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NORMAL DERIVATIONS IN OPERATOR ALGEBRAS

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Let A be a ring with involution (briefly, a $*$-ring), $\delta: A \to A$ a derivation of A, that is, $\delta(x+y) = \delta x + \delta y$ and $\delta(xy) = (\delta x)y + x(\delta y)$ for all x, y in A. When A is a $*$-algebra (over the complex field) one requires also that δ be a linear mapping. For each $a \in A$ we write δ_a for the inner derivation implemented by a: $\delta_a x = [a, x] = ax - xa$. For a derivation δ of A, the adjoint δ^* of δ is the derivation of A defined by the formula $\delta^* x = -(\delta(x^*))^*$; the purpose of the minus sign is to validate the formula $(\delta_a)^* = \delta_{a^*}$. Note also that $\ker(\delta^*) = (\ker \delta)^*$.

In the first part of the paper we explore the relationships between several plausible definitions of normality for a derivation δ of a $*$-ring, with particular attention to C^*-algebras and von Neumann algebras; in the second part, we discuss derivations of certain algebras of "unbounded operators" affiliated with AW^*-algebras.

Here are some natural candidates for the definition of "normal derivation" (there seems to be no compelling reason for making a definitive choice):

(N_1) $\ker \delta = \ker(\delta^*)$;

(N_2) $\delta^*\delta = \delta\delta^*$;

(N_3) there exist a $*$-ring B containing A as a $*$-subring, and an element $b \in B$, such that b is normal ($b^*b = bb^*$) and $\delta = \delta_b | A$ (that is, $\delta x = [b, x]$ for all $x \in A$);

(N_4) there exist a $*$-ring B containing A as a $*$-subring, and an element $b \in B$, such that $\delta = \delta_b | A$ and $A \cap \{b\}'$ is a $*$-subring of A, where $\{b\}'$ denotes the commutant of b in B;

(N_5) there exist a $*$-ring B containing A as a $*$-subring, and an element $b \in B$, such that $\delta = \delta_b | A$ and $\{b\}'$ is a $*$-subring of B.

The most natural condition is (N_2), which mimics the definition of normality for an element of a $*$-ring. The remaining conditions are motivated by the well-known theorem of B. Fuglede: if $\delta = \delta_a$, $a \in A$, then (N_1) means that $xa = ax$ if and only if $xa^* = a^*x$; thus, when A is a $*$-algebra of operators in a Hilbert space, δ_a satisfies (N_1) if and only if a is normal (Fuglede's theorem [13, Prob. 152]). Here are some elementary relations between these conditions:

THEOREM 1. $(N_5) \Rightarrow (N_4) \Rightarrow (N_1)$ *and* $(N_5) \Rightarrow (N_3) \Rightarrow (N_2)$.

PROOF. Let δ be a derivation of the $*$-ring A. It is obvious that (N_5) implies (N_3) and (N_4). (N_4) implies (N_1) because $\ker \delta = A \cap \{b\}'$.

Assume (N_3): Then $(\delta_b)^* | A = \delta^*$, therefore $\delta^*\delta - \delta\delta^* = [(\delta_b)^*, \delta_b] | A = \delta_{[b^*, b]} | A = 0$, whence (N_2). (Incidentally, it suffices that $b^*b - bb^*$ commute with every element of A.) ∎

When A is a C^*-algebra, we shall see that all but one of the implications in Theorem 1 can be reversed. A device in the proof is the following result of C. R. Putnam (we offer an alternative proof, for imitation in Theorem 7):

LEMMA [15, p. 5]. *If x is an operator in a Hilbert space H, such that x commutes with $x^*x - xx^*$, then $x^*x = xx^*$.*

PROOF. Let $z = x^*x - xx^*$ and let $B = \{z\}'$ be the set of all operators in H that commute with z; then B is a von Neumann algebra containing x, and z is a self-adjoint element of the center of B. The assertion is that $z = 0$. Assume to the contrary that $z \neq 0$; then there exists a projection h in the center of B such that zh is invertible in Bh and such that either $zh \geqq 0$ or $zh \leqq 0$. Interchanging x and x^* if necessary, we can suppose that $zh \geqq 0$; then $zh = r^2$ with r in the center of B, $r^* = r$, r invertible in Bh. If s is the inverse of r in Bh, then $h = r^2s^2 = (zh)s^2 = (xs)^*(xs) - (xs)(xs)^*$; since xs belongs to the Banach algebra Bh, this contradicts a theorem of A. Wintner (the unity element of a Banach algebra is never a commutator [cf. 13, Prob. 182]). ∎

THEOREM 2. *If A is a C^*-algebra, then*

$$(N_2) \Leftrightarrow (N_3) \Leftrightarrow (N_5) \Rightarrow (N_4) \Leftrightarrow (N_1) .$$

PROOF. It will suffice to show that $(N_1) \Rightarrow (N_4)$ and $(N_2) \Rightarrow (N_5)$. Let δ be a derivation of A.

