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THE OUTRADIUS OF THE TEICHMÜLLER SPACE

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1. Let D be the complement of the closed unit disc in the Riemann sphere $\hat{C} = C \cup \{\infty\}$. We set $k(z) = z/(1-z)^2$, which is the Koebe extremal function and plays an important role in the theory of conformal mappings. It is a schlicht meromorphic function in D and has the Schwarzian derivative $[k](z) = (k''(z)/k'(z))' - (1/2)(k''(z)/k'(z))^2 = -6/(1-z^2)^2$.

Let ρ be the Poincaré metric of D. We denote by $B(D)$ the Banach space of holomorphic functions ϕ defined in D which satisfy the growth condition

$$\begin{aligned}||\phi|| &= \sup\{\rho(z)^{-2}|\phi(z)|;\ z \in D\} \\ &= \sup\{(|z|^2-1)^2|\phi(z)|;\ z \in D\} < \infty\ .\end{aligned}$$

Let Γ be a Fuchsian group acting on D. We denote by $B(D, \Gamma)$ the closed subspace of $B(D)$ consisting of those $\phi \in B(D)$ which satisfy the functional equation

$$\phi(T(z))(T'(z))^2 = \phi(z)\ , \quad T \in \Gamma\ .$$

This space $B(D, \Gamma)$ is finite-dimensional if and only if Γ is a finitely generated Fuchsian group of the first kind.

2. First we prove the following.

THEOREM 1. *If $[k]$ belongs to $B(D, \Gamma)$ for a Fuchsian group Γ acting on D, then the limit set $\Lambda(\Gamma)$ of Γ is empty or consists of two points.*

PROOF. We define Γ^* as the set of all Möbius transformations T leaving D invariant and satisfying the functional equation

$$[k \circ T] = ([k] \circ T)(T')^2 = [k]\ . \tag{1}$$

Since a Möbius transformation T leaving D invariant is of the form

$$T(z) = \varepsilon \frac{z-\alpha}{1-\bar{\alpha}z}\ ,$$

where $|\varepsilon| = 1$ and $|\alpha| < 1$, the equation (1) can be written as

$$(2)\qquad \left[\frac{\varepsilon(1-|\alpha|^2)}{1-\varepsilon^2\alpha^2}\right]^2\left(1-\frac{\varepsilon+\bar{\alpha}}{1+\varepsilon\alpha}z\right)^{-2}\left(1+\frac{\varepsilon-\bar{\alpha}}{1-\varepsilon\alpha}z\right)^{-2}=(1-z^2)^{-2}.$$

If T is a transformation belonging to Γ^*, then (2) yields

$$\left[\frac{\varepsilon(1-|\alpha|^2)}{1-\varepsilon^2\alpha^2}\right]^2=1$$

and

$$\frac{\varepsilon+\bar{\alpha}}{1+\varepsilon\alpha}=\frac{\varepsilon-\bar{\alpha}}{1-\varepsilon\alpha}=\pm 1.$$

Hence we have $\varepsilon=\pm 1$ and $\alpha=\bar{\alpha}$. Therefore Γ^* consists of transformations of the following two types:

$$T_1(r)(z)=\frac{z-r}{1-rz},\quad -1<r<1,$$

$$T_2(s)(z)=-\frac{z-s}{1-sz},\quad -1<s<1.$$

Here $T_1(r)$ is a hyperbolic transformation and $T_2(s)$ is an elliptic transformation of order two. It is easily seen that Γ^* is a group.

Now Γ is a subgroup of Γ^*. If Γ contains only elliptic transformations, Γ is an elliptic cyclic group of order two. Indeed, we have

$$T_2(s_1)\circ T_2(s_2)=T_1((s_2-s_1)/(1-s_1s_2)).$$

Hence $\Lambda(\Gamma)$ is empty. If Γ contains a hyperbolic transformation, $\Lambda(\Gamma)$ is the closure of the set of the fixed points of hyperbolic transformations in Γ. Hence $\Lambda(\Gamma)$ consists of two points, for the fixed points of $T_1(r)$ are 1 and -1 for any r.

3. In this section we state an application of Theorem 1.

The universal Teichmüller space $T(1)$ may be defined as the set of functions $\phi\in B(D)$ which are Schwarzian derivatives of schlicht meromorphic functions in D admitting quasiconformal extensions to $\hat{C}$. It is well known that $T(1)$ is a bounded domain in $B(D)$.

Let Γ be a Fuchsian group acting on D. The Teichmüller space of Γ, $T(\Gamma)$, may be defined as the connected component of $T(1)\cap B(D,\Gamma)$ which contains the origin of $B(D,\Gamma)$. For a Fuchsian group Γ with $\dim T(\Gamma)>0$, we define the outradius $o(\Gamma)$ of $T(\Gamma)$ by

$$o(\Gamma)=\sup\{\|\phi\|;\phi\in T(\Gamma)\}.$$

It follows from well-known results of Nehari, Earle, and Hille that

$$2 < o(\Gamma) \leqq 6 \text{ and } o(1) = 6 .$$

By using Theorem 1 we can obtain the following, which we shall prove in § 5.

