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# $\sigma$-HYPERSURFACES IN A LOCALLY SYMMETRIC ALMOST HERMITIAN MANIFOLD

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**0. Introduction.** Let $\tilde{M}^{n+p}$ be an $n+p$-dimensional $C^\infty$ Riemannian manifold with metric tensor $\tilde{g}$ and Levi-Civita connection $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}^{n+p}$ is given by $\tilde{R}(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$ for any $X, Y \in \mathfrak{X}(\tilde{M})$ where $\mathfrak{X}(\tilde{M})$ is the Lie algebra of $C^\infty$ vector fields in $\tilde{M}^{n+p}$.

Moreover, let $M^n$ be an $n$-dimensional submanifold immersed in $\tilde{M}^{n+p}$. Then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \qquad \text{for any} \quad X, Y \in \mathfrak{X}(M) \tag{0.1}$$

where $\nabla_X Y$ and $\sigma$ denote the component of $\tilde{\nabla}_X Y$ tangent to $M^n$ and the second fundamental form of $M^n$ in $\tilde{M}^{n+p}$, respectively. It is well known that $\nabla$ is the covariant differentiation of $M^n$ and $\sigma$ is a symmetric covariant tensor field of degree 2 with values in the normal bundle $T(M)^\perp$ of $M^n$ where $T(M)$ denotes the tangent bundle of $M^n$.

We have further

$$\tilde{\nabla}_X \xi_\alpha = -A_\alpha X + \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta \qquad (\alpha = 1, 2, \cdots, p) \tag{0.2}$$

where $\{\xi_\alpha\}$ is a local orthonormal frame field for $T(M)$ and $-A_\alpha X$ is the tangential component of $\tilde{\nabla}_X \xi_\alpha$.

Let $\nabla'$ be the covariant differentiation with respect to the connection in $T(M) \oplus T(M)^\perp$. Then we have

$$(\nabla'_X \sigma)(Y, Z) = (\tilde{\nabla}_X \sigma(Y, Z))^\perp - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{0.3}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

If $\nabla'_X \sigma = 0$ for any $X \in \mathfrak{X}(M)$, then the second fundamental form is said to be parallel. $M^n$ is said to be curvature invariant if $\tilde{R}(X, Y)Z$ belongs to $\mathfrak{X}(M)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

Next, let $\tilde{M}^{2m+2q}$ be a $2m+2q$-dimensional almost Hermitian manifold with an almost Hermitian structure $(\tilde{J}, \tilde{g})$. Then a $2m$-dimensional invariant submanifold $M^{2m}$ of $\tilde{M}^{2m+2q}$ is said to be a $\sigma$-submanifold if the second fundamental form $\sigma$ is complex bilinear, i.e., (0.4) $\sigma(JX, Y) = \sigma(X, JY) = \tilde{J}\sigma(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ ,

where $J$ is the induced almost complex structure on $M^{2m}$. In particular, if $M^{2m}$ is a $\sigma$-hypersurface of $\tilde{M}^{2m+2}$, then the condition (0.4) is equivalent to $B = JA$ and $AJ = -JA$, where $A$ and $B$ are the second fundamental tensors with respect to any unit normal vectors $\xi$ and $\tilde{J}\xi$ to $M^{2m}$, respectively.

An almost Hermitian manifold $\tilde{M}$ is called an *$O$-space (or quasi-Kähler manifold) [2] if

$$(\tilde{\nabla}_X \tilde{J})Y + (\tilde{\nabla}_{\tilde{J}X} \tilde{J})\tilde{J}Y = 0 \tag{0.5}$$

for any $X, Y \in \mathfrak{X}(\tilde{M})$ and $\tilde{M}$ is called a $K$-space (or Tachibana space or nearly Kähler manifold) if

$$(\tilde{\nabla}_X \tilde{J})Y + (\tilde{\nabla}_Y \tilde{J})X = 0 \quad (\text{or equivalently } (\tilde{\nabla}_X \tilde{J})X = 0) \tag{0.6}$$

for any $X, Y \in \mathfrak{X}(\tilde{M})$.

It is well known that a Kähler manifold is a $K$-space and a $K$-space is an *$O$-space. It is also well known that an invariant hypersurface of a $K$-space or an *$O$-space is a $K$-space or an *$O$-space respectively. Moreover, we know that an invariant hypersurface of a Kähler manifold or a $K$-space is a $\sigma$-hypersurface (see for example [3]).

The following theorem is well known.