View A as a C^*-algebra of operators in a Hilbert space and let B be the closure of A for the weak operator topology. By a theorem of R. V. Kadison and S. Sakai [cf. 11, Ch. III, §9, n° 3, Cor. of Th. 1], there exists $b \in B$ such that $\delta_b | A = \delta$.

Assume (N_1): Then $A \cap \{b\}' = \ker \delta$ is a $*$-ring by hypothesis, whence (N_4).

Assume (N_2): Then the element $b^*b - bb^*$ of B commutes with every element of A, hence with every element of B, hence with b; therefore b is normal by the lemma, and $\{b\}'$ is a $*$-ring by Fuglede's theorem, whence (N_5). ∎

The unilateral implication in Theorem 2 cannot be reversed; a counterexample is provided by the unilateral shift [cf. 2, p. 82]:

EXAMPLE 1. Let H be the Hilbert space of square-integrable functions on the unit circle that are analytic in the sense that their Fourier coefficients with negative index vanish [9]. Let A be the C^*-algebra of all compact operators in H, B the algebra of all operators in H. Let $u \in B$ be the unilateral shift operator defined via the canonical orthonormal basis of H and define a derivation δ in the ideal A of B by the formula $\delta x = ux - xu$. One has $\ker \delta = A \cap \{u\}'$; but $\{u\}'$ is the set of all analytic Toeplitz operators [9, Th. 7] and the only compact Toeplitz operator is 0 [9, Cor. of Th. 4]; thus $\ker \delta = \{0\}$ and (N_4) holds trivially. However, (N_2) does not hold, since it would imply that the operator $u^*u - uu^* \in A' = A''' = B'$ is a scalar multiple of the identity, which it is not. It follows (see Theorem 2) that, even for a C^*-algebra, $(N_4) \not\Rightarrow (N_5)$, $(N_1) \not\Rightarrow (N_2)$, etc. For a von Neumann algebra, all five conditions coalesce:

THEOREM 3. *If A is a C^*-algebra all of whose derivations are inner, then the conditions* (N_1)-(N_5) *are equivalent.*

PROOF. It will suffice to show that $(N_1) \Rightarrow (N_5)$. Say $\delta = \delta_a$, $a \in A$. If δ satisfies (N_1), then from $\delta a = 0$ we infer that $\delta(a^*) = 0$, thus a is normal, therefore the commutant of a in A is a $*$-subring of A (Fuglede's theorem); thus (N_5) holds with $B = A$. ■

Examples of C^*-algebras all of whose derivations are inner: any von Neumann algebra [11, Ch. III, §9, n° 3, Th. 1]; more generally, any AW^*-algebra [14]; any simple C^*-algebra with unity [20, Th. 4.1.11].

C^*-algebras are algebras of bounded operators in Hilbert space; one avenue for further exploration is to move to algebras of unbounded operators. The next results make a slight incursion into the problem. If A is a finite AW^*-algebra, we write C for the regular ring of A ([4], [8, Ch. 8]). {Theorems 4 and 6 appear to hold for A an arbitrary AW^*-algebra, with C the ring of "measurable operators" ([18], [19], [7]), but I have not checked every detail. At any rate, C is regular if and only if A is finite [18, Th. 6.2]; the special interest of the finite case is that C is then the maximal ring of right quotients of A ([12, p. 158, Th. 2 and p. 160, Th. 2], [16]).}

LEMMA. *Let A be a finite AW^*-algebra, C its regular ring. If δ is a derivation of C such that $\delta | A = 0$, then $\delta = 0$.*

PROOF. Let $x \in C$ and let e_n be a sequence of projections in A, with supremum 1, such that $xe_n \in A$ for all n [8, §48, Prop. 1]; then $0 = \delta(xe_n) = (\delta x)e_n + x(\delta e_n) = (\delta x)e_n$ for all n, therefore $\delta x = 0$. ■

