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A GALOIS CORRESPONDENCE IN A VON NEUMANN ALGEBRA

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1. Introduction. Recently, several authors showed that there exist Galois correspondences between a class of subgroups of a locally compact automorphism group of a von Neumann algeba A and a class of von Neumann subalgebras of A (cf. [7], [14], [15], and [19]).

The Galois theory for von Neumann algebras was initiated by M. Nakamura and Z. Takeda, who established the Galois theory for II_1-factors and finite groups of outer automorphisms by observing the close analogy between theories of classical simple algebras and of II_1-factors ([16], [17], [18], and others).

Subsequently, by M. Henle [12] and Y. Haga and Z. Takeda [11], the theory due to M. Nakamura and Z. Takeda was generalized as follows:

Let A be a von Neumann algebra, G a countable discrete group of freely acting automorphisms of A and B the fixed point algebra of A under G. Then, M. Henle established a Galois correspondence between the class of all subgroups of G and a class of certain von Neumann subalgebras of A, under the condition that there exist mutually orthogonal projections $p_g (g \in G)$ in A such that $\sum_{g \in G} p_g = 1$ and $g(p_h) = p_{hg^{-1}} (g, h \in G)$. And, as a consequence of a generalization of the Dye correspondence ([9]), Y. Haga and Z. Takeda established a Galois correspondence between the class of all intermediate von Neumann subalgebras and the class of all Z-full subgroups of the Z-full group determined by G, under the condition that there exists a representation of A on some Hilbert space $\mathfrak{K}$ having the following properties:

(i) G has a unitary representation u_g on $\mathfrak{K}$ such that $g(a) = u_g a u_g^*$ for all $a \in A$,

(ii) A' has a faithful normal finite trace invariant under G,

(iii) B' is isomorphic to the crossed product $G \otimes A'$ by an isomorphic mapping $A' \leftrightarrow \pi(A')$ and $u_g \leftrightarrow \lambda(g)$ for all $g \in G$.

In this paper, for a von Neumann algebra A and a discrete group G of freely acting automorphisms of A, we shall discuss a Galois correspondence between the class of all subgroups of G and a class of certain von Neumann subalgebras of A under the condition that there exists a faithful normal expectation of B' onto A', where B is the fixed point algebra of A under G. This condition may seem to be spatial. But it is an algebraical condition (Lemma 3) and it is implied by the Henle condition. Especially, in the case where G is an automorphism group of A induced by a unitary group, the condition is equivalent to the condition (ii) in the Haga-Takeda correspondence (Lemma 2).

A main result in this paper is the following:

Let H be a group of automorphisms of A which commute with each element in G. If H is ergodic on the center of A and there exists a faithful normal expectation of B' onto A', then there exists a Galois correspondence between the class of all subgroups of G and the class of all globally H-invariant von Neumann subalgebras C of A containing B such that there exists a faithful normal expectation of B' onto C' (Theorem 7).

2. Crossed product. In this section, we shall state some notations and properties with respect to the crossed product.

Throughout this note, we shall treat a von Neumann algebra acting on a separable Hilbert space and a countable discrete group of automorphisms.

Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and G a group of automorphisms of A. Denote by $G \otimes \mathfrak{H}$ the Hilbert space of $\mathfrak{H}$-valued square summable functions on G. We shall define a faithful normal representation π of A into $G \otimes \mathfrak{H}$ and a unitary representation λ of G into $G \otimes \mathfrak{H}$ as the following;

$$(\pi(a)\xi)(h) = h^{-1}(a)\xi(h) \qquad (a \in A,\ h \in G,\ \xi \in G \otimes \mathfrak{H})$$
$$(\lambda(g)\xi)(h) = \xi(g^{-1}h) \qquad (g,\ h \in G,\ \xi \in G \otimes \mathfrak{H})\ ,$$

where $\xi(h)$ is the value of ξ at h. Then we have the relation

$$\lambda(g)\pi(a)\lambda(g)^* = \pi(g(a)) \qquad (a \in A,\ g \in G)\ .$$

We shall denote by $G \otimes A$ the von Neumann algebra generated by $\{\pi(a); a \in A\}$ and $\{\lambda(g); g \in G\}$, and call it the *crossed product* of A by G. For a subgroup K of G, we shall denote by $N(K)$ the von Neumann algebra on $G \otimes \mathfrak{H}$ generated by $\{\pi(a); a \in A\}$ and $\{\lambda(k); k \in K\}$. For a subset S of G, we shall denote by χ_S the characteristic function of S on G.

