Therefore X is an AWNR. □

By 2-4, we have

COROLLARY 4-2. AWR's *are the same as* MAR's.

For AWNR, the following theorem holds. This was established by S. A. Bogatyi [2] in the case of compacta.

THEOREM 4-3. *Movable* AWNR's *are the same as* MANR's.

PROOF. By 3-2, MANR's are the same as strongly movable metric spaces and then clearly these are movable and AWNR's.

Let X be a movable AWNR. By 4-1, X is a weak retract of an ANR P. For each $U \in \mathrm{Nbd}(X, P)$, X is also a weak retract of U. From the movability of X, there is a $U_0 \in \mathrm{Nbd}(X, U)$ satisfying the condition $\mathrm{m}(X, U)$. For each $V \in \mathrm{Nbd}(X, P)$, there is a continuous map $r: U \to V \cap U$ such that $r|X = \mathrm{id}$. Since $V \cap U$ is an ANR, $r|V' \sim i_{V\cap U, V'}$ for some $V' \in \mathrm{Nbd}(X, V \cap U)$. By the Homotopy Extension Theorem, there is a continuous map $r': U \to V \cap U$ such that $r'|V' = \mathrm{id}$. From the condition $\mathrm{m}(X, U)$ of U_0, there is a continuous map $f: U_0 \to V'$ such that $i_{U,V'}f \sim i_{U,U_0}$. Since $i_{V\cap U,V'}f|X = r'i_{U,V'}fi_{U_0,X} \sim r'i_{U,X} = i_{V\cap U,X}$, by the Homotopy Extension Theorem, there is a continuous map $h: U_0 \to V \cap U$ such that $h|X = \mathrm{id}$ and $h \sim i_{V\cap U,V'}f$, so $i_{U,V\cap U}h \sim i_{U,V'}f \sim i_{U,U_0}$. Thus U_0 satisfies the condition $\mathrm{sm}(X, U)$. Therefore X is strongly movable. □

The concept of AWR is shape invariant by 4-2. Here we will prove that the concept of AWNR is hereditary shape invariant in all metric spaces.

THEOREM 4-4. *Let* $\mathrm{Sh}\, X \geqq \mathrm{Sh}\, Y$. *If X is an* AWNR, *then Y is so.*

PROOF. By 4-1, X is a weak retract of some ANR P. Let Q be an ANR containing Y as a closed subset. Since $\mathrm{Sh}\, X \geqq \mathrm{Sh}\, Y$, there are mutations $F: \mathrm{Nbd}(X, P) \to \mathrm{Nbd}(Y, Q)$ and $G: \mathrm{Nbd}(Y, Q) \to \mathrm{Nbd}(X, P)$ such that $FG \sim I_{(Y,Q)}$. There is a $g: W \to P$ in G (from (M-2)). Then it is enough to show that Y is a weak retract of W in Q, because W is an ANR.

Let $V \in \mathrm{Nbd}(Y, Q)$. There are an $f: U \to V$ in F (from (M-2)) and a continuous map $r: P \to U$ such that $r|X = \mathrm{id}$. Since U is an ANR, $r|U' \sim i_{U,U'}$ for some $U' \in \mathrm{Nbd}(X, U)$. By the Homotopy Extension Theorem, there is a continuous map $r': P \to U$ such that $r'|U' = \mathrm{id}$. There is a $g': V' \to U'$ in G (from (M-2)), and then $i_{P,U'}g'i_{V',Y} \sim gi_{W,Y}$ (from (M-3)). Thus $fr'gi_{W,Y} \sim fr'i_{P,U'}g'i_{V',Y} = fi_{U,U'}g'i_{V',Y}$. Since $FG \sim I_{(Y,Q)}$, $fi_{U,U'}g'i_{V',Y} \sim i_{V,Y}$, that is, $fr'g|Y \sim i_{V,Y}$. By the Homotopy Extension Theorem, there is a continuous map $r'': W \to V$ such that $r''|Y = \mathrm{id}$. Therefore Y is a weak retract of W in Q. □

REMARK 4-5. Recently, K. Tsuda [26] extended the notion of AWNR for arbitrary topological spaces which is equivalent to the notion of*M-condition* introduced by T. Watanabe [27] and he proved that AWNR is a hereditary shape invariant for arbitrary topological spaces. And he show that AWNR is a concept different from movability; there exists a non-movable locally compact separable metrizable AWNR.