THEOREM 4. *Let A be a finite AW^*-algebra, C its regular ring. The derivations δ of C such that $\delta(A) \subset A$ are the inner derivations $\delta = \delta_a$ with $a \in A$.*

PROOF. Suppose δ is a derivation of C with $\delta(A) \subset A$. Then $\delta | A$ is a derivation of A, hence is inner by an element $a \in A$ [14]: $\delta x = ax - xa$ for all $x \in A$. Then $\delta - \delta_a$ vanishes on A, hence is identically zero by the Lemma. ■

PROBLEM. Is every derivation of C inner? At any rate, it is easy to exhibit inner derivations δ of C such that $\delta(A) \not\subset A$:

EXAMPLE 2. With A and C as in Theorem 4, write $Z(C) = C \cap C'$ for the center of C. Suppose $c \in C$. In order that $\delta_c(A) \subset A$, it is necessary and sufficient that $c \in A + Z(C)$. {Proof: If $\delta_c(A) \subset A$ then by Theorem 4, $\delta_c = \delta_a$ for some $a \in A$; thus $\delta_{c-a} = 0$, $c - a \in Z(C)$.} One can identify $Z(C)$ with the regular ring of the center $Z(A)$ of A [4, Th. 9.2]; thus if A is factorial (i.e., has scalar center) then so is C, in which case $A + Z(C) = A$. In particular, if A is a factor of type II_1, then $A + Z(C) = A \neq C$; thus for $c \in C$ one has $\delta_c(A) \subset A$ if and only if $c \in A$. Here is a characterization of the finite AW^*-algebras A (necessarily of type I) such that every inner derivation of C is implemented by an element of A:

THEOREM 5. *Let A be a finite AW^*-algebra, C its regular ring, $Z(C)$ the center of C. The following conditions on A are equivalent*:

(a) *A is a Lie ideal of C (that is, $[C, A] \subset A$);*

(b) *$A + Z(C) = C$;*

(c) *$A = F \oplus K$ with F finite-dimensional and K abelian.*

PROOF. (a)$\Rightarrow$(b): Let $c \in C$. By (a), $\delta_c(A) \subset A$, therefore $c \in A + Z(C)$ as remarked in Example 2.

(b)$\Rightarrow$(a): Obvious.

For any finite AW^*-algebra B, let us write C_B for its regular ring; it is easy to see that $B = C_B$ if and only if B is finite-dimensional. For any ring B, write $Z(B)$ for the center of B.

(c)$\Rightarrow$(b): Suppose that the algebra A is a direct sum $A = F \oplus K$ with F a finite-dimensional algebra and K an abelian algebra. Then [6, p. 177, Lemma] $C_A = C_F \oplus C_K = F \oplus C_K$, where C_K is abelian, therefore $Z(C_A) = Z(F) \oplus C_K$; thus $A + Z(C_A) = (F \oplus K) + (Z(F) \oplus C_K) = F \oplus C_K = C_A$.

(b) $\Rightarrow$ (c): For a finite AW^*-algebra B, consider the condition

$$(*) \qquad B + Z(C_B) \neq C_B .$$

REMARK 1. If e is a projection of A such that the "corner" eAe of A satisfies $(*)$, then A satisfies $(*)$. For, writing $C = C_A$, we have $C_{eAe} = eCe$ [4, Th. 9.4] and $Z(eCe) = eZ(C)e$. {Proof: One has $Z(eCe) = Z(C_{eAe}) = C_{Z(eAe)} = C_{eZ(A)e} = eC_{Z(A)}e = eZ(C)e$ by [4, Ths. 9.2 and 9.4] and [8, §6, Cor. 2 of Prop. 4].} If one had $A + Z(C) = C$, it would follow that $eAe + eZ(C)e = eCe$, thus $eAe + Z(eCe) = eCe$, that is, $eAe + Z(C_{eAe}) = C_{eAe}$, contrary to supposition.

In particular, if some direct summand of A satisfies $(*)$, then so does A.

REMARK 2. If A is infinite-dimensional (in other words, if $A \neq C$) and n is an integer $\geqq 2$, then the algebra A_n of $n \times n$ matrices over A satisfies $(*)$. For, writing $C = C_A$, C_n is the regular ring of A_n (cf. [5, p. 43, Remark 2], [4, Section 11], [8, §52, Prop. 3]); since the elements of $Z(C_n)$ are diagonal matrices, $A_n + Z(C_n)$ consists of matrices whose off-diagonal elements are in A, hence it cannot exhaust C_n.