The Hilbert space $G \otimes \mathfrak{H}$ is identified with $\mathfrak{H} \otimes l^2(G)$. On the other hand, for each g in G, put

$$\varepsilon_g(h) = \delta_g^h = \begin{cases} 1 & (g = h) \\ 0 & (g \neq h)\ , \end{cases}$$

then the Hilbert space $G \otimes \mathfrak{H}$ is identifiable with the direct sum $\sum_{g \in G} \oplus$ $(\mathfrak{H} \otimes \varepsilon_g)$ of subspace $\mathfrak{H} \otimes \varepsilon_g (g \in G)$. For each g in G and η in $\mathfrak{H}$, put

$$J_g \eta = \eta \otimes \varepsilon_g ,$$

then J_g is an isometry of $\mathfrak{H}$ onto $\mathfrak{H} \otimes \varepsilon_g$. Every x in $L(G \otimes \mathfrak{H})$ has a matrix representation with an operator on $\mathfrak{H}$ as each matrix element

$$(x)_{g,h} = J_g^* x J_h ,$$

where $L(\mathfrak{K})$ is the algebra of all (bounded linear) operators on the Hilbert space $\mathfrak{K}$. Especially, we have that

$$(\pi(a))_{g,h} = \delta_g^h g^{-1}(a) \qquad (a \in A,\ g,\ h \in G)$$

and

$$(\lambda(k))_{g,h} = \delta_g^{kh} \qquad (g,\ k,\ h \in G) .$$

Furthermore, as a modification of Zeller-Meier [22], we have a characterization of $N(K)$ by using the matrix representation as the following:

PROPOSITION 1. *Let A be a von Neumann algebra, G a group of automorphisms of A and K a subgroup of G. Then the von Neumann algebra $N(K)$ is the set of all elements x in $L(G \otimes \mathfrak{H})$ having the matrix form*

$$(x)_{g,h} = g^{-1}(x(gh^{-1})) ,$$

where, for each g, $x(g)$ is an element of A with $\chi_K(g)x(g) = x(g)$.

PROOF (cf. [22; p. 206]). Let B be the set of operators x on $G \otimes \mathfrak{H}$ having the matrix form

$$(x)_{g,h} = g^{-1}(x(gh^{-1})) ,$$

where, for each g, $x(g)$ is an element of A with $\chi_K(g)x(g) = x(g)$. Then it is clear that B is a von Neumann algebra and $N(K) \subset B$.

The necessary and sufficient condition that x in $L(G \otimes \mathfrak{H})$ commutes with every element of $N(K)$ (that is $x \in N(K)'$) is that

$$(x)_{g,h} a = g^{-1}h(a)(x)_{g,h} \qquad (g,\ h \in G,\ a \in A)$$

and

$$(x)_{k^{-1}g,h} = (x)_{g,kh} \qquad (g,\ h \in G,\ k \in K) .$$

In order to show that $B \subset N(K)$, it is sufficient to show that $xy = yx$ for each x in $N(K)'$ and y in B. By the definition of B, the (g, h)-component

$(y)_{g,h}$ of y is represented by

$$(y)_{g,h} = g^{-1}(y(gh^{-1})) ,$$

where $y(g)$ is an A-valued function on G such that $\chi_K(g)y(g) = y(g)$. Then we have that

$$\begin{aligned}(xy)_{g,h} &= \sum_{l\in G}(x)_{g,l}(y)_{l,h} = \sum_{l\in G}(x)_{g,l}l^{-1}(y(lh^{-1}))\\ &= \sum_{l\in G} g^{-1}(y(lh^{-1}))(x)_{g,l}\\ &= \sum_{m\in G} g^{-1}(y(gm^{-1}))(x)_{g,gm^{-1}h}\\ &= \sum_{m\in G} g^{-1}(y(gm^{-1}))(x)_{m,h} = (yx)_{g,h} .\end{aligned}$$

Thus we have that $B = N(K)$.