THEOREM 2. *If Γ is a finitely generated Fuchsian group of the first kind, then $o(\Gamma)$ is strictly less than* 6.

According to a result of Chu [3], the value 6 in the above theorem cannot be replaced by a smaller constant.

4. In this section we prove two lemmas necessary in § 5.

LEMMA 1 (Bers [2], Proposition 8). *The set of Schwarzian derivatives of schlicht meromorphic functions in D is closed in $B(D)$.*

LEMMA 2. *Let f be a schlicht meromorphic function defined in D and $[f]$ its Schwarzian derivative. Assume that*

$$||[f]|| = \rho(z_0)^{-2}|[f](z_0)| = 6$$

for some point $z_0 \in D$. Then there exists a Möbius transformation S leaving D invariant such that $[f] = [k \circ S]$.

PROOF. We follow carefully an argument of Nehari [6]. First we set $U(z) = (1 - \bar{z}_0 z)/(z - z_0)$ if $|z_0| < \infty$ and $U(z) = z$ if $z_0 = \infty$. For a suitably chosen Möbius transformation η, $F = \eta \circ f \circ U^{-1}$ is expanded in D as follows:

$$F(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots .$$

Using the formula $[f] = [\eta \circ f] = [F \circ U] = ([F] \circ U)(U')^2$, we have $\rho(z_0)^{-2}|[f](z_0)| = 6|b_1|$. Then our assumption means $|b_1| = 1$. Hence it follows from the classical Bieberbach's area theorem that $b_n = 0 \, (n = 2, 3, \cdots)$. Therefore we have $[F](z) = -6b_1/(z^2 - b_1)^2$. If we set $T(z) = \varepsilon z$ $(b_1 = \varepsilon^{-2})$, then $S = T \circ U$ is a required transformation.

5. PROOF OF THEOREM 2. Suppose that $o(\Gamma) = 6$. Then there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ in $T(\Gamma)$ such that $\lim_{n\to\infty} ||\phi_n|| = 6$. Since $\dim T(\Gamma) = \dim B(D, \Gamma) < \infty$, we may assume that $\lim_{n\to\infty} \phi_n = \phi$ for some $\phi \in B(D, \Gamma)$ with $||\phi|| = 6$. We see from Lemma 1 that ϕ is the Schwarzian derivative of a schlicht meromorphic function defined in D.

Now let N be a normal polygon for Γ relative to D, $\bar{N}$ its closure in $\hat{\boldsymbol{C}}$ and ∂D the unit circle in $\boldsymbol{C}$. Since Γ is a finitely generated Fuchsian group of the first kind, $\partial D \cap \bar{N}$ consists of at most finitely many points, say $\zeta_1, \zeta_2, \cdots, \zeta_m$, which are so-called parabolic cusps of Γ. As

ϕ is a cusp form for Γ, it holds that $\lim_{\bar{N}\cap D\ni z, z\to\zeta_i}\phi(z) = 0$ $(i = 1, 2, \cdots, m)$. Hence we see $\lim_{\bar{N}\cap D\ni z, z\to\zeta_i}\rho(z)^{-2}|\phi(z)| = 0$ $(i = 1, 2, \cdots, m)$. On the other hand, we have $||\phi|| = \sup\{\rho(z)^{-2}|\phi(z)|; z\in\bar{N}\cap D\}$. Therefore it follows that $||\phi|| = \rho(z_0)^{-2}|\phi(z_0)|$ for some point $z_0\in\bar{N}\cap D$. We conclude by Lemma 2 that there is a Möbius transformation S leaving D invariant such that $\phi = [k\circ S]$. It can be seen easily that $[k\circ S]$ belongs to $B(D, \Gamma)$ if and only if $[k]$ belongs to $B(D, S\Gamma S^{-1})$. Therefore Theorem 1 implies that $\Lambda(S\Gamma S^{-1})$ (and hence $\Lambda(\Gamma)$ also) is empty or consists of two points. This contradicts our assumption that $\Lambda(\Gamma)$ coincides with ∂D, and the theorem is proved.

6. Let K be the one-dimensional subspace of $B(D)$ which is spanned by $[k]$. The fact that $o(1) = 6$ is proved by considering the intersection of K and $T(1)$ (see Chu [3]).