5. Product of an MAR and a locally compact metric space. Y. Kodama [18] showed that if Y is an FAR (i.e. a compact MAR), then for any metric space X, the projection of the space $X \times Y$ onto X induces a shape equivalence (Corollary 2 in [18]) and that the product of an MAR (resp. an MANR) and an FAR (resp. a pointed FANR) is also an MAR (resp. an MANR) (Theorem 2 in [18]). In this section, we will prove an analogous result: If Y is an MAR and X is a locally compact metric space, then the projection of $X \times Y$ onto X induces a shape equivalence. And as its corollary, we shall see that the product of an MAR and a locally compact MAR (resp. a locally compact MANR) is an MAR (resp. MANR).

The result of Michael's paper [20] is our tool. Let $\mathscr{P}$ be a topological property of spaces. We say that a space X *has property $\mathscr{P}$ locally* if each point of X has a neighborhood in X which has property $\mathscr{P}$. A property $\mathscr{P}$ is *F-hereditary* for X if every subspace X' of X satisfies condition (F-1) below, and if every closed subspace X' of X also satisfies conditions (F-2) and (F-3): (The term "closed" will mean "closed with respect to X'" and similarly for "open" and "interior").

(F-1): If X' has property $\mathscr{P}$, then every closed subset of X' has property $\mathscr{P}$.

(F-2): If X' is the union of two closed sets, both of which have property $\mathscr{P}$, and whose interiors cover X', then X' has property $\mathscr{P}$.

(F-3): If X' is the union of disjoint collection of open subsets all of which have property $\mathscr{P}$, then X' has property $\mathscr{P}$.

THEOREM 5-1 ([20] Theorem 5.5). *Let X be a paracompact Hausdorff space and $\mathscr{P}$ a property which is F-hereditary for X. If X has property $\mathscr{P}$ locally, then X has property $\mathscr{P}$.*

Let Y be an MAR and a closed subset of an AR Q and let $y_0 \in Y$. For any space X, let $i_X: X \to X \times Y$ be the injection defined by $i_X(x) = (x, y_0)$ for each $x \in X$, let $p_X: X \times Q \to X$ be the projection, and let $j_{U,X} =$

$i_{U,X\times Y}i_X$, $q_{X,U} = p_X i_{X\times Q,U}$ for each $U \in \mathrm{Nbd}\,(X \times Y, X \times Q)$. We will define the topological property $p(Y, Q)$ of X as follows: *X has property $p(Y, Q)$ if for each $U \in \mathrm{Nbd}\,(X \times Y, X \times Q)$ there exist $U_0 \in \mathrm{Nbd}\,(X \times Y, U)$ and an X-preserving homotopy $f\colon U_0 \times I \to U$ (i.e., $q_{X,U}f_t = q_{X,U_0}$ for each $t \in I$) such that $f_0 = i_{U,U_0}$ and $f_1 j_{U_0,X} q_{X,U_0} = f_1$.*

PROPOSITION 5-2. *The property $p(Y, Q)$ is F-hereditary for a metric space X.*

PROOF. (F-1): Let X_0 be a subset of X which has property $p(Y, Q)$, and X_1 a closed subset of X_0. For each $U \in \mathrm{Nbd}\,(X_1 \times Y, X_1 \times Q)$, $U' = U \cup (X_0 \backslash X_1) \times Q \in \mathrm{Nbd}\,(X_0 \times Y, X_0 \times Q)$, so there are a $U'_0 \in \mathrm{Nbd}\,(X_0 \times Y, U')$ and an X_0-preserving homotopy $f'\colon U'_0 \times I \to U'$ such that $f'_0 = i_{U',U'_0}$ and $f'_1 j_{U'_0,X_0} q_{X_0,U'_0} = f'_1$. Note that $U_0 \equiv U'_0 \cap X_1 \times Q \in \mathrm{Nbd}\,(X_1 \times Y, X_1 \times Q)$. Restricting f' to $U_0 \times I$, we can obtain an X_1-preserving homotopy $f\colon U_0 \times I \to U$ such that $f_0 = i_{U,U_0}$ and $f_1 j_{U_0,X_1} q_{X_1,U_0} = f_1$. Therefore X_1 has property $p(Y, Q)$.