Let A be a von Neumann algebra and B a von Neumann subalgebra of A. Then a positive linear mapping e of A onto B is called an *expectation* of A onto B if e satisfies

$$e(1) = 1$$

and

$$e(ab) = e(a)b \qquad (a \in A,\ b \in B) .$$

An expectation e of A onto B is *faithful* if $e(x^*x) = 0$ implies $x = 0$, and e is *normal* if $e(x_\alpha) \uparrow e(x)$ for every net $\{x_\alpha\}$ of selfadjoint elements of A with $x_\alpha \uparrow x$.

Let A be a von Neumann algebra and G a group of automorphisms of A then it is well known that there exists a faithful normal expectation e of $G\otimes A$ onto $\pi(A) = N(\{1\})$ with $e(\lambda(g)) = 0$ for each $g \neq 1$ (the identity) in G.

This result is generalized as the following:

PROPOSITION 2. *Let A, G, and K be the same in Proposition* 1. *Then there exists a faithful normal expectation e_K of $G\otimes A$ onto $N(K)$ such that*

$$e_K(\lambda(g)) = 0 \quad \textit{for each} \quad g \notin K .$$

PROOF. Let $\{g_\iota\}_{\iota\in I}$ be the family of representative elements of left cosets of K in G and p_ι a projection onto $\sum_{h\in Kg_\iota} \mathfrak{H}\otimes\varepsilon_h$. Then $\{p_\iota\}_{\iota\in I}$ is a family of mutually orthogonal projections and $\sum_{\iota\in I} p_\iota = 1$. Put

$$e_K(x) = \sum_{\iota\in I} p_\iota x p_\iota \qquad (x \in L(G\otimes\mathfrak{H})) ,$$

then e_K is a faithful normal expectation of $L(G \otimes \mathfrak{H})$ onto $\{p_\iota; \iota \in I\}'$ (cf. [2]). Denote by the same notation e_K the restriction of e_K to $G \otimes A$. We shall show that $e_K(x) \in N(K)$ for every x in $G \otimes A$. The (g, h)-component $(p_\iota)_{g,h}$ of the matrix representation of p_ι satisfies that

$$(p_\iota)_{g,h} = \delta_g^h \chi_{Kg_\iota}(h) .$$

Hence we have that

$$\begin{aligned}(e_K(x))_{g,h} &= \sum_{\iota \in I} (p_\iota x p_\iota)_{g,h} = \sum_{\iota \in I} \chi_{Kg_\iota}(g)\chi_{Kg_\iota}(h)(x)_{g,h} \\ &= \chi_K(gh^{-1})(x)_{g,h} .\end{aligned}$$

It implies that

$$(e_K(x))_{g,h} = g^{-1}(\chi_K(gh^{-1})x(gh^{-1}))$$

for x in $G \otimes A$ the (g, h)-component $(x)_{g,h}$ of which is represented by A-valued function $x(g)$ on G as $(x)_{g,h} = g^{-1}(x(gh^{-1}))$.
Put

$$y(g) = \chi_K(g)x(g) \quad \text{for each} \quad g \quad \text{in} \quad G ,$$

then, $y(g)$ is an A-valued function on G satisfying that

$$\chi_K(g)y(g) = y(g)$$

and

$$g^{-1}(y(gh^{-1})) = (e_K(x))_{g,h} .$$

Therefore, we have that $e_K(x) \in N(K)$ by Proposition 1.

On the other hand, we have that $e_K(x) = x$ for each x in $N(K)$, so that e_K is a faithful normal expectation of $G \otimes A$ onto $N(K)$.

