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THE PREPONDERANTLY CONTINUOUS DENJOY INTEGRAL

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1. Introduction. The author has defined the approximately continuous Denjoy integral (AD-integral) which is more general than the Denjoy integral in the wide sense and whose indefinite integral is approximately continuous ([2]). The object of this paper is to generalize the AD-integral by means of the notion of preponderant continuity ([5]).

2. The preponderantly continuous Denjoy integral. We begin by reproducing the definition of the AD-integral for completeness. A real valued function $F(x)$ is said to be (ACG) on the interval $[a, b]$ if $[a, b]$ is a union of countably many closed sets on each of which $F(x)$ is absolutely continuous.

DEFINITION 1 ([2], p. 715). An extended real valued function $f(x)$ is said to be AD-integrable on $[a, b]$ if there exists a function $F(x)$ which is approximately continuous, (ACG) on $[a, b]$ and

$$\mathrm{AD}\, F(x) = f(x) \qquad \text{a.e.},$$

where by AD we mean the approximate derivative. The definite AD-integral of $f(x)$ over $[a, b]$ is defined as $F(b) - F(a)$.

Given a function F defined in a neighborhood of a point x_0, we shall call preponderant upper limit of F at x_0 the lower bound of all the numbers y ($+\infty$ included) for which the set $\{x: F(x) > y\}$ has upper density at x_0 less than 1/2, and we denote this limit by $\mathrm{pr}\limsup_{x\to x_0} F(x)$. The preponderant lower limit of F at x_0, $\mathrm{pr}\liminf_{x\to x_0} F(x)$, is analogously defined. The function F has a preponderant limit at x_0, $\mathrm{pr}\lim_{x\to x_0} F(x)$, if the preponderant upper and lower limits at x_0 coincide. Also a function is preponderantly continuous at a point if the function is equal to its preponderant limit at that point.

Let F be a measurable function defined on a neighborhood of a point x_0 and E be a measurable set with density at x_0 more than 1/2. Then it follows easily that if $\lim_{x\to x_0} F(x) = l (x \in E)$ then $\mathrm{pr}\lim_{x\to x_0} F(x) = l$. Hence the preponderant limit (resp. continuity) is a generalization of the approximate limit (resp. continuity).

We also remark that the class of all approximately continuous functions defined on $[a, b]$, written apC, is a linear space but the family of all preponderantly continuous functions, pC, is not so.

LEMMA 1. *If a non-void closed set E is a union of countably many closed sets E_k, then there exist an interval (l, m) containing points of E and an integer n such that $(l, m)\cdot E \subset E_n$.*

For the proof, see for example [6], p. 143.

LEMMA 2 ([1], p. 543). *Let $\mathscr{F}$ be a system of open intervals in $I_0 = (a, b)$ satisfying the following conditions*:

(i) *If $I_k \in \mathscr{F} (k = 1, 2, \cdots, n)$ and $(\bigcup_{k=1}^{n} \bar{I}_k)^\circ = I$ is an open interval, then $I \in \mathscr{F}$, where $\bar{I}_k$ is the closure of I_k, and E° is the open kernel of E.*

(ii) *If $I \in \mathscr{F}$ and $J \subset I$ (J: open interval), then $J \in \mathscr{F}$.*

(iii) *Let I be an open interval. If any open interval J with $\bar{J} \subset I$ belongs to $\mathscr{F}$, then $I \in \mathscr{F}$.*

(iv) *If $\mathscr{F}_1$ is a sub-system of $\mathscr{F}$ such that $\mathscr{F}_1$ does not cover I_0, then there exists an $I \in \mathscr{F}$ such that $\mathscr{F}_1$ does not cover I.*

Then $I_0 \in \mathscr{F}$.

LEMMA 3. *If $F(x)$ is absolutely continuous on $[a, b]$ and if $F'(x) \geqq 0$ a.e., then $F(x)$ is non-decreasing on $[a, b]$.*

THEOREM 1. *If $f(x)$ is preponderantly continuous, (ACG) on $[a, b]$ and* $\mathrm{AD}\, F(x) \geqq 0$ *a.e., then $F(x)$ is non-decreasing on $[a, b]$.*

PROOF. Let $\mathscr{F}$ be the system of all open intervals on (a, b) in each of which F is non-decreasing. Then $\mathscr{F}$ satisfies evidently the conditions (i), (ii), and (iii) in Lemma 2. If we show that $\mathscr{F}$ also satisfies the condition (iv), then the open interval (a, b) is contained in $\mathscr{F}$ by Lemma 2, and therefore by preponderant continuity, F is non-decreasing on $[a, b]$.

Let $\mathscr{F}_1$ be a sub-system of $\mathscr{F}$ which does not cover the interval (a, b), and E be the set of points not covered by $\mathscr{F}_1$. Then E is a closed set. If $[p, q]$ is any closed interval which contains no points of E, then we can show by the method of repeated bisection that F is non-decreasing on $[p, q]$. It follows from the preponderant continuity of F that F is non-decreasing in the closure of each contiguous interval of E with respect to (a, b). Since F is (ACG) on $[a, b]$, the interval is the sum of a countable number of closed sets E_k, on each of which F is absolutely continuous. It follows from Lemma 1 that there exist an interval (l, m) and a natural number n such that $(l, m)\cdot E \subset E_n$. Hence F is absolutely continuous in $[l, m]\cdot E$. Since F is non-decreasing in the closure of each contiguous interval of $[l, m]\cdot E$ with respect to (l, m), we have $F'(x) \geqq 0$ a.e. on $[l, m]$. By Lemma 3, F is non-decreasing on

$[l, m]$. Hence $(l, m) \in \mathscr{F}$. But $\mathscr{F}_1$ cannot cover the interval (l, m) since (l, m) contains points of E. Thus we have (iv).