(F-2): Let X_0 be a closed subset of X, let X_1 and X_2 be closed subsets of X_0 which have property $p(Y, Q)$ and let $X_0 = \mathrm{int}_{X_0} X_1 \cup \mathrm{int}_{X_0} X_2$. For each $U \in \mathrm{Nbd}\,(X_0 \times Y, X_0 \times Q)$, $U' = U \cap X_1 \times Q \in \mathrm{Nbd}\,(X_1 \times Y, X_1 \times Q)$, so there are a $U'_0 \in \mathrm{Nbd}\,(X_1 \times Y, U')$ and an X_1-preserving homotopy $f'\colon U'_0 \times I \to U'$ such that $f'_0 = i_{U',U'_0}$ and $f'_1 j_{U'_0,X_1} q_{X_1,U'_0} = f'_1$. Since $U'' = (U'_0 \cap X_2 \times Q) \cup (U \cap (X_0 \backslash X_1) \times Q) \in \mathrm{Nbd}\,(X_2 \times Y, X_2 \times Q)$, there are $U''_0 \in \mathrm{Nbd}\,(X_2 \times Y, U'')$ and an X_2-preserving homotopy $f''\colon U''_0 \times I \to U''$ such that $f''_0 = i_{U'',U''_0}$ and $f''_1 j_{U''_0,X_2} q_{X_2,U''_0} = f''_0$. Since $X_0 \backslash \mathrm{int}_{X_0} X_2 \subset \mathrm{int}_{X_0} X_1$, there is an open subset W of X_0 such that $X_0 \backslash \mathrm{int}_{X_0} X_2 \subset W \subset \mathrm{cl}_{X_0} W \subset \mathrm{int}_{X_0} X_1$. Then there are continuous maps $k_1, k_2\colon X_0 \to I$ such that $k_1(X_0 \backslash \mathrm{int}_{X_0} X_1) = 0$, $k_1(\mathrm{cl}_{X_0} W) = 1$, $k_2(X_0 \backslash \mathrm{int}_{X_0} X_2) = 0$ and $k_2(X_0 \backslash W) = 1$. Put $U_0 = U''_0 \cup (U'_0 \cap (X_0 \backslash X_2) \times Q) \in \mathrm{Nbd}\,(X_0 \times Y, U)$ and define an X_0 preserving homotopy $f\colon U_0 \times I \to U$ by

$$f(x, y, t) = \begin{cases} f'(x, y, t) & \text{if } x \in X_0 \backslash X_2 \\ f'(f''(x, y, k_2(x)t), k_1(x)t) & \text{if } x \in X_1 \cap X_2 \\ f''(x, y, t) & \text{if } x \in X_2 \backslash X_1 . \end{cases}$$

Then $f_0 = i_{U,U_0}$ and $f_1 j_{U_0,X_0} q_{X_0,U_0} = f_1$. Therefore X_0 has property $p(Y, Q)$.

(F-3): trivial. □

PROPOSITION 5-3. *Locally compact metric spaces have property $p(Y, Q)$.*

PROOF. Let X be a locally compact metric space. By 5-1, it is sufficient to see that X has property $p(Y, Q)$ locally. For each $x \in X$, there is a compact neighbourhood V of x in X. For each $U \in \mathrm{Nbd}(V \times Y, V \times Q)$, there is a $U' \in \mathrm{Nbd}(Y, Q)$ such that $V \times U' \subset U$. By 2-4, there is a $U'_0 \in \mathrm{Nbd}(Y, U')$ which is contractible in U'. Put $U_0 = V \times U'_0$. We can easily construct a desired X-preserving homotopy $f: U_0 \times I \to U$. Then X has property $p(Y, Q)$ locally. □

THEOREM 5-4. *Let Y be an* MAR. *For any locally compact metric space X, the projection $p: X \times Y \to X$ induces a shape equivalence.*

PROOF. Let Y be embedded in some AR Q as a closed subset. Consider the collection of continuous maps

$$F = \left\{ f_1 j_{U_0,X}: X \to U \;\middle|\; \begin{array}{l} U, U_0 \in \mathrm{Nbd}(X \times Y, X \times Q),\ U_0 \subset U, \\ f: U_0 \times I \to U \text{ is an } X\text{-preserving homotopy} \\ \text{such that } f_1 j_{U_0,X} q_{X,U_0} = f_1 \end{array} \right\}.$$

From the property $p(Y, Q)$ of X, it is easy to see that $F: \mathrm{Nbd}(X, X) \to \mathrm{Nbd}(X \times Y, X \times Q)$ is a mutation such that $PF = \{\mathrm{id}_X\}$, $FP \sim I_{(X \times Y, X \times Q)}$, where $P: \mathrm{Nbd}(X \times Y, X \times Q) \to \mathrm{Nbd}(X, X)$ is the mutation consisting of all projections $q_{X,U}: U \to X$, $U \in \mathrm{Nbd}(X \times Y, X \times Q)$. From 1-2, we obtain the theorem. □

COROLLARY 5-5. *Let X be a locally compact metric space and Y an* MAR. *$X \times Y$ is an* MAR, *an* MANR, *movable, or an* AWNR *if and only if X is so.*

In Theorem 5-4, we do not know whether local compactness of X is essential or not:

PROBLEM. Let Y be an MAR. For any metric space X, does the projection $p: X \times Y \to X$ induce a shape equivalence?

