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AN INVARIANT OF SYSTEMS IN THE ERGODIC THEORY

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1. Introduction. The following theorem is due to V. A. Rohlin and Ja. G. Sinai (cf. [3], [4]).

THEOREM. *Let* $(\Omega, \mathscr{F}, P)$ *be a separable complete probability space and* T *an invertible measure preserving transformation on* Ω. *Then there exists a sub-*σ*-field* $\mathscr{F}_0$ *such that*

$$\mathscr{F}_0 \subset T\mathscr{F}_0 , \quad \bigvee_{i\in Z} T^i\mathscr{F}_0 = \mathscr{F}, \quad \bigcap_{i\in Z} T^i\mathscr{F}_0 = \mathscr{P}(T) ,$$

where $\mathscr{P}(T)$ *denotes Pinsker's field and* $\boldsymbol{Z}$ *the set of all integers.*

Such a sub-σ-field $\mathscr{F}_0$ plays an important role in studies of mixing properties of transformations, but it is not uniquely determined by the transformation. The purpose of this note is to investigate the pair $(T, \mathscr{F}_0)$ which we call a system and to characterize the relation between the transformation and the sub-σ-field. Let $(S, \mathscr{G}_0)$ be another system defined on the same probability space. If there exists an invertible measure preserving transformation $R: \Omega \to \Omega$ such that $RT = SR$ and $R\mathscr{F}_0 = \mathscr{G}_0$, then these systems $(T, \mathscr{F}_0)$ and $(S, \mathscr{G}_0)$ are said to be isomorphic or more precisely, system-isomorphic. Our problem is to find a metric invariant of the system under system-isomorphy. We shall give an invariant employing a result due to J. de Sam Lazaro and P. A. Meyer [5], but this invariant is of "spectral" nature and is not complete with respect to the system-isomorphy, so it remains still a problem to find an invariant of "spatial" character.

In §§2 and 3 we shall construct two representations by applying the method given in [5], and further we shall make the theorem more precise so as to show the uniqueness of the representation. Then, this theorem determines the required invariant which we call the multiplicity of the system.

In §4 we shall show that the multiplicity of a Bernoulli system is equal to the dimension of the space of all squarely integrable functions which have zero expectations and are measurable with respect to the independent generator of this system.

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2. The first representation of systems. In this paragraph we treat the result in [5] restricting to the case of discrete time. The discussions will be considerably simplified.

Let $\mathscr{H}$ be an infinite dimensional real separable Hilbert space, U a unitary operator on $\mathscr{H}$ and $\mathscr{H}_0$ a closed subspace of $\mathscr{H}$. We denote by $\mathscr{H}_i$ the closed subspaces $U^i\mathscr{H}_0$, for any $i \in \boldsymbol{Z}$.

DEFINITION 2.1 (cf. [5]). A pair $(U, \mathscr{H}_0)$ is called a situation if the following conditions are satisfied:

(a) $\mathscr{H}_i \subset \mathscr{H}_{i+1}$ for any $i \in \boldsymbol{Z}$,

(b) $\bigvee_{i\in \boldsymbol{Z}} \mathscr{H}_i = \mathscr{H}$.

In addition, if

(c) $\mathscr{H}_{-\infty} \equiv \bigcap_{i\in \boldsymbol{Z}} \mathscr{H}_i = \{0\}$,

the situation is called purely non-deterministic.

It is clear that a purely non-deterministic situation is not trivial. In fact, if it is trivial $\mathscr{H} = \mathscr{H}_{-\infty}$, then $\mathscr{H} = \{0\}$ which is absurd.

Hereafter we consider a fixed situation $(U, \mathscr{H}_0)$.

DEFINITION 2.2 (cf. [5]). A sequence $X = (x_i)_{i\in \boldsymbol{Z}}$ in $\mathscr{H}$ is called a helix for the situation $(U, \mathscr{H}_0)$, if the following conditions are satisfied:

(a) $x_0 = 0$,

(b) $x_i - x_{i-1} \in \mathscr{H}_i$ for any $i \in \boldsymbol{Z}$,

(c) $x_i - x_{i-1} \in \mathscr{H}_{i-1}^{\perp}$ for any $i \in \boldsymbol{Z}$ where $\perp$ indicates the orthogonal complement in $\mathscr{H}$,

(d) $U(x_i - x_{i-1}) = x_{i+1} - x_i$ for any $i \in \boldsymbol{Z}$.

A helix $X \equiv 0$ is called a trivial helix. For two helices $X = (x_i)_{i\in \boldsymbol{Z}}$ and $X' = (x'_i)_{i\in \boldsymbol{Z}}$ we write $X = X'$ if $x_i = x'_i$ for all $i \in \boldsymbol{Z}$. We denote by $X + X'$, the helix $(x_i + x'_i)_{i\in \boldsymbol{Z}}$ and cX the helix $(cx_i)_{i\in \boldsymbol{Z}}$ where c is a constant. The sequence $d_i = x_i - x_{i-1}$, $i \in \boldsymbol{Z}$ is called the helix-difference of X. The closed linear span of $(d_i)_{i\in \boldsymbol{Z}}$ is denoted by $\mathscr{H}_X$.

DEFINITION 2.3. Two helices X and X' for a situation are said to be mutually orthogonal if $\mathscr{H}_X$ and $\mathscr{H}_{X'}$ are mutually orthogonal in $\mathscr{H}$, that is, if the corresponding helix-differences at time 1, $d_1 = x_1$ and $d'_1 = x'_1$ are orthogonal in $\mathscr{H}$.