Let $\mathscr{L}$ denote a linear space contained in pC.

DEFINITION 2. A function $f(x)$ defined on $[a, b]$ is said to be integrable in the preponderantly continuous Denjoy sense on $[a, b]$ with respect to $\mathscr{L}$ or PD $(\mathscr{L})$-integrable on $[a, b]$ if there exists a function $F \in \mathscr{L} \cap (ACG)$ such that

$$\mathrm{AD}\, F(x) = f(x) \qquad \text{a.e.} .$$

The function $F(x)$ is called an indefinite PD $(\mathscr{L})$-integral and the definite integral on $[a, b]$, denoted by $(\mathrm{PD}\,(\mathscr{L}))\int_a^b f(x)dx$, is defined as $F(b) - F(a)$.

The definition of this integral requires a uniqueness theorem, namely that if F_1 and F_2 both satisfy the condition of Definition 2 then

$$F_1(b) - F_1(a) = F_2(b) - F_2(a) .$$

This is guaranteed by Theorem 1, since $F_1 - F_2 \in \mathscr{L} \cap (ACG)$ and $\mathrm{AD}\,(F_1 - F_2) = 0$ a.e..

We can prove the following fundamental properties for the PD $(\mathscr{L})$-integral as usual ([2]).

THEOREM 2. (i) *The class of all* PD $(\mathscr{L})$*-integrable functions on* $[a, b]$ *is a linear space and the* PD $(\mathscr{L})$*-integral is a linear functional on it.*

(ii) *If f is* PD $(\mathscr{L})$*-integrable on* $[a, b]$, *then f is* PD $(\mathscr{L})$*-integrable on any subinterval* $[\alpha, \beta]$, *and the* PD $(\mathscr{L})$*-integral is an additive function of an interval on* $[a, b]$.

(iii) *If f is* PD $(\mathscr{L})$*-integrable on* $[a, b]$ *and non-negative almost everywhere, then f is Lebesgue integrable.*

(iv) *If* $\{f_n\}$ *is a sequence of* PD $(\mathscr{L})$*-integrable functions converging to a function f and if* g_1, g_2 *are* PD $(\mathscr{L})$*-integrable with* $g_1 \leqq f_n \leqq g_2 (n = 1, 2, \cdots)$, *then f is also* PD $(\mathscr{L})$*-integrable and*

$$\lim_{n\to\infty} (\mathrm{PD}\,(\mathscr{L}))\int_a^b f_n(x)dx = (\mathrm{PD}\,(\mathscr{L}))\int_a^b f(x)dx .$$

We remark that if we put $\mathscr{L} = apC$ in Definition 2, then the PD (apC)-integral is the AD-integral in Definition 1.

Next we shall show that our integral is strictly more general than the AD-integral.

THEOREM 3. *There exist a function f defined on* $[a, b]$ *and a linear space* $\mathscr{L}$ *contained in* pC *such that f is* PD $(\mathscr{L})$*-integrable but not* AD*-integrable on* $[a, b]$.

PROOF. Let $I_n = [1/(4n+1), 1/4n] = [a_n, b_n] (n = 1, 2, \cdots)$ be a sequence of closed intervals on $[0, 1]$. If we put $E = \bigcup_{n=1}^{\infty} I_n$, then the set E has 1/4 right density at 0.

Let $\varphi_n(x) (n = 1, 2, \cdots)$ be a sequence of functions defined on $[0, 1]$ as follows:

$$\begin{aligned} \varphi_n(x) &= \sin^2 \{(x - a_n)\pi/(b_n - a_n)\} \quad \text{for} \quad x \in I_n , \\ &= 0 \qquad \text{elsewhere} . \end{aligned}$$

Finally we define $F(x) = \sum_{n=1}^{\infty} \varphi_n(x)$. Then $F(x)$ is continuous on $[0, 1]$ except at $x = 0$, where $F(x)$ is preponderantly continuous but not approximately continuous, for $\lim_{x \to 0} F(x) = 0 = F(0) (x \in E^c)$ and the set E^c has 3/4 right density at 0.

Since $\varphi_n(x)$ is absolutely continuous on the closed interval I_n and is zero elsewhere, $F(x)$ is (ACG) on $[0, 1]$ and is differentiable in the ordinary sense everywhere except at 0.

Let $\mathscr{L}$ be the linear space spanned by the function $F(x)$ and the space apC on $[0, 1]$. If we put

$$\begin{aligned} f(x) &= F'(x) \qquad (x \neq 0) \\ &= 0 \qquad\quad (x = 0) , \end{aligned}$$

then it follows from Definition 2 that the function $f(x)$ is PD$(\mathscr{L})$-integrable on $[0, 1]$. However $f(x)$ is not AD-integrable on $[0, 1]$. Suppose that f is AD-integrable. Then there exists a function $G(x)$ which is approximately continuous, (ACG) on $[0, 1]$ and AD $G(x) = f(x)$ a.e.. Since AD $(F(x) - G(x)) = 0$ a.e. and $F - G \in \mathscr{L} \cap (ACG)$, we have from Theorem 1 that $F - G$ is constant on $[0, 1]$. This contradicts the fact that G is approximately continuous at 0 but F is not so.

REMARK. The preponderant derivative of a function F at x_0 is defined as the preponderant limit of the difference quotient $(F(x) - F(x_0))/(x - x_0)$ as x approaches x_0, and this derivative is a generalization of the approximate derivative. However the integral does not increase when the preponderant derivative is used instead of the approximate derivative in Definition 2. In fact, an (ACG) function defined on an interval is approximately differentiable almost everywhere and is therefore preponderantly differentiable almost everywhere.

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