THEOREM A (B. Smyth [4], T. Takahashi [5]). *Let $M^{2m}$ be an invariant hypersurface of a Kähler manifold $\tilde{M}^{2m+2}$ of constant holomorphic sectional curvature. If $M^{2m}$ is an Einstein (or Ricci parallel) manifold, then $M^{2m}$ is locally symmetric.*

A Kähler manifold $\tilde{M}^{2m+2}$ of constant holomorphic sectional curvature is locally symmetric and its invariant hypersurface $M^{2m}$ is curvature invariant (see for example [4]). What will become of this theorem if we replace the assumption of being of constant holomorphic sectional curvature by being locally symmetric? Our main result is

THEOREM. *Let $M^{2m}$ be a $\sigma$-hypersurface of a locally symmetric *$O$-space $\tilde{M}^{2m+2}$. If $M^{2m}$ is Ricci parallel and curvature invariant, then $M^{2m}$ is locally symmetric.*

COROLLARY. *Let $M^{2m}$ be an invariant hypersurface of a locally symmetric Kähler manifold $\tilde{M}^{2m+2}$. If $M^{2m}$ is Ricci parallel and curvature invariant, then $M^{2m}$ is locally symmetric.*

This generalizes Theorem A.

**1. Submanifolds of a Riemannian manifold.** Let $M^n$ be a submanifold immersed in a Riemannian manifold $\tilde{M}^{n+p}$ and put

$$\sigma(X, Y) = \sum_{\alpha=1}^{p} h_\alpha(X, Y)\xi_\alpha . \tag{1.1}$$

Then we have

$$\tilde{g}(\sigma(X, Y), \xi_\alpha) = g(A_\alpha X, Y) = g(X, A_\alpha Y) = h_\alpha(X, Y) \tag{1.2}$$

for any $X, Y \in \mathfrak{X}(M)$ and $\xi_\alpha \in \mathfrak{X}(M)^\perp$ where $g$ denotes the induced metric tensor on $M^n$.

The following two lemmas are well known (see for example [1]).

LEMMA 1.1. *Let $\tilde{R}$ and $R$ be the curvature tensors of $\tilde{M}^{n+p}$ and $M^n$ respectively. Then we have*

$$\begin{aligned}\tilde{R}(X, Y)W = R(X, Y)W &- \sum_{\alpha=1}^{p} h_\alpha(Y, W)A_\alpha X + \sum_{\alpha=1}^{p} h_\alpha(X, W)A_\alpha Y \\ &+ \sum_{\alpha=1}^{p}\Big[(\nabla_X h_\alpha)(Y, W) - (\nabla_Y h_\alpha)(X, W) + \sum_{\beta=1}^{p} h_\beta(Y, W)s_{\beta\alpha}(X) \\ &- \sum_{\beta=1}^{p} h_\beta(X, W)s_{\beta\alpha}(Y)\Big]\xi_\alpha ,\end{aligned} \tag{1.3}$$

$$s_{\alpha\beta}(X) + s_{\beta\alpha}(X) = 0 \qquad (\alpha, \beta = 1, 2, \cdots, p) \tag{1.4}$$

*for any $X, Y, W \in \mathfrak{X}(M)$.*

LEMMA 1.2. *$\nabla'_X\sigma = 0$ is equivalent to $\nabla_X A_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)A_\beta$ $(\alpha = 1, 2, \cdots, p)$.*

The following lemma which plays an important role in proving the main theorem is also easily verified (see for example [6], p. 99, where $\nabla'_X\sigma = 0$ means $\nabla_d h^x_{cb} = 0$).

LEMMA 1.3. *Let $M^n$ be a submanifold immersed in a locally symmetric Riemannian manifold $\tilde{M}^{n+p}$. If the second fundamental form is parallel, then $M^n$ is locally symmetric.*

**2. Invariant hypersurfaces of an almost Hermitian manifold.** Let $M^{2m}$ be an invariant hypersurface of an almost Hermitian manifold $\tilde{M}^{2m+2}$. Then putting

$$\begin{aligned}A = A_1,\ B = A_2,\ \xi = \xi_1,\ J\xi = \xi_2,\ h(X, Y) = h_1(X, Y),\\ k(X, Y) = h_2(X, Y),\ s(X) = s_{12}(X),\ t(X) = s_{21}(X)\\ \text{for any}\quad X, Y \in \mathfrak{X}(M)\end{aligned} \tag{2.1}$$

and rewriting (0.1), (0.2) and (1.4), we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi + k(X, Y)\tilde{J}\xi , \tag{2.2}$$