Clearly $\mathscr{H}_X$ is invariant under U and orthogonal to $\mathscr{H}_{-\infty}$.

If X is a non-trivial helix, then as is easily seen from (d),

$$\|x_{i+1} - x_i\| = (\text{constant}) \text{ for any } i \in \boldsymbol{Z},$$

which is denoted by $\|X\|$. If $\|X\| = 1$, X is called a normalized helix.

In order to establish a representation theorem of situations, we begin with the following lemma.

LEMMA 2.1. *If there is no non-trivial helix for a situation, then the situation is trivial.*

PROOF. Suppose that the situation is non-trivial, that is, $\mathscr{H}_{-\infty} \neq \mathscr{H}$. Then $\mathscr{H}_i \neq \mathscr{H}_{i+1}$ for all $i \in \boldsymbol{Z}$. Therefore there exists a non-zero element $x \in \mathscr{H}_1 \cap \mathscr{H}_0^{\perp}$. We put $x_0 = 0$, $x_i = \sum_{k=1}^{i} U^{k-1}x$ $(i > 0)$, $x_i = -U^i x_{-i}$ $(i < 0)$. Then $X = (x_i)_{i \in Z}$ is a non-trivial helix, which contradicts our assumption.

THEOREM 2.1. *Let $(U, \mathscr{H}_0)$ be any situation. Then there exists a family of at most countable mutually orthogonal normalized helices $\mathscr{X} = (X^{(n)})$ such that*

$$\mathscr{H} = \mathscr{H}_{-\infty} \oplus \sum_n \oplus \mathscr{H}_{X^{(n)}} \tag{1}$$

and hence for any $x \in \mathscr{H}$

$$x = x_{-\infty} + \sum_n \sum_{i \in Z} a_i^{(n)} d_i^{(n)} \tag{2}$$

where $x_{-\infty} \in \mathscr{H}_{-\infty}$, $(d_i^{(n)})_{i \in Z}$ is the helix-difference of $X^{(n)}$, $(a_i^{(n)})_{i \in Z} \in l^2(\boldsymbol{Z})$ and

$$\|x\|^2 = \|x_{-\infty}\|^2 + \sum_n \sum_{i \in Z} a_i^{(n)2} .$$

PROOF. Let $\mathscr{X} = (X^{(n)})$ be a maximal family of mutually orthogonal normalized helices for the situation $(U, \mathscr{H}_0)$. It is at most countable since $\mathscr{H}$ is separable.

Define

$$\mathscr{K} = (\mathscr{H}_{-\infty} \oplus \sum_n \oplus \mathscr{H}_{X^{(n)}})^{\perp} ,$$

which is a subspace of $\mathscr{H}$ and invariant under U and orthogonal to $\mathscr{H}_{-\infty}$. We put $\mathscr{K}_i = (\mathscr{K} \cap \mathscr{H}_i)$ $(i \in \boldsymbol{Z})$ and consider the induced situation $(U, \mathscr{K}_0)$ in $\mathscr{K}$, which is purely non-deterministic, since

$$\mathscr{K}_{-\infty} \equiv \bigcap_{i \in Z} \mathscr{K}_i = \mathscr{K} \cap \bigcap_{i \in Z} \mathscr{H}_i = \mathscr{K} \cap \mathscr{H}_{-\infty} = \{0\} .$$

On the other hand, the maximality of $\mathscr{X}$ implies $\mathscr{K} \cap (\mathscr{H}_1 \ominus \mathscr{H}_0) = \emptyset$ and so $\mathscr{K}_1 \ominus \mathscr{K}_0 = \emptyset$, hence there is no non-trivial helix for this induced situation. Therefore from Lemma 2.1, $\mathscr{K} = \mathscr{K}_{-\infty}$, i.e., $\mathscr{K} = \{0\}$. Thus we get (1).

The family of helices $\mathscr{X} = (X^{(n)})$ with the property (1) is called a base for the situation.

Now we shall apply the above result to measure preserving transformations. Let $(\Omega, \mathscr{F}, P)$ be a separable complete probability space which is isomorphic to the unit interval of R^1 with the Lebesgue measure, and

T an automorphism of Ω, that is, a bimeasurable measure preserving bijection. Let $\mathscr{F}_0$ be a complete sub-σ-field of $\mathscr{F}$. We denote by $\mathscr{F}_i$ the sub-σ-fields $T^i\mathscr{F}_0$ for all $i \in \boldsymbol{Z}$.

DEFINITION 2.4. A pair $(T, \mathscr{F}_0)$ is called a system if the following conditions are satisfied:

(a) $\mathscr{F}_i \subset \mathscr{F}_{i+1}$ for any $i \in \boldsymbol{Z}$,

(b) $\bigvee_{i \in \boldsymbol{Z}} \mathscr{F}_i = \mathscr{F}$.

If

(c) $\mathscr{F}_{-\infty} \equiv \bigcap_{i \in \boldsymbol{Z}} \mathscr{F}_i$ is the trivial σ-field, the system $(T, \mathscr{F}_0)$ is called a Kolmogorov system, or simply a K-system.