Suppose now that $A + Z(C) = C$. Write A as the sum of a type I algebra and a type II algebra [8, §15, Th. 2]. The type II summand must be zero; otherwise, it could be written as a 2×2 matrix algebra over a type II algebra [8, §19, Cor. of Th. 1], it would satisfy $(*)$ by Remark 2, hence A would satisfy $(*)$ by Remark 1, contrary to supposition. Thus A is of type I. Decompose A into homogeneous summands, each of which is a full matrix algebra over an abelian AW^*-algebra:

$$A = K_1^1 \oplus K_2^2 \oplus K_3^3 \oplus \cdots ,$$

where K_n^n is the algebra of $n \times n$ matrices over an abelian algebra K^n [8, §18, Th. 2]. Since A does not satisfy $(*)$, no summand of A can satisfy $(*)$ (Remark 1); therefore K^n is finite-dimensional for all $n \geqq 2$ (Remark 2). It will suffice to show that the number of summands is finite. Every nonzero summand for $n \geqq 2$ contains a pair of nonzero, orthogonal equivalent projections; if there were infinitely many nonzero summands, there would exist in A a pair of orthogonal equivalent projections f, g with fAf infinite-dimensional; then, for the projection $e = f + g$, the corner $eAe = (fAf)_2$ of A would satisfy $(*)$ (Remark 2), therefore A would satisfy $(*)$ (Remark 1), contrary to supposition. ■

REMARK 1. One can regard C as an algebra of "unbounded operators" affiliated with A [4, Section 2]. We remark that E. Christensen has considered the derivations of a concretely represented C^*-algebra that are implemented by unbounded operators "affiliated" with the C^*-algebra [10, Prop. 2.1].

REMARK 2. If B is a C^*-algebra with unity and A is a C^*-subalgebra of B such that $uAu^* = A$ for every unitary element u of B, then A is a Lie ideal of B, that is, $ab - ba \in A$ for all $a \in A$ and $b \in B$ (in other words, A is invariant under every inner derivation of B) [1, Prop. 5.2]. The situation is quite different for a finite AW^*-algebra A and its regular overring C: every unitary element u of C belongs to A [4, Th. 5.2], hence satisfies $uAu^* = A$; but A is a Lie ideal of C only under the conditions of Theorem 5.

To explore the conditions (N_1)-(N_5) for derivations of C, one wants to know whether Fuglede's theorem holds in C. Here is a fragmentary result (an improvement on [6, Ths. 5 and 7]):

THEOREM 6. *Let A be a finite AW^*-algebra, C its regular ring. If $az = za$, where $z \in C$ is normal and $a \in A$, then $az^* = z^*a$.*

PROOF. The proof is inspired by an argument of M. Rosenblum [17]. Write $z = \langle z_n, e_n \rangle$ with z_n, e_n lying in a commutative AW^*-subalgebra of A, (e_n) being a sequence of projections with $e_n \uparrow 1$ and $z_n e_m = z_m e_m$ for $m < n$ [4, Cor. 4.1]. {For the case of an arbitrary AW^*-algebra, see [18, Th. 5.2].} Replacing z_n by $z_n e_n$, we can suppose that $z_n e_n = z_n$. Then $ze_n = z_n$ [8, §48, Prop. 1], therefore

$$(e_n a e_n) z_n = e_n (az) e_n = e_n (za) e_n = z_n (e_n a e_n) ;$$

by Fuglede's theorem (in A), $(e_n a e_n) z_n^* = z_n^* (e_n a e_n)$, thus $e_n (az^*) e_n = e_n (z^* a) e_n$ for all n, whence $az^* = z^*a$. ■

COROLLARY. *Let A be a finite AW^*-algebra, C its regular ring. If z_1, z_2 are normal elements of C and if $b \in A$ satisfies $bz_1 = z_2 b$, then $bz_1^* = z_2^* b$.*

PROOF. The algebra C_2 of 2×2 matrices over C is the regular ring of A_2 [5]; apply Theorem 6 to the elements $a \in A_2$, $z \in C_2$ given by the matrices

$$\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \quad \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

[cf. 6, Th. 2]. ■

PROBLEM. In Theorem 6, need one assume that $a \in A$? {In other words, in the jargon of [6], is C an FT-ring (hence a PT-ring)? It would suffice, by the argument in the proof of Theorem 6, to show that $xz = zx$ implies $xz^* = z^*x$ when $z \in A$ is normal and $x \in C$.} A possibly more tractable special case: if y and z are normal elements of C such that $yz = zy$, does it follow that $yz^* = z^*y$?