$$(2.3)\qquad \tilde{\nabla}_X\xi = -AX + s(X)\tilde{J}\xi\ ,\quad \tilde{\nabla}_X(\tilde{J}\xi) = -BX + t(X)\xi\ ,$$

$$(2.4)\qquad s(X) + t(X) = 0\ ,$$

respectively. Here $A$ and $B$ are symmetric tensors with respect to $g$ and from (1.2), we have

$$(2.5)\qquad h(X, Y) = g(AX, Y)\ ,\qquad k(X, Y) = g(BX, Y)\ .$$

Moreover, the equation (1.3) becomes

$$\begin{aligned}(2.6)\qquad \tilde{R}(X, Y)W &= R(X, Y)W - [h(Y, W)AX - h(X, W)AY] \\ &- [k(Y, W)BX - k(X, W)BY] + [(\nabla_X h)(Y, W) - (\nabla_Y h)(X, W) \\ &+ k(Y, W)t(X) - k(X, W)t(Y)]\xi + [(\nabla_X k)(Y, W) \\ &- (\nabla_Y k)(X, W) + h(Y, W)s(X) - h(X, W)s(Y)]\tilde{J}\xi\ .\end{aligned}$$

From (2.6) the following well known lemma follows.

LEMMA 2.1. *Let $M^{2m}$ be an invariant hypersurface of an almost Hermitian manifold $\tilde{M}^{2m+2}$. If $M^{2m}$ is curvature invariant, then we have*

$$\begin{aligned}(2.7)\qquad \tilde{R}(X, Y)W &= R(X, Y)W - [g(AY, W)AX - g(AX, W)AY] \\ &- [g(BY, W)BX - g(BX, W)BY]\ ,\end{aligned}$$

$$(2.8)\qquad (\nabla_X A)Y - (\nabla_Y A)X - s(X)BY + s(Y)BX = 0\ ,$$

$$(2.9)\qquad (\nabla_X B)Y - (\nabla_Y B)X + s(X)AY - s(Y)AX = 0\quad (\textit{Codazzi equation})$$

*for any $X, Y, W \in \mathfrak{X}(M)$.*

For an almost Hermitian manifold $\tilde{M}^{2m+2}$, the Ricci tensor $\tilde{S}(Y, W)$ is given by

$$(2.10)\qquad \tilde{S}(Y, W) = \sum_{i=1}^{m}\tilde{g}(\tilde{R}(e_i, Y)W, e_i) + \sum_{i=1}^{m}\tilde{g}(\tilde{R}(Je_i, Y)W, Je_i) + L(Y, W)$$

for any $Y, W \in \mathfrak{X}(M)$, where $L(Y, W) = \tilde{g}(\tilde{R}(\xi, Y)W, \xi) + \tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \tilde{J}\xi)$ and $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$ is an orthonormal frame field defined on an open set $U$ of $M^{2m}$. It is easily seen that $L(Y, W)$ is a symmetric tensor field of type (0, 2) on $M^{2m}$.

LEMMA 2.2. *Let $M^{2m}$ be an invariant hypersurface of a locally symmetric almost Hermitian manifold $\tilde{M}^{2m+2}$. If $M^{2m}$ is curvature invariant, then we have*

$$\begin{aligned}(2.11)\qquad (\nabla_X L)(Y, W) &= k(X, Y)\tilde{S}(\tilde{J}\xi, W) + k(X, W)\tilde{S}(\tilde{J}\xi, Y) \\ &+ h(X, Y)\tilde{S}(\xi, W) + h(X, W)\tilde{S}(\xi, Y)\end{aligned}$$

*for any $X, Y, W \in \mathfrak{X}(M)$.*

PROOF. We have

$$(\nabla_X L)(Y, W) = X(L(Y, W)) - L(\nabla_X Y, W) - L(Y, \nabla_X W) . \tag{2.12}$$

For the first term of the right hand side of (2.12), we have

$$X(L(Y, W)) = X(\tilde{g}(\tilde{R}(\xi, Y)W, \xi)) + X(\tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \tilde{J}\xi)) . \tag{2.13}$$

Since $M^{2m}$ is curvature invariant, we have

$$\tilde{g}(\tilde{R}(AX, Y)W, \xi) = 0 , \quad \tilde{g}(\tilde{R}(\xi, Y)W, AX) = -\tilde{g}(\tilde{R}(W, AX)Y, \xi) = 0 .$$