Denote by $\mathscr{H} = L_0^2(\Omega)$ the class of all squarely integrable real random variables with zero expectations, which is an infinite dimensional Hilbert space under the inner product $(x, y) = E[xy]$ for $x, y \in \mathscr{H}$. For each $i \in \boldsymbol{Z}$, let $\mathscr{H}_i$ be the subspace of $\mathscr{H}$ consisting of all elements measurable with respect to $\mathscr{F}_i$. We define a unitary operator U on $\mathscr{H}$ by $(Ux)(\omega) = x(T^{-1}\omega)$ for $x \in \mathscr{H}$. Then a pair $(U, \mathscr{H}_0)$ corresponding to a system $(T, \mathscr{F}_0)$ is obviously a situation and the situation corresponding to a K-system is purely non-deterministic.

A helix for the corresponding situation is also called a helix for the system, and other nomenclatures will be introduced in the same way.

Thus for any system we have the representations (1) and (2).

Further, it is known that the unitary operator U induced by a non-trivial system $(T, \mathscr{F}_0)$ has a Lebesgue spectrum with infinite multiplicity in $\mathscr{H}_{-\infty}^{\perp}$. Thus we have

THEOREM 2.2. *For any non-trivial system, the number of helices which form a base is really countably infinite.*

3. The second representation of K-systems. As the helices determine only the space $\mathscr{H}_{-\infty}^{\perp}$ by Theorem 2.1, we simplify our discussion by assuming that $\mathscr{H}_{-\infty} = \{0\}$. Thus throughout this paragraph we deal with K-systems. In the following we often omit the expression "a.s." for simplicity.

The method applied in this section are analogous to the ones used in martingale theory. First we note that a helix has the martingale property, namely, for a helix $X = (x_i)_{i \in \boldsymbol{Z}}$, $(x_{i+j} - x_j, \mathscr{F}_{i+j})_{i \geqq 0}$ is a martingale for a fixed $j \in \boldsymbol{Z}$. In fact, by the orthogonality of the increments of helices in Definition 2.2. (c), we have

$$\begin{aligned} E[x_{(i+1)+j} - x_j \mid \mathscr{F}_{i+j}] &= E[(x_{i+j+1} - x_{i+j}) + (x_{i+j} - x_j) \mid \mathscr{F}_{i+j}] \\ &= E[x_{i+j+1} - x_{i+j} \mid \mathscr{F}_{i+j}] + x_{i+j} - x_j \\ &= x_{i+j} - x_j \,. \end{aligned}$$

Let $(T, \mathscr{F}_0)$ be a K-system.

DEFINITION 3.1. Given two helices $X = (x_i)_{i \in Z}$ and $X' = (x'_i)_{i \in Z}$ for $(T, \mathscr{F}_0)$, we define a random variable

$$\langle X, X' \rangle = E[x_1 x'_1 | \mathscr{F}_0]$$

and if $X = X'$ we write simply $\langle X \rangle$ instead of $\langle X, X \rangle$. Obviously $\langle X, X' \rangle$ is measurable with respect to $\mathscr{F}_0$ and $\langle X, X' \rangle = \langle X', X \rangle$. Further for another helix $Y = (y_i)_{i \in Z}$, we see easily,

$$\langle X + Y, X' \rangle = \langle X, X' \rangle + \langle Y, X' \rangle .$$

Two helices X and X' are called strictly orthogonal if $\langle X, X' \rangle = 0$, and in this case X and X' are orthogonal in the sense of §2, since

$$E[x_1 x'_1] = E[E[x_1 x'_1 | \mathscr{F}_0]] = 0 .$$

For any helices X and X', the random variable $\langle X, X' \rangle$ is clearly integrable. Now we define a signed measure on $\mathscr{F}_0$ with density $\langle X, X' \rangle$:

$$\mu_{\langle X, X' \rangle}(A) = \int_A \langle X, X' \rangle dP = \int_A x_1 x'_1 dP$$

for any $A \in \mathscr{F}_0$; we write $\mu_{\langle X \rangle} \equiv \mu_{\langle X, X \rangle}$. Then $\mu_{\langle X, X' \rangle}$ is absolutely continuous with respect to $\mu_{\langle X \rangle}$ and $\mu_{\langle X' \rangle}$. Indeed, for any $A \in \mathscr{F}_0$,

$$\begin{aligned} |\mu_{\langle X, X' \rangle}(\mathrm{A})| &\leqq \int_A |x_1 x'_1| dP \leqq \left(\int_A |x_1|^2 dP\right)^{1/2} \left(\int_A |x'_1|^2 dP\right)^{1/2} \\ &= (\mu_{\langle X \rangle}(A))^{1/2} (\mu_{\langle X' \rangle}(A))^{1/2} . \end{aligned}$$

DEFINITION 3.2. Let X be a helix for a K-system $(T, \mathscr{F}_0)$ and ν a random variable measurable with respect to $\mathscr{F}_0$ and squarely integrable with respect to $\mu_{\langle X \rangle}$. We define a process $Y = (y_i)_{i \in Z}$: $y_0 = 0$, $y_i = \sum_{k=1}^{i} (\nu \circ T^{-(k-1)}) d_k$ $(i > 0)$ and $y_i = -y_{-i} \circ T^{-i}$ $(i < 0)$ where $(d_i)_{i \in Z}$ is the helix-difference of X. Then Y is clearly a helix for $(T, \mathscr{F}_0)$. We call Y a helix-transform of X by ν and write $Y = \nu * X$.