6. Union of an MAR and a metric space with an MAR intersection. In this section, using the technique of infinite-dimensional manifolds, we will show that the inclusion of a metric space X into a union $X \cup Y$ of X and an MAR Y induces a shape equivalence if X and Y are closed in $X \cup Y$ and if $X \cap Y$ is an MAR.

Let E be a linear metric space (LMS) which is homeomorphic ($\cong$) to the countable infinite product E^ω of itself or to the subspace $E_f\omega = \{(x_i) \in E^\omega \mid x_i = 0 \text{ for almost all } i \in N\}$ of E^ω. A space X is *E-stable* if $X \times E \cong X$. *Every E-manifold* (i.e., *manifold modelled on E*) *is E-stable* (the Schori's Stability Theorem [23]). A subset K of an E-stable space X is *E-deficient* in X if there is a homeomorphism $h: X \to X \times E$ such that $h(K) \subset X \times \{0\}$. If K is E-deficient in X, then for each open subset U of X, $U \cap K$ is E-deficient in U (e.g., this follows from Lemma 2.1 in [22]). First, we will show the following lemmas.

LEMMA 6-1. *Let $E \cong E^\omega$ or $\cong E_f^\omega$ be an* AR, LMS *and Y a metric space. If $Y \cap E$ is an* MAR *and an E-deficient closed subset of E and if Y and E are closed in $Y \cup E$, then the inclusion $i: Y \to Y \cup E$ induces a shape equivalence.*

PROOF. Let $J: \mathrm{Nbd}(Y, Y \cup E) \to \mathrm{Nbd}(Y \cup E, Y \cup E)$ be the mutation consisting of inclusions. We must show that J is a shape equivalence. Consider the collection of continuous maps

$$G = \left\{ g: Y \cup E \to U \,\middle|\, \begin{array}{l} U \in \mathrm{Nbd}(Y, Y \cup E),\ g|Y = \mathrm{id}, \\ g|E: E \to U \cap E \text{ is an open embedding} \end{array} \right\}.$$

We will show that $G: \mathrm{Nbd}(Y \cup E, Y \cup E) \to \mathrm{Nbd}(Y, Y \cup E)$ is a mutation. (M-1): trivial. (M-2): For each $U \in \mathrm{Nbd}(Y, Y \cup E)$, $U \cap E \in \mathrm{Nbd}(Y \cap E, E)$. Since $Y \cap E$ is an MAR (an AWR), there is a continuous map $r: E \to U \cap E$ such that $r|Y \cap E = \mathrm{id}$. By Corollary 4.3 in [22], there is an open embedding $g': E \to U \cap E$ such that $g'|Y \cap E = r|Y \cap E = \mathrm{id}$. Then we can obtain an embedding $g: Y \cup E \to U$ defined by $g|E = g'$ and $g|Y = \mathrm{id}$. (M-3): Let $g, g': Y \cup E \to U$ be in G. Since $Y \cap E$ is an MAR and since $g(E) \cap g'(E) \in \mathrm{Nbd}(Y \cap E, E)$, there is a continuous map $r: E \to g(E) \cap g'(E)$ such that $r|Y \cap E = \mathrm{id}$. Since $g(E) \cong E$ is an AR, there is a homotopy $f: E \times I \to g(E)$ such that $f_t|Y \cap E = \mathrm{id}$ for each $t \in I$, $f_0 = r$ and $f_1 = g|E$. Then g is homotopic to $\bar{r}: Y \cup E \to U$ defined by $\bar{r}|Y = \mathrm{id}$ and $\bar{r}|E = r$ because the homotopy $\bar{f}: (Y \cup E) \times I \to U$ defined by

$$\bar{f}(x, t) = \begin{cases} x & \text{if } x \in Y \\ f(x, t) & \text{if } x \in E . \end{cases}$$

connects g and $\bar{r}$. Similarly, g' is also homotopic to $\bar{r}$.