First we state a result of Hille [4]. We set

$$f(z) = \left(\frac{z-1}{z+1}\right)^{\delta}, \quad \delta = (1-\alpha)^{1/2},$$

where $f(\infty) = 1$ and the square root is 1 for $\alpha = 0$. Then

$$[f](z) = \frac{2\alpha}{(1-z^2)^2} = -\frac{\alpha}{3}[k](z)$$

and f is schlicht in D if and only if α lies in the interior or on the boundary of the cardioid

$$(3) \qquad \alpha = -2e^{\sqrt{-1}\theta} - e^{2\sqrt{-1}\theta}, \quad -\pi < \theta \leqq \pi.$$

Let V be the interior of the cardioid (3) and R the right half-plane $\{z = x + \sqrt{-1}y \in \boldsymbol{C}; x > 0\}$ of the complex plane $\boldsymbol{C}$. We set $\zeta = (z-1)/(z+1)$ and $w = g(\zeta) = f(z) = \zeta^{\delta}$.

Kalme [5] showed the following theorem by proving that $f(z) = \zeta^{\delta}$ is quasiconformally extendable to $\hat{\boldsymbol{C}}$ for $\alpha\in V$.

THEOREM 3. *The set* $\{\alpha\in\boldsymbol{C}; -(\alpha/3)[k]\in T(1)\}$ *coincides with the interior of the cardioid* (3).

Here we shall give an alternative proof of this theorem.

The universal Teichmüller space $T(1)$ can be defined as the set of functions $\phi\in B(D)$ which are Schwarzian derivatives of schlicht meromorphic functions in D whose images of D are bordered by quasi-circles. Furthermore, Ahlfors [1] gave geometric characterization of quasi-circles. Therefore, we have only to show that the boundary of the domain $f(D) = g(R)$ is a quasi-circle for any $\alpha\in V$.

Now, for any $\alpha\in V$, the boundary of the domain $g(R)$ is a Jordan curve given by

$$w = g(y) = \exp\left[\left(\mu \log|y| - \frac{\nu\pi}{2}\operatorname{sign}(y)\right) + \sqrt{-1}\left(\nu \log|y| + \frac{\mu\pi}{2}\operatorname{sign}(y)\right)\right], \quad -\infty < y < \infty, \tag{4}$$

where sign (y) is the sign of y and $\delta = \mu + \sqrt{-1}\nu$ $(\mu > 0)$. Hence, by Ahlfors's result [1], we have to show that there exists a constant M satisfying

$$\left|\frac{g(y_1) - g(y_2)}{g(y_1) - g(y_3)}\right| \leqq M \tag{5}$$

for any $y_1, y_2, y_3 (y_1 < y_2 < y_3)$. There occur seven cases according as $y_i (i = 1, 2, 3)$ is positive, zero or negative. Here we show (5) only in the case $0 < y_1 < y_2 < y_3$ and omit the details for the other cases. If $0 < y_1 < y_2 < y_3$, then (4) gives

$$\left|\frac{g(y_1) - g(y_2)}{g(y_1) - g(y_3)}\right|^2 = \frac{h(y_2/y_1)}{h(y_3/y_1)}, \tag{6}$$

where

$$h(x) = 1 + x^{2\mu} - 2x^{\mu}\cos(\nu \log x), \quad x > 1.$$

It can be seen easily that $h(x)$ has the following properties: (i) $h(x) > 0$ for $x > 1$ (which follows from the fact that a curve given by (4) is a Jordan curve) and (ii) $h(x)$ is monotone increasing on both intervals $(1, 1 + \varepsilon)$ and (N, ∞) for a sufficiently small $\varepsilon > 0$ and a sufficiently large $N > 0$. Therefore, by (6), (5) holds for some constant M.

7. REMARKS. 1. The proof of Theorem 3, mainly due to Hille [4], contains the proof of the fact that $o(1) = 6$ (see Chu [3]).

2. Theorem 3 and the proof of Theorem 1 imply that $o(\Gamma) = 6$ for a Fuchsian group Γ which is a subgroup of the group Γ^* introduced in the proof of Theorem 1.

3. According to Theorem 1, no points of $K \cap T(1)$ except the origin of $B(1)$ belong to any Teichmüller space $T(\Gamma)$ with dim $T(\Gamma) < \infty$.

4. Let Γ be a Fuchsian group acting on D and A a Möbius transformation leaving D invariant. Then the mapping χ which takes $\phi \in B(D, \Gamma)$ to $(\phi \circ A)(A')^2 \in B(D, A\Gamma A^{-1})$ is a norm-preserving linear isomorphism and the image $\chi(T(\Gamma))$ of $T(\Gamma)$ under χ coincides with $T(A\Gamma A^{-1})$. In particular, we have $o(A\Gamma A^{-1}) = o(\Gamma)$.

5. By setting $\Gamma = 1$ in Remark 4, we see that Theorem 3 also

holds if we substitute $[k\circ A]$ for $[k]$, where A is a Möbius transformation leaving D invariant.

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