For any helix X and X', $\langle \nu * X, X' \rangle = \nu \langle X, X' \rangle$, since

$$E[(\nu x_1) x'_1 | \mathscr{F}_0] = \nu E[x_1 x'_1 | \mathscr{F}_0] .$$

Now we show the representation theorem of helices by strictly orthogonal helices. First we prove the existence part of the representation.

THEOREM 3.1. *For any K-system $(T, \mathscr{F}_0)$, there is a family of at most countable strictly orthogonal non-trivial helices $\mathscr{X} = (X^{(n)})$ such that for any helix X, there exist helix-transforms $\nu^{(n)} * X^{(n)}$ with*

$$X = \sum_n \nu^{(n)} * X^{(n)} \tag{1}$$

where

$$\sum_n \int_\Omega \nu^{(n)2} d\mu_{\langle X^{(n)}\rangle} < \infty .$$

PROOF. Let $\mathscr{X} = (X^{(n)})$ be a maximal family of strictly orthogonal non-trivial helices. Take any helix X and let $\nu^{(n)}$ be the Radon-Nikodym derivative of $\mu_{\langle X, X^{(n)}\rangle}$ with respect to $\mu_{\langle X^{(n)}\rangle}$. Remark that

$$\langle X, X^{(n)}\rangle = \nu^{(n)}\langle X^{(n)}\rangle .$$

We have

$$\begin{aligned} 0 &\leqq E[(x_1 - \sum_n \nu^{(n)} x_1^{(n)})^2 | \mathscr{F}_0] = E[x_1^2 | \mathscr{F}_0] - 2\sum_n \nu^{(n)} E[x_1 x_1^{(n)} | \mathscr{F}_0] \\ &\qquad + \sum_n \nu^{(n)2} E[x_1^{(n)2} | \mathscr{F}_0] \\ &= \langle X\rangle - 2\sum_n \nu^{(n)}\langle X, X^{(n)}\rangle + \sum_n \nu^{(n)2}\langle X^{(n)}\rangle \\ &= \langle X\rangle - \sum_n \nu^{(n)2}\langle X^{(n)}\rangle . \end{aligned}$$

Therefore

$$\sum_n \nu^{(n)2}\langle X^{(n)}\rangle \leqq \langle X\rangle$$

and so

$$\sum_n \int_\Omega \nu^{(n)2} d\mu_{\langle X^{(n)}\rangle} \leqq E[\langle X\rangle] < \infty .$$

Thus we have helix-transforms $\nu^{(n)} * X^{(n)}$ $(n \geqq 1)$ and we can consider a helix $\sum_n \nu^{(n)} * X^{(n)}$.

Put $Y \equiv X - \sum_n \nu^{(n)} * X^{(n)}$, then for any n

$$\begin{aligned} \langle Y, X^{(n)}\rangle &= \langle X, X^{(n)}\rangle - \sum_m \nu^{(m)}\langle X^{(m)}, X^{(n)}\rangle \\ &= \langle X, X^{(n)}\rangle - \nu^{(n)}\langle X^{(n)}\rangle \\ &= \nu^{(n)}\langle X^{(n)}\rangle - \nu^{(n)}\langle X^{(n)}\rangle \\ &= 0 . \end{aligned}$$

Hence Y is strictly orthogonal to all $X^{(n)}$. By the maximality of $\mathscr{X}$, Y is trivial.

As the above proof indicates, the family $(X^{(n)})$ may be considered to consist of normalized helices.

Now we shall show that the above representation is unique in the following sense.

THEOREM 3.2. *We can choose the family* $\mathscr{X}$ *in Theorem* 3.1. *so that*

$$(2) \qquad \mu_{\langle X^{(n)}\rangle} \succ \mu_{\langle X^{(n+1)}\rangle}$$

for all n where $\succ$ denotes the absolute continuity relation of measures.

Further, if another family $\mathscr{Y} = (Y^{(n)})$ is also one stated in Theorem 3.1. *and satisfies the property* (2), *then* $\mu_{\langle X^{(n)}\rangle} \sim \mu_{\langle Y^{(n)}\rangle}$ *for all n where $\sim$ denotes the equivalence relation of measures.*

In the above theorem, the number of non-trivial helices in $\mathscr{X}$ coincides with that of helices in $\mathscr{Y}$. We call such a family $\mathscr{X}$ a strict base of non-trivial helices for the system. The number of non-trivial helices in a strict base of helices for a K-system is called the multiplicity of this K-system $(T, \mathscr{F}_0)$, and denoted by $M(T, \mathscr{F}_0)$.

PROOF OF THEOREM 3.2. Let $\mathscr{X} = (X^{(n)})$ be the family stated in Theorem 3.1. with the representation (1) for any helix X. As we can employ the helices $X^{(n)}/\|X^{(n)}\|$ instead of $X^{(n)}$, we assume that each helix $X^{(n)}$ is normalized. Put

$$X^{(0)} = \sum_n X^{(n)}/n\ ,$$

which is a helix. Then $\mu_{\langle X^{(n)}\rangle}$ is absolutely continuous with respect to $\mu_{\langle X^{(0)}\rangle}$, since

$$\langle X^{(0)}\rangle = \sum_n \langle X^{(n)}\rangle/n^2\ .$$

Hence there exists a non-negative Radon-Nikodym derivative $\phi^{(n)}$ such that

$$d\mu_{\langle X^{(n)}\rangle} = \phi^{(n)} d\mu_{\langle X^{(0)}\rangle} \tag{3}$$

for all n.