Now we must show that $GJ \sim I_{(Y, Y \cup E)}$ and $JG \sim I_{(Y \cup E, Y \cup E)}$. Let $g: Y \cup E \to U$ be in G and $U' \in \mathrm{Nbd}(Y, Y \cup E)$. Because $g(E)$ is an AR, $g i_{Y \cup E, U'} | U \cap U' \cap g(E): U \cap U' \cap g(E) \to g(E)$ is homotopic to the inclusion $i_{g(E), U \cap U' \cap g(E)}$ fixing $Y \cap E$. By the same argument as (M-3) above, $g i_{Y \cup E, U'} i_{U', U \cap U' \cap (g(E) \cup Y)} \sim i_{U, U \cap U' \cap (g(E) \cup Y)}$. Similarly, $i_{Y \cup E, U} g \sim \mathrm{id}$, because E is an AR. Thus $GJ \sim I_{(Y, Y \cup E)}$ and $JG \sim I_{(Y \cup E, Y \cup E)}$. □

LEMMA 6-2. *Let P be an* AR *and X a metric space. If X and P are closed in $X \cup P$ and if $X \cap P$ is an* MAR, *then the inclusion $i: X \to X \cup P$ induces a shape equivalence.*

PROOF. By the Arens-Eelles' theorem ([1], [21] and [24]), there exists a Banach space B which contains an isometric copy P' of P such that P' is linearly independent in B and closed in span P'. Then span P' is an AR, LMS with dens $P \leqq$ dens (span P') and P is an AR admitting a closed embedding into span P'. By the Toruńczyk's result (Theorem 3.1 in [25]), put $E = (\text{span } P')^\omega_f$, then $P \times E \cong E$. Since $X \times \{0\} \cap P \times E = (X \cap P) \times \{0\}$ is an MAR and an E-deficient closed subset of $P \times E$ $(\cong E \cong E^\omega_f)$, the inclusion $i: X \times \{0\} \to X \times \{0\} \cup P \times E$ induces a shape equivalence by 6-1. Since the projection $p: X \times \{0\} \cup P \times E \to X \times \{0\} \cup P \times \{0\} = (X \cup P) \times \{0\}$ is a homotopy equivalence, the inclusion $pi: X \times \{0\} \to (X \cup P) \times \{0\}$ induces a shape equivalence. □

THEOREM 6-3. *Let X be a metric space and Y an* MAR. *If X and Y are closed in $X \cup Y$ and if $X \cap Y$ is an* MAR, *then the inclusion $i: X \to X \cup Y$ induces a shape equivalence.*

PROOF. Let P be an AR such that P contains Y as a closed subset and $X \cap P = X \cap Y$. By 6-2, the inclusions $j: X \to X \cup P$ and $j': X \cup Y \to X \cup P$ induce shape equivalences. Since $j = j'i$, it is easy to see that i induces a shape equivalence. □

Informally, J. Ono has proved that two closed subspaces X_1 and X_2 of an FANR $X = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ an FAR are FANR's. This result is extended as follows:

COROLLARY 6-4. *Let X_1 and X_2 be closed subsets of a metric space $X = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ an* MAR. *If X is an* MANR, *an* AWNR *or movable, then both X_1 and X_2 are so.*

PROOF. Let P be an AR such that $X \cap P = X_0$ is closed in P. It is easy to see that $X_i \cup P$ is a retract of $X \cup P$ for $i = 1, 2$. By 6-2, $\operatorname{Sh} X \cup P = \operatorname{Sh} X$ and $\operatorname{Sh} X_i \cup P = \operatorname{Sh} X_i$ for $i = 1, 2$. Hence if X is an MANR, an AWNR or movable, then $X \cup P$ and then its retracts $X_1 \cup P$ and $X_2 \cup P$ are so, therefore X_1 and X_2 are so. □

Moreover, for AWNR, the following holds:

COROLLARY 6-5. *If X is a union of two closed* AWNR's *X_1 and X_2 with $X_0 = X_1 \cap X_2$ an* AWR (i.e., *an* MAR), *then X is an* AWNR.