Our final theorem pertains to the normality of inner derivations of C.

LEMMA. *Let A be a finite AW^*-algebra, C its regular ring. The equation $x^*x - xx^* = 1$ has no solution in C.*

PROOF. Assume to the contrary that $x \in C$ satisfies $x^*x - xx^* = 1$. Then $x^*x = 1 + xx^*$ shows that the right projection of x is 1, therefore x is invertible in C. Write $x = ur$ with $u \in A$ unitary and $r \geqq 0$ [4, Cor. 7.4]. Then $r^2 = 1 + ur^2u^*$, thus $ur^2u^* = r^2 - 1$; transforming again by u, one has $u^2r^2(u^2)^* = ur^2u^* - 1 = r^2 - 2$. Inductively, $u^nr^2(u^n)^* = r^2 - n$, therefore $r^2 + n$ is unitarily equivalent to r^2, therefore $(r^2 + n)^{-1}$ is unitarily equivalent to $(r^2)^{-1}$; but $r^2 + n \geqq n$, whence $(r^2 + n)^{-1} \leqq 1/n$ [6, Th. 6]; thus $(r^2 + n)^{-1}$ and $(r^2)^{-1}$ are in A [4, Lemma 5.1] and $||(r^2)^{-1}|| = ||(r^2 + n)^{-1}|| \leqq 1/n$ for all n, which is absurd. ■

PROBLEM. Is the equation $yx - xy = 1$ solvable in C? {If A is a finite AW^*-algebra of type I, then C has a center-valued trace [6, Th. 5] and the answer is obviously negative. The equation $yx - xy = 1$ is not solvable in a Banach algebra (Wintner's theorem [13, Prob. 182]).}

Analogous to Putnam's theorem (lemma to Theorem 2), we have:

THEOREM 7. *Let A be a finite AW^*-algebra, C its regular ring, $x \in C$. If x commutes with $x^*x - xx^*$, then x is normal.*

PROOF. Write $z = x^*x - xx^*$ and let $D = \{z\}'$ be the commutant of z in C; then D is a $*$-subalgebra of C containing x, $D = D''$, and z is a self-adjoint element of the center of D. Let $B = D \cap A$; then B is an AW^*-subalgebra of A, whose regular ring may be identified with D [4, Th. 9.3]. Thus, dropping down to B, D and changing notation, we can suppose that the element $z = x^*x - xx^*$ belongs to the center of C.

One then has $z = \langle z_n, e_n \rangle$ for suitable elements z_n, e_n in the center of A, the e_n being projections such that $e_n \uparrow 1$ and $z_ne_n = z_n$ [4, Th. 4.2]. It will suffice to show that $z_n = 0$ for all n. Assume to the contrary that $z_n \neq 0$ for some n. Then $(xe_n)^*(xe_n) - (xe_n)(xe_n)^* = ze_n = z_n \neq 0$. Let h be a central projection such that z_nh is invertible in Ah and such that either $z_nh \geqq 0$ or $z_nh \leqq 0$. Interchanging x and x^* if necessary, we can suppose $z_nh \geqq 0$. Since $z_ne_n = z_n$, necessarily $h \leqq e_n$. Then $(xh)^*(xh) - (xh)(xh)^* = z_nh$; dropping down to Ah and changing notation, we can suppose that $z \in A$ and that z is positive, central and invertible. Write $z = r^2$ with r self-adjoint, central and invertible and let s be the inverse of r; then $(xs)^*(xs) - (xs)(xs)^* = zs^2 = r^2s^2 = 1$, contradicting the lemma. ■

COROLLARY. *Let A be a finite AW^*-algebra, C its regular ring, $x \in C$, and $\delta = \delta_x$ the inner derivation of C implemented by x. Then $\delta^*\delta = \delta\delta^*$ if and only if x is normal.*

PROOF. If x is normal, then $\delta^*\delta - \delta\delta^* = \delta_{[x^*,x]} = 0$. Conversely, if $\delta^*\delta - \delta\delta^* = 0$, that is, if $\delta_{[x^*,x]} = 0$, then $[x^*, x]$ belongs to the center of C, therefore x is normal by Theorem 7. ■

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