Now let us define a sequence of $\mathscr{F}_0$-measurable sets; For any m, n such that $m \leqq n$,

$$A_n = \{\omega \in \Omega \mid \phi^{(n)}(\omega) > 0\}$$

$$A_{mn} = \left\{\omega \in A_n \,\middle|\, \sum_{k=1}^{n} I_{A_k}(\omega) = m\right\}$$

where I_A denotes the indicator of A.

Then for each m, $(A_{mn})_{m\leqq n}$ is a disjoint family and for each n.

$$A_n = \bigcup_{m=1}^{n} A_{mn} \qquad \text{(disjoint union)}\ .$$

We put

$$\bar{X}^{(m)} = \sum_{n\geqq m} (1/n) I_{A_{mn}} * X^{(n)}$$

for all m. We shall prove that the family $(\bar{X}^{(m)})$ is a desired one.

Orthogonality. For any $m, m', m > m'$

$$\langle \bar{X}^{(m)}, \bar{X}^{(m')}\rangle = \langle \bar{X}^{(m)}, \sum_{m'\leqq n<m}(1/n)I_{A_{m'n}} * X^{(n)}\rangle + \sum_{n\geqq m}(1/n^2)I_{A_{mn}}I_{A_{m'n}}\langle X^{(n)}\rangle ,$$

where the first term of the right hand side vanishes by the strict orthogonality of $(X^{(n)})$ and the second term vanishes by the disjointness of A_{mn} and of $A_{m'n}$ for each n. Namely, $\langle X^{(m)}, X^{(m')}\rangle = 0$ for $m \neq m'$.

Absolute continuity. Define for each m

$$\psi^{(m)} = \begin{cases} \phi^{(n)}/n^2 & \text{on } A_{mn},\ n \geqq m \\ 0 & \text{otherwise} , \end{cases}$$

then $\operatorname{supp} \psi^{(m)} = \bigcup_{n\geqq m} A_{mn}$ and from (3)

$$\begin{aligned}\langle \bar{X}^{(m)}\rangle &= \sum_{n\geqq m}(1/n^2)I_{A_{mn}}\langle X^{(n)}\rangle = \sum_{n\geqq m}(1/n^2)\phi^{(n)}I_{A_{mn}}\langle X^{(0)}\rangle \\ &= \psi^{(m)}\langle X^{(0)}\rangle .\end{aligned}$$

Since

$$\begin{aligned}\operatorname{supp} \psi^{(m)} &= \bigcup_{n\geqq m} A_{mn} = \left\{\omega \in \Omega \,\middle|\, \sum_k I_{A_k} \geqq m\right\} \\ &\supset \operatorname{supp} \psi^{(m+1)}\end{aligned}$$

we have $\mu\langle \bar{X}^{(m)}\rangle \succ \mu\langle \bar{X}^{(m+1)}\rangle$ for all m.

Representation. Let X be any helix for the system, then Theorem 3.1. implies

$$X = \sum_n \nu^{(n)} * X^{(n)} .$$

Since

$$X^{(n)} = I_{A_n} * X^{(n)}$$

by (3) and for all $n, m, n \geqq m$.

$$I_{A_{mn}} * \bar{X}^{(m)} = (1/n)I_{A_{mn}} * X^{(n)}$$

by the disjointness of $(A_{mn})_{n\geqq m}$, we have

$$\begin{aligned}X &= \sum_n \nu^{(n)} * \left(n \sum_{m\leqq n} I_{A_{mn}} * \bar{X}^{(m)}\right) \\ &= \sum_m \left(\sum_{n\geqq m} n\nu^{(n)} I_{A_{mn}}\right) * \bar{X}^{(m)} \\ &= \sum_m \lambda^{(m)} * \bar{X}^{(m)} \quad \text{(say)} .\end{aligned}$$

Finally we prove the uniqueness following the method of [1]. Let $\mathscr{X} = (X^{(n)})$ and $\mathscr{Y} = (Y^{(n)})$ be the families stated in the theorem with the property (2). Define $\theta^{(n)}$ by

$$d\mu_{\langle Y^{(n)}\rangle} = \theta^{(n)} d\mu_{\langle Y^{(1)}\rangle}$$

for all n. By Theorem 3.1. for a helix $X^{(n)}$, there exists $(\nu^{(k,n)})_k$ of $\mathscr{F}_0$-

measurable random variables such that

$$X^{(n)} = \sum_k \nu^{(k,n)} * Y^{(k)}$$

for all n. Hence for any m, n

$$d\mu_{\langle X^{(n)}, X^{(m)}\rangle} = \sum_k \nu^{(k,n)}\nu^{(k,m)} d\mu_{\langle Y^{(k)}\rangle} = \sum_k \nu^{(k,n)}\nu^{(k,m)}\theta^{(k)} \mu d_{\langle Y^{(1)}\rangle} \tag{4}$$

and clearly $\mu_{\langle X^{(1)}\rangle} \prec \mu_{\langle Y^{(1)}\rangle}$. By symmetry we can conclude that $\mu_{\langle X^{(1)}\rangle} \sim \mu_{\langle Y^{(1)}\rangle}$. For $n > 1$, we shall prove