PROOF. We can easily find AR's $P_i (i = 0, 1, 2)$ such that X_i is a closed subset of P_i for $i = 0, 1, 2$ and $P_0 = P_1 \cap P_2$. Then $P = P_1 \cup P_2$ is an AR. For $i = 1, 2$, since $\operatorname{Sh} X_i \cup P_0 = \operatorname{Sh} X_i$ by 6-2, $X_i \cup P_0$ is an AWNR. Hence $X_i \cup P_0$ is a weak retract of some $W_i \in \operatorname{Nbd}(X_i \cup P_0, P_i)$. We shall show that $X \cup P_0$ is a weak retract of $W \equiv W_1 \cup W_2 \in \operatorname{Nbd}(X \cup P_0, P)$. For each $U \in \operatorname{Nbd}(X \cup P_0, P)$, there is a continuous map $r_i: W_i \to U$ such that $r_i | X_i \cup P_0 = \text{id}$ $(i = 1, 2)$. Since $W_1 \cap W_2 = P_0$, we can define a continuous map $r: W \to U$ by $r | W_i = r_i$ for $i = 1, 2$. Since $r | X \cup P_0 = \text{id}$, $X \cup P_0$ is a weak retract of an ANR W. By 4-1, $X \cup P_0$ is an AWNR. Since $\text{Sh}\, X \cup P_0 = \text{Sh}\, X$ by 6-2, X is an AWNR. □

REMARK 6-6. The movable version of this corollary is not true. In fact, C. Cox [10] constructed a non-movable compactum which is a one-point union of two movable compacta. The MANR version is not known yet. Recently, K. Tsuda [26] proved that a union of two compact AWNR's X_1 and X_2 with $X_1 \cap X_2$ a pointed FANR is an AWNR. And J. Dydak, S. Nowak and M. Strok [11] proved that a union of two pointed FANR X_1 and X_2 with $X_1 \cap X_2$ a pointed FANR is also a pointed FANR.

For MAR, we give an elementary proof.

THEOREM 6-7. *Let X_1 and X_2 be two closed subspaces of a metric space $X = X_1 \cup X_2$ and let $X_0 = X_1 \cap X_2$.*

(i) *If X_0, X_1 and X_2 are* MAR's, *then X is also an* MAR.

(ii) *If X and X_0 are* MAR'*s then both X_1 and X_2 are* MAR'*s.*

PROOF. Let P_i $(i = 0, 1, 2)$ be AR's such that X_i is a closed subsets of P_i and $P_0 = P_1 \cap P_2$ and let $P = P_1 \cup P_2$. Then P is an AR and X is a closed subset of P. Recall that MAR and AWR are the same concept (4-2).

(i): For each $U \in \text{Nbd}(X, P)$, there are $U_i \in \text{Nbd}(X_i, U \cap P_i)$ and continuous maps $r_i: P_i \to U \cap P_i$ such that $r_i | U_i = \text{id}$ $(i = 1, 2)$. Put $U_0 = U_1 \cap U_2 \in \text{Nbd}(X_0, U \cap P_0)$. There is a continuous map $r_0: P_0 \to U_0$ such that $r_0 | X_0 = \text{id}$. Since P_i are AR's, there are continuous maps $r_i': P_i \to P_i$ such that $r_i' | P_0 = i_{P_i, U_0} r_0$ and $r_i' | X_i = \text{id}$. Put $r_i'' = r_i r_i': P_i \to U \cap P_i$. Then $r_i'' | P_0 = r_i r_i' | P_0 = r_i i_{P_i, U_0} r_0 = i_{P_i, U_0} r_0$ and $r_i'' | X_i = r_i r_i' | X_i = r_i' | X_i = \text{id}$. Define a continuous map $r: P \to U$ by $r | P_i = r_i''$ $(i = 1, 2)$. Then $r | X = \text{id}$. Thus X is a weak retract of an AR P, therefore X is an AWR, i.e., an MAR.

(ii): For each $U_1 \in \text{Nbd}(X_1, P_1)$, there is a continuous map $r_0: P_2 \to U_1$ such that $r_0 | X_0 = \text{id}$. Since U_1 is an ANR, there are a $U \in \text{Nbd}(X \cup P_2, P)$ and a continuous map $r_0': U \to U_1$ such that $r_0' | P_2 = r_0$ and $r_0' | X_1 = \text{id}$. On the other hand, there is a continuous map $r: P \to U$ such that $r | X = \text{id}$. Then $r_0' r: P \to U_1$ is a continuous map such that $r_0' r | X_1 = r_0' | X_1 = \text{id}$. Thus X_1 is a weak retract of an AR P, therefore X_1 is an AWR, i.e., an MAR. Similarly, X_2 is also an MAR. □

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