Hence, making use of (2.2) and (2.3), we have

$$\begin{aligned} &X(\tilde{g}(\tilde{R}(\xi, Y)W, \xi)) \\ &\quad = s(X)\tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \xi) + \tilde{g}(\tilde{R}(\xi, \nabla_X Y)W, \xi) \\ &\quad\quad + k(X, Y)\tilde{g}(\tilde{R}(\xi, \tilde{J}\xi)W, \xi) + \tilde{g}(\tilde{R}(\xi, Y)\nabla_X W, \xi) \\ &\quad\quad + k(X, W)\tilde{g}(\tilde{R}(\xi, Y)\tilde{J}\xi, \xi) + s(X)\tilde{g}(\tilde{R}(\xi, Y)W, \tilde{J}\xi) . \end{aligned} \tag{2.14}$$

For the third term of the right hand side of (2.14), making use of $\tilde{g}(\tilde{R}(e_i, \tilde{J}\xi)W, e_i) = 0$ and $\tilde{g}(\tilde{R}(Je_i, \tilde{J}\xi)W, Je_i) = 0$, we have

$$\begin{aligned} k(X, Y)\tilde{g}(\tilde{R}(\xi, \tilde{J}\xi)W, \xi) &= k(X, Y)\Big[ \sum_{i=1}^{m} \tilde{g}(\tilde{R}(e_i, \tilde{J}\xi)W, e_i) \\ &\quad + \sum_{i=1}^{m} \tilde{g}(\tilde{R}(Je_i, \tilde{J}\xi)W, Je_i) + \tilde{g}(\tilde{R}(\xi, \tilde{J}\xi)W, \xi) + \tilde{g}(\tilde{R}(\tilde{J}\xi, \tilde{J}\xi)W, \tilde{J}\xi)\Big] \\ &= k(X, Y)\tilde{S}(\tilde{J}\xi, W) . \end{aligned}$$

Similarly, for the fifth term, we have

$$k(X, W)\tilde{g}(\tilde{R}(\xi, Y)\tilde{J}\xi, \xi) = k(X, W)\tilde{S}(\tilde{J}\xi, Y) .$$

Thus, (2.14) turns out to be

$$\begin{aligned} &X(\tilde{g}(\tilde{R}(\xi, Y)W, \xi)) \\ &\quad = s(X)[\tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \xi) + \tilde{g}(\tilde{R}(\xi, Y)W, \tilde{J}\xi)] + \tilde{g}(\tilde{R}(\xi, \nabla_X Y)W, \xi) \\ &\quad\quad + \tilde{g}(\tilde{R}(\xi, Y)\nabla_X W, \xi) + k(X, Y)\tilde{S}(\tilde{J}\xi, W) + k(X, W)\tilde{S}(\tilde{J}\xi, Y) . \end{aligned} \tag{2.15}$$

Similarly, for the second term of the right hand side of (2.13), we have

$$\begin{aligned} &X(\tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \tilde{J}\xi)) \\ &\quad = -s(X)[\tilde{g}(\tilde{R}(\xi, Y)W, \tilde{J}\xi) + \tilde{g}(\tilde{R}(\tilde{J}\xi, Y)W, \xi)] \\ &\quad\quad + \tilde{g}(\tilde{R}(\tilde{J}\xi, \nabla_X Y)W, \tilde{J}\xi) + \tilde{g}(\tilde{R}(\tilde{J}\xi, Y)\nabla_X W, \tilde{J}\xi) \\ &\quad\quad + h(X, Y)\tilde{S}(\xi, W) + h(X, W)\tilde{S}(\xi, Y) . \end{aligned} \tag{2.16}$$

Consequently, by (2.15) and (2.16), (2.12) turns out to be

$$\begin{aligned} (\nabla_X L)(Y, W) &= \tilde{g}(\tilde{R}(\xi, \nabla_X Y)W, \xi) + \tilde{g}(\tilde{R}(\xi, Y)\nabla_X W, \xi) \\ &\quad + \tilde{g}(\tilde{R}(\tilde{J}\xi, \nabla_X Y)W, \tilde{J}\xi) + \tilde{g}(\tilde{R}(\tilde{J}\xi, Y)\nabla_X W, \tilde{J}\xi) - L(\nabla_X Y, W) \end{aligned}$$

$$- L(Y, \nabla_X W) + k(X, Y)\tilde{S}(\tilde{J}\xi, W) + k(X, W)\tilde{S}(\tilde{J}\xi, Y)$$
$$+ h(X, Y)\tilde{S}(\xi, W) + h(X, W)\tilde{S}(\xi, Y) = k(X, Y)\tilde{S}(\tilde{J}\xi, W)$$
$$+ k(X, W)\tilde{S}(\tilde{J}\xi, Y) + h(X, Y)\tilde{S}(\xi, W) + h(X, W)\tilde{S}(\xi, Y) ,$$

because of the definition of $L(Y, W)$.