$$\mu_{\langle X^{(n)}\rangle} \sim \mu_{\langle Y^{(n)}\rangle}$$

by induction. Assume that this equivalence holds for $n = 1, 2, \cdots, r$. By the Lebesgue decomposition of measure $\mu_{\langle X^{(r+1)}\rangle}$ with respect to $\mu_{\langle Y^{(r+1)}\rangle}$, we have $\mu_{\langle X^{(r+1)}\rangle} = \mu + \mu'$ where $\mu \prec \mu_{\langle Y^{(r+1)}\rangle}$ and μ' is singular with respect to $\mu_{\langle Y^{(r+1)}\rangle}$, that is, there exists $B \in \mathscr{F}_0$ such that

$$\mu'(B^c) = 0\,, \qquad \mu_{\langle Y^{(r+1)}\rangle}(B) = 0\,. \tag{5}$$

If we can show that $\mu'(B) = 0$, then μ' is a null measure and we get $\mu_{\langle X^{(r+1)}\rangle} = \mu \prec \mu_{\langle Y^{(r+1)}\rangle}$, and by symmetry we can conclude that $\mu_{\langle X^{(r+1)}\rangle} \sim \mu_{\langle Y^{(r+1)}\rangle}$ which completes the induction.

Assume, for contrary, that $\mu'(B) > 0$. Then

$$\int_B \frac{d\mu_{\langle X^{(r+1)}\rangle}}{d\mu_{\langle Y^{(1)}\rangle}} d\mu_{\langle Y^{(1)}\rangle} = \mu_{\langle X^{(r+1)}\rangle}(B) \geqq \mu'(B) > 0\,.$$

Put

$$B_0 = B \cap \left\{\frac{d\mu_{\langle X^{(r+1)}\rangle}}{d\mu_{\langle Y^{(1)}\rangle}} > 0\right\}\,,$$

then $\mu_{\langle X^{(r+1)}\rangle}(B_0) > 0$ and $\mu_{\langle Y^{(1)}\rangle}(B_0) > 0$.

On the other hand, $\mu_{\langle Y^{(r+1)}\rangle}(B_0) = 0$ from (5) and since $\mu_{\langle Y^{(k)}\rangle} \prec \mu_{\langle Y^{(r+1)}\rangle}$ for $k \geqq r+1$ we get

$$\int_{B_0} \theta^{(k)} d\mu_{\langle Y^{(1)}\rangle} = \mu_{\langle Y^{(k)}\rangle}(B_0) = 0\,.$$

Therefore $\theta^{(k)} = 0$, $\mu_{\langle Y^{(1)}\rangle}$-a.s. on B_0 for all $k \geqq r+1$.

Then from (4)

$$\frac{d\mu_{\langle X^{(n)}, X^{(m)}\rangle}}{d\mu_{\langle Y^{(1)}\rangle}} = \sum_k \nu^{(k,n)}\nu^{(k,m)}\theta^{(k)} = \sum_{k=1}^{r} \nu^{(k,n)}\nu^{(k,m)}\theta^{(k)}\,, \qquad \mu_{\langle Y^{(1)}\rangle}\text{-a.s. on } B_0\,, \tag{6}$$

for any n, m. Since

$$\mu_{\langle Y^{(1)}\rangle} \sim \mu_{\langle X^{(1)}\rangle} , \qquad \mu_{\langle X^{(n)}\rangle} \succ \mu_{\langle X^{(r+1)}\rangle}$$

for any $n, 1 \leqq n \leqq r$, we have from the definition of B_0

$$\frac{d\mu_{\langle X^{(n)}\rangle}}{d\mu_{\langle Y^{(1)}\rangle}} > 0 , \qquad \mu_{\langle Y^{(1)}\rangle}\text{-a.s. on } B_0 \tag{7}$$

for any $n, 1 \leqq n \leqq r + 1$. Now by (6) and (7)

$$\sum_{k=1}^{r} \nu^{(k,n)2}\theta^{(k)} > 0 , \qquad \mu_{\langle Y^{(1)}\rangle}\text{-a.s. on } B_0$$

for any $n, 1 \leqq n \leqq r + 1$, and

$$\sum_{k=1}^{r} \nu^{(k,n)}\nu^{(k,m)}\theta^{(k)} = 0 , \qquad \mu_{\langle Y^{(1)}\rangle}\text{-a.s. on } B_0$$

for any $n, m, n \neq m, 1 \leqq n, m \leqq r + 1$.

Thus we get $r + 1$ non-zero vectors

$$a_n = [\nu^{(k,n)}(\omega)\sqrt{\theta^{(k)}(\omega)},\ k = 1, 2, \cdots, r] \qquad (1 \leqq n \leqq r + 1)$$

for some $\omega \in B_0$ on the r-dimensional Euclidean space, which are mutually orthogonal. This contradiction shows that $\mu'(B) = 0$ and the proof of Theorem 3.2. is completed.

Our multiplicity is not an invariant under the ordinary isomorphy in the ergodic theory but it is an invariant under the isomorphy in the sense of the following definition.

DEFINITION 3.3. Two K-systems $(T, \mathscr{F}_0)$ and $(S, \mathscr{G}_0)$ are said to be system-isomorphic if there exists an automorphism R of Ω such that $RT = SR$ and $R\mathscr{F}_0 = \mathscr{G}_0$.