The necessary and sufficient condition that $gkg_\iota \in Kg_\iota$ for each $g \in G$ $\iota \in I$ and $k \in K$ is that g belongs to K. It implies that $p_\iota \lambda(g) p_\iota = 0$ for every $g \notin K$. Thus we have that

$$e_K(\lambda(g)) = 0 \quad \text{for every} \quad g \notin K .$$

3. Correspondence between subgroups and subalgebras in a crossed product. Let A be a von Neumann algebra and α an automorphism of A. Then α is called *freely acting* on A if each $a \in A$ satisfying $ab = \alpha(b)a$ for all $b \in A$ is 0 [13] (also, cf. [3]). A group G of automorphisms of A is called *freely acting* on A if each $g(\neq 1)$ in G is freely acting on A. A von Neumann subalgebra C of the crossed product $G \otimes A$ of a von Neumann algebra A by an automorphism group G is called an *intermediate von Neumann subalgebra* of $G \otimes A$ if C contains $\pi(A)$.

In this section, we shall discuss a correspondence between subgroups and intermediate von Neumann subalgebras of a crossed product. We need the following lemma, which is a variation of [6, Lemma 1.5.6] (also, cf. [11, Lemma 5] and [16, Proof of Theorem 2]).

LEMMA 1. *Let A be a von Neumann algebra, B a von Neumann subalgebra of A with $B' \cap A \subset B$, C a von Neumann subalgebra of A containing B and e an expectation of A onto C. If a unitary operator u in A satisfies the condition $uBu^* = B$, then $e(u)$ has the following properties;*

(1) *$e(u)$ is a partial isometry,*

(2) *the initial projection p and the final projection q of $e(u)$ are contained in the center of B,*

(3) *$e(u) = up = qu$.*

PROOF. For all $x \in B$, we have that

$$e(u)x = e(uxu^*u) = uxu^*e(u) .$$

Then $u^*e(u) \in B' \cap A$, so $u^*e(u) \in B \cap B' \subset C$. Hence

$$e(u)e(u)^*e(u) = e(u)u^*e(u) = e(uu^*e(u)) = e(u) ,$$

that is, $e(u)$ is a partial isometry. Also, we have that

$$p = e(u)^*e(u) = e(u^*e(u)) = u^*e(u) ,$$

so that $p \in B' \cap B$. Similarly, we have that $e(u)u^* \in B' \cap A$ and that $q = e(u)e(u)^* = e(u)u^* \in B' \cap B$.

If an automorphism group G of a von Neumann algebra A is freely acting on A, then for each intermediate von Neumann subalgebra B,

$$G \otimes A \cap B' \subset G \otimes A \cap \pi(A)' \subset \pi(A) \subset B .$$

Hence, Lemma 1 is applicable to an intermediate von Neumann subalgebra in the crossed product by a freely acting automorphism group.

THEOREM 3. *Let A be a von Neumann algebra and G a group of freely acting automorphisms of A. If there exists a group H of automorphisms of $G \otimes A$ satisfying the following conditions:*

(1) *$\pi(A)$ is globally invariant under H,*

(2) *H is ergodic on $\pi(A \cap A')$*

and

(3) *$h(\lambda(g)) = \lambda(g)$ for each $g \in G$ and each $h \in H$,*

then there exists a one-to-one correspondence between the class of all subgroups K of G and the class of all H-invariant intermediate von Neumann subalgebras N of $G \otimes A$ such that there exists a faithful normal expectation of $G \otimes A$ onto N in such a way that *subgroup* $K \longmapsto N = N(K)$

intermediate subalgebra $N \longmapsto K = K(N) = \{g \in G;\ \lambda(g) \in N\}$.

PROOF. For a subgroup K, $N(K)$ satisfies the condition that $N(K)$ is globally invariant under H and that there exists a faithful normal expectation of $G \otimes A$ onto $N(K)$, by Proposition 2. Also, by Proposition 2, we have $K(N(K)) = K$.