LEMMA 2.3. *Let $M^{2m}$ be a curvature invariant σ-hypersurface of an *O-space $\tilde{M}^{2m+2}$. Then we have*

$$(2.17) \qquad (\nabla_X J)AY = 0 , \qquad A(\nabla_X J)Y = 0 ,$$

$$(2.18) \qquad (\nabla_X A)JY = - J(\nabla_X A)Y$$

*for any $X, Y \in \mathfrak{X}(M)$.*

PROOF. Substituting $B = JA$ into (2.9), we have

$$J(\nabla_X A)Y + (\nabla_X J)AY - J(\nabla_Y A)X - (\nabla_Y J)AX + s(X)AY - s(Y)AX = 0 .$$

Applying $-J$, we have

$$(2.19) \qquad (\nabla_X A)Y - (\nabla_Y A)X - s(X)JAY + s(Y)JAX - J[(\nabla_X J)AY - (\nabla_Y J)AX] = 0 .$$

Comparing (2.8) with (2.19), we have

$$(2.20) \qquad (\nabla_X J)AY = (\nabla_Y J)AX .$$

Replacing $Y$ by $JY$, we have

$$(2.21) \qquad (\nabla_X J)AJY = (\nabla_{JY} J)AX .$$

Then, using $J^2 = -I$, we have

$$(\nabla_X J)JAJY = (\nabla_{JY} J)JAX$$

or by $JA = -AJ$

$$(2.22) \qquad (\nabla_X J)AY = (\nabla_{JY} J)JAX .$$

Thus, forming the sum (2.20) + (2.22), by (0.5) we have

$$(2.23) \qquad (\nabla_X J)AY = 0 \qquad \text{for any } X, Y \in \mathfrak{X}(M) .$$

Since $A$ and $J$ are symmetric and skew-symmetric respectively, the other formula of (2.17) follows immediately from (2.23). For (2.18), by (2.17) and $JA = -AJ$, we have

$$(\nabla_X A)JY = \nabla_X(AJY) - A\nabla_X(JY) = -\nabla_X(JAY) - A\nabla_X(JY)$$
$$= -(\nabla_X J)AY - J(\nabla_X A)Y - JA(\nabla_X Y) - A(\nabla_X J)Y - AJ(\nabla_X Y)$$
$$= -J(\nabla_X A)Y .$$

**3. Proof of Theorem.** By Lemma 1.3, it is sufficient to show that $\nabla'_X\sigma = 0$ or by Lemma 1.2,

$$\nabla_X A = s(X)JA\ , \qquad \nabla_X(JA) = -s(X)A\ . \tag{3.1}$$

From (2.7) it follows that the linear endomorphism of $T_y(M)(y \in M^{2m})$ determined by $X \mapsto \tilde{R}(X, Y)W$ has the trace

$$\operatorname{trace}(X \mapsto \tilde{R}(X, Y)W) = S(Y, W) + 2g(A^2Y, W) \tag{3.2}$$

for any $X, Y, W \in T_y(M)$, where $S(Y, W)$ is the Ricci tensor of $M^{2m}$.

Thus, by (2.10), we have

$$\tilde{S}(Y, W) = S(Y, W) + 2g(A^2Y, W) + L(Y, W)\ . \tag{3.3}$$

On the other hand, taking account of the fact that the Ricci tensor $\tilde{S}$ of the locally symmetric manifold $\tilde{M}^{2m+2}$ is parallel, we have

$$\begin{aligned} X\tilde{S}(Y, W) &= \tilde{S}(\nabla_X Y + h(X, Y)\xi + k(X, Y)\tilde{J}\xi, W) + \tilde{S}(Y, \nabla_X W + h(X, W)\xi \\ &\quad + k(X, W)\tilde{J}\xi) \\ &= \tilde{S}(\nabla_X Y, W) + \tilde{S}(Y, \nabla_X W) + h(X, Y)\tilde{S}(\xi, W) + k(X, Y)\tilde{S}(\tilde{J}\xi, W) \\ &\quad + h(X, W)\tilde{S}(Y, \xi) + k(X, W)\tilde{S}(Y, \tilde{J}\xi)\ . \end{aligned}$$