If and only if X is a helix for $(T, \mathscr{F}_0)$, then $X \circ R^{-1}$ is a helix for $(S, \mathscr{G}_0)$. Therefore the multipicity is an invariant under system-isomorphy. A relation between the multiplicity and the ordinary isomorphy is given as follows. By the definition, a K-automorphism S has a sub-σ-field $\mathscr{G}_0$ such that $(S, \mathscr{G}_0)$ is a K-system.

THEOREM 3.3. *Let $(T, \mathscr{F}_0)$ be a K-system and S a K-automorphism. If $M(S, \mathscr{G}_0) \neq M(T, \mathscr{F}_0)$ for any $\mathscr{G}_0$ such that $(S, \mathscr{G}_0)$ is a K-system, then T and S are not ordinarily isomorphic. Namely if two K-automorphisms T and S are ordinarily isomorphic, then we can find two sub-σ-fields $\mathscr{F}_0$ and $\mathscr{G}_0$ such that $(T, \mathscr{F}_0)$ and $(S, \mathscr{G}_0)$ are K-systems and $M(T, \mathscr{F}_0) = M(S, \mathscr{G}_0)$.*

4. The second representation of B-systems.

DEFINITION 4.1. Let T be an automorphism of Ω and $\mathscr{A}$ a sub-σ-

field of $\mathscr{F}$. The pair $(T, \mathscr{A})$ is called a B-system if

(a) $(T^i\mathscr{A})_{i\in Z}$ is an independent sequence of sub-σ-fields,

(b) $\bigvee_{i\in Z} T^i\mathscr{A} = \mathscr{F}$.

If we put $\mathscr{F}_0 = \bigvee_{i<0} T^i\mathscr{A}$, then $(T, \mathscr{F}_0)$ is clearly a K-system. Indeed, the conditions (a) and (b) in Definition 2.3 hold as we see easily, and (c) is derived from the Kolmogorov's zero-one law. Thus we can apply the representation theorem in §3 to B-systems.

THEOREM 4.1. *Let $(T, \mathscr{A})$ be a B-system and $(T, \mathscr{F}_0)$ the K-system deduced from $(T, \mathscr{A})$. Then the multiplicity $M(T, \mathscr{F}_0)$, or simply $M(T, \mathscr{A})$, is equal to the dimension of $L_0^2(\mathscr{A})$, the subspace of all $\mathscr{A}$-measurable random variables in $L_0^2(\Omega)$.*

For the proof we need the following lemma.

LEMMA 4.1. *Let $(\Omega, \mathscr{B}, P)$ be a product space of separable probability spaces $(\Omega_1, \mathscr{B}_1, P_1)$ and $(\Omega_2, \mathscr{B}_2, P_2)$. Let $x \in L_0^2(\Omega)$ satisfy*

$$\int_{\Omega_2} x(\cdot, \omega_2)dP_2(\omega_2) = 0 \tag{1}$$

in $L^2(\Omega_1)$. If $(x^{(n)})$ is a complete orthonormal system of $L_0^2(\Omega_2)$, then

$$x = \sum_n \nu^{(n)}x^{(n)} \tag{2}$$

in $L^2(\Omega)$, where

$$\nu^{(n)}(\cdot) = \int_{\Omega_2} x(\cdot, \omega_2)x^{(n)}(\omega_2)dP_2(\omega_2) .$$

PROOF. For almost all $\omega_1 \in \Omega_1$, $x(\omega_1, \cdot) \in L^2(\Omega_2)$, and

$$x(\omega_1, \cdot) = y_1(\omega_1) + \sum_n \nu^{(n)}(\omega_1)x^{(n)}(\cdot) \quad \text{in} \quad L^2(\Omega_2) , \tag{3}$$

where

$$y_1(\cdot) = \int_{\Omega_2} x(\cdot, \omega_2)dP_2(\omega_2) = 0 \quad \text{in} \quad L^2(\Omega_1)$$

and

$$\nu^{(n)}(\cdot) = \int_{\Omega_2} x(\cdot, \omega_2)x^{(n)}(\omega_2)dP_2(\omega_2) .$$

By Fubini's theorem, we see that the equality (2) holds in $L^2(\Omega)$-sense.

PROOF OF THEOREM 4.1. Let $(T, \mathscr{F}_0)$ be the K-system deduced from the B-system $(T, \mathscr{A})$ and $(x^{(n)})$ a complete orthonormal system of $L_0^2(\mathscr{A})$. We define helices $(X^{(n)})$ as follows:

$$X^{(n)} = (x_i^{(n)})_{i \in Z} ,$$
$$x_0^{(n)} = 0 , \qquad x_i^{(n)} = \sum_{k=1}^{i} x^{(n)} \circ T^{-(k-1)} \quad (i > 0) ,$$
$$x_i^{(n)} = -x_{-i}^{(n)} \circ T^{-i} \quad (i < 0) .$$

They are indeed helices because $x_1^{(n)} \in L_0^2(\mathscr{F}_1)$ and $E[x_1^{(n)} | \mathscr{F}_0] = E[x^{(n)} | \mathscr{F}_0] = E[x^{(n)}] = 0$ by the independence of $\mathscr{A}$ and $\mathscr{F}_0$. Since

$$(4) \qquad \langle X^{(n)}, X^{(m)} \rangle = E[x_1^{(n)} x_1^{(m)} | \mathscr{F}_0] = E[x^{(n)} x^{(m)}] = \delta_{nm} ,$$

$(X^{(n)})$ is a family of strictly orthogonal normalized helices for $(T, \mathscr{F}_0)$.