Conversely, let N be an H-invariant intermediate von Neumann subalgebra of $G \otimes A$ such that there exists a faithful normal expectation e of $G \otimes A$ onto N. Then, by Lemma 1, for each $g \in G$, there exists a central projection $a(g)$ in A such that

$$e(\lambda(g))\lambda(g)^* = \pi(a(g)) .$$

Since N is globally invariant under H, for each $h \in H$, $h \circ e \circ h^{-1}$ is a faithful normal expectation of $G \otimes A$ onto N. So, we have, for each $h \in H$ and $x \in G \otimes A$,

$$h(e(x)) = e(h(x))$$

by [6, Theorem 1.5.5(a)] (also cf. [21]).
By condition (3), we have, for all $g \in G$ and all $h \in H$,

$$h(e(\lambda(g))\lambda(g)^*) = e(\lambda(g))\lambda(g)^* .$$

Hence, by condition (2), we have that

$$e(\lambda(g)) = \lambda(g) \quad \text{or} \quad 0 .$$

Now, we shall show that $N(K(N)) = N$. For each x in $G \otimes A$, there exists a net $x_\alpha = \sum_{g \in G} \lambda(g)\pi(a_g^\alpha)$ with $a_g^\alpha = 0$ except a finite g such that x_α converges to x in σ-weak operator topology. Then by the normality of e, $e(x_\alpha) = \sum_{g \in G} e(\lambda(g))\pi(a_g^\alpha)$ converges to $e(x)$ in σ-weak operator topology.

On the other hand, we have that

$$(*) \qquad e(\lambda(g)) = 0 \quad \text{for all} \quad g \notin K(N) .$$

Hence we have $e(x_\alpha) \in N(K(N))$. Therefore, we have that $e(x) \in N(K(N))$ for all $x \in G \otimes A$, that is, $N \subset N(K(N))$. The converse inclusion is clear. Then we have $N = N(K(N))$.

Professor Y. Nakagami was very kind to pointing out that the free action of G does not need for $K(N(K)) = K$.

In Theorem 3, assume that the von Neumann algebra A is a factor. Then the group consisting only the identity automorphism of $G \otimes A$ satisfies the conditions (1), (2) and (3). So that we have the following corollary, which is a generalization of [5, Theorem 6] and [16, Theorem 2].

COROLLARY 4. *Let A be a factor and G a group of outer automorphisms of A. Then there exists a one-to-one correspondence between the class of all subgroups K of G and the class of all intermediate von Neumann subalgebras N of $G \otimes A$ such that there exists a faithful normal expectation of $G \otimes A$ onto N in the same way as in Theorem* 3.

In Theorem 3, we assume that the group G is freely acting and that there exists an automorphism group H of $G \otimes A$ satisfying the three conditions (1), (2) and (3). Such assumptions are used in order to show that a faithful normal expectation e of $G \otimes A$ onto globally H-invariant intermediate von Neumann subalgebra N of $G \otimes A$ satisfies the condition (*). Therefore, we have the following proposition.

PROPOSITION 5. *Let A be a von Neumann algebra and G a group of (not necessarily freely acting) automorphisms of A. Then there exists a one-to-one correspondence between the class of all subgroups K of G and the class of all intermediate von Neumann subalgebras N of $G \otimes A$ such that there exists a faithful normal expectation e of $G \otimes A$ onto N satisfying $e(\lambda(g)) = 0$ for all $\lambda(g) \notin N$ in the same way as in Theorem* 3.

REMARK 1. Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, G a group of automorphisms of A and H a group of automorphisms of A with the following properties:

(i) H is ergodic on the center of A,

(ii) $gh = hg$ for all $g \in G$ and all $h \in H$,

and

(iii) H has a unitary representation u_h on $\mathfrak{H}$ such that $h(x) = u_h x u_h^*$ for all $h \in H$ and $x \in A$.

Consider a unitary representation $u_h \otimes 1 (h \in H)$ into the Hilbert space $G \otimes \mathfrak{H} = \mathfrak{H} \otimes l^2(G)$. Then, for each $x \in G \otimes A$ and each $h \in H$, we have

$$((u_h \otimes 1)x(u_h \otimes 1)^*)_{g,k} = h(g^{-1}(x(gk^{-1}))) = g^{-1}(h(x(gk^{-1})))$$

for all $g, k \in G$, where $(x)_