Consequently, operating $\nabla_X$ on both sides of (3.3) and making use of Lemma 2.2 and $\nabla_X S = 0$, we have

$$\nabla_X A^2 = 0\ . \tag{3.4}$$

Then, let us consider the distributions $D^\alpha(\alpha = 1, 2, \cdots, l)$ on a neighborhood $U(x)$ of each point $x \in M^{2m}$ defined by

$$D^\alpha(y) = \{X \in T_y(M);\ A^2X = \lambda_\alpha^2 X\}\ ,$$

where $\lambda_\alpha$ are nonnegative constant with $\lambda_\alpha \neq \lambda_\beta(\alpha \neq \beta)$ and $y \in U(x)$. By (3.4), $D^\alpha(\alpha = 1, 2, \cdots, l)$ is parallel and

$$T_y(M) = D^1(y) \oplus \cdots \oplus D^l(y)$$

on $U(x)$. Furthermore, $D^\alpha(\alpha = 1, 2, \cdots, l)$ are invariant under $J$ by virtue of $JA = -AJ$ and therefore $JA^2 = A^2J$.

Hence, we can take the distributions $D^\alpha_+, D^\alpha_-(\alpha = 1, 2, \cdots, l)$ on $U(x)$ given by

$$D^\alpha_+(y) = \{X \in T_y(M);\ AX = \lambda_\alpha X\}\ ,$$
$$D^\alpha_-(y) = \{X \in T_y(M);\ AX = -\lambda_\alpha X\}\ .$$

Then we have

$$D^{\alpha}(y) = D^{\alpha}_{+}(y) \oplus D^{\alpha}_{-}(y),\ D^{\alpha}_{-}(y) = JD^{\alpha}_{+}(y),\ D^{\alpha}_{+}(y) = JD^{\alpha}_{-}(y) .$$

By (3.4), we have

$$0 = (\nabla_X(AA))Y = A(\nabla_X A)Y + (\nabla_X A)AY ,$$

from which it follows that if $Y \in D^{\alpha}_{+}(y)$, then

$$A(\nabla_X A)Y = -\lambda_{\alpha}(\nabla_X A)Y$$

which means that $(\nabla_X A)Y \in D^{\alpha}_{-}(y)$. Similarly, if $Y \in D^{\alpha}_{-}(y)$, then $(\nabla_X A)Y \in D^{\alpha}_{+}(y)$. Moreover, as is easily seen, if $Y \in D^{\alpha}_{+}(y)$ or $Y \in D^{\alpha}_{-}(y)$, then $(JA)Y \in D^{\alpha}_{-}(y)$ or $(JA)Y \in D^{\alpha}_{+}(y)$, respectively.

Consequently, if $X \in D^{\alpha}_{-}(y)$ and $Y \in D^{\beta}_{+}(y)(\beta = 1, 2, \cdots, l)$, then from the Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X - s(X)JAY + s(Y)JAX = 0 ,$$

we have

$$(3.5) \qquad (\nabla_X A)Y = s(X)JAY .$$

Similarly, when $X \in D^{\alpha}_{+}(y)$ and $Y \in D^{\beta}_{-}(y)$, we also have (3.5).

Next, we consider the case where $X \in D^{\alpha}_{-}(y)$ and $Y \in D^{\beta}_{-}(y)$. $D^{\beta}_{+}(y) = JD^{\beta}_{-}(y)$ means that if $Y \in D^{\beta}_{-}(y)$, then $JY \in D^{\beta}_{+}(y)$. Therefore since $X \in D^{\alpha}_{-}(y)$ and $JY \in D^{\beta}_{+}(y)$, by (3.5) we have

$$(\nabla_X A)JY = s(X)JA(JY) = s(X)AY$$

or by (2.18)

$$-J(\nabla_X A)Y = s(X)AY ,$$

from which we have (3.5).

Similarly, when $X \in D^{\alpha}_{+}(y)$ and $Y \in D^{\beta}_{+}(y)$, we also have (3.5). For the other formula of (3.1), making use of (2.17) and (3.5), we have

$$\begin{aligned} \nabla_X(JA)Y &= (\nabla_X J)AY + J(\nabla_X A)Y \\ &= J(\nabla_X A)Y = Js(X)JAY = -s(X)AY . \end{aligned}$$

Thus, we have $\nabla'_X \sigma = 0$. Consequently, the proof of Theorem is complete.

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