Now let $X = (x_i)_{i \in Z}$ be any helix for $(T, \mathscr{F}_0)$. By the definition of helices,

$$x_1 \in L_0^2(\mathscr{F}_1) = L_0^2(\mathscr{F}_0 \vee \mathscr{A}) , \qquad x_1 \in L_0^2(\mathscr{F}_0)^{\perp} ,$$

hence for any $A \in \mathscr{F}_0$.

$$(5) \qquad \int_A x_1 dP = 0 .$$

By the independence of $\mathscr{A}$ and $\mathscr{F}_0$ we can consider $(\Omega, \mathscr{F}_0 \vee \mathscr{A}, P)$ as the product space of $(\Omega, \mathscr{F}_0, P)$ and $(\Omega, \mathscr{A}, P)$. Then (5) implies (1) in Lemma 4.1 for x_1 of the helix X, and hence we get

$$x_1 = \sum_n \nu^{(n)} x^{(n)} \quad \text{in} \quad L^2(\Omega, \mathscr{F}_0 \vee \mathscr{A}, P) .$$

This means

$$(6) \qquad X = \sum_n \nu^{(n)} * X^{(n)} .$$

Since $\mu_{\langle X^{(n)} \rangle} = P$ for all n, from (4), all the measures $\mu_{\langle X^{(n)} \rangle}$ are mutually equivalent, and so the condition (2) of Theorem 3.2 is satisfied for the representation (6). Therefore $M(T, \mathscr{A}) = M(T, \mathscr{F}_0) =$ the dimension of $L_0^2(\mathscr{A})$ by the uniqueness theorem. q.e.d.

In a B-system $(T, \mathscr{A})$, if $\mathscr{A}$ is generated by a finite partition α, $M(T, \mathscr{A}) = \alpha^{\#} - 1$ where $\alpha^{\#}$ is the number of atoms of α. We often write $M(\alpha)$ instead of $M(T, \mathscr{A})$.

SUPPLEMENT. We shall give here a proof of Theorem 2.2 using the term of helix-transforms and we shall show that a K-system has the Lebesgue spectrum with infinite multiplicity.

To prove the existence of family of countably infinite mutually orthogonal helices, it is sufficient to show that for any non-trivial helix X, the dimension of $L^2(d\mu_{\langle X \rangle})$ is infinite. Indeed if (ν_k) is a complete orthonormal base of $L^2(d\mu_{\langle X \rangle})$, the helix-transforms $(\nu_k * X)$ is a family of countably infinite mutually orthogonal non-trivial helices. Denote by S the support of the measure $\mu_{\langle X\rangle}$, i.e., $S=\{\langle X\rangle\neq 0\}$. Since X is non-trivial, $P(S)>0$. Then the measure space $(S, \mathscr{F}_0\cap S, \mu_{\langle X\rangle})$ has no atom. (For any $A\in\mathscr{F}_0\cap S$, $\mu_{\langle X\rangle}(A)=\int_A\langle X\rangle dP>0$, that is, $P(A)>0$, there exists $B\in\mathscr{F}_0\cap S$, $B\subset A$, $P(B)>0$, that is, $\mu_{\langle X\rangle}(B)>0$.) Hence $(\Omega, \mathscr{F}_0, \mu_{\langle X\rangle})$ has also no atom, which means that the space $L^2(d\mu_{\langle X\rangle})$ has the infinite dimension.

Thus Theorem 2.2 is proved without the term of the spectrum of the systems. Further, the helix-differences of the countably infinite family $\mathscr{H}$ in Theorem 2.2 is a base of spectrum with infinite multiplicity for a non-trivial system.

EXAMPLE 1. For Meshalkin's isomorphic two systems α, β such that

$$\text{distr.}(\alpha)=(1/4,\ 1/4,\ 1/4,\ 1/4)\ ,$$
$$\text{distr.}(\beta)=(1/2,\ 1/8,\ 1/8,\ 1/8,\ 1/8)$$

where distr.(α) denotes the distribution of the partition α, we have $M(\alpha)=3$, $M(\beta)=4$. But it is well-known that these two systems are ordinary isomorphic.

EXAMPLE 2. For Baker's transformation T on the two dimensional torus,

$$T(x, y)=(2x, y/2)\quad \bmod 1\ ,\quad 0<x, y\leqq 1\ ,$$

put

$$\alpha=[\{(x, y)\,|\,0<y\leqq 1/2\}\ ,\qquad \{(x, y)\,|\,1/2<y\leqq 1\}]\ ,$$

then $(T, \bar{\alpha})$ is a B-system, where $\bar{\alpha}$ denotes the sub-σ-field generated by α. By Theorem 4.1, $M(T, \bar{\alpha})=1$.

As a helix $H=(h_i)_{i\in Z}$ for $(T, \bar{\alpha})$ we define

$$h_1(x, y)=\begin{cases}1 & 0<y\leqq 1/2\\ -1 & 1/2<y\leqq 1\ .\end{cases}$$

Then any random variable $x\in L_0^2(\bar{\alpha})\ominus L_0^2(T^{-1}(\bar{\alpha})$ can be written as $x=\nu h_1$ where ν is a squarely integrable $T^{-1}\bar{\alpha}$-measurable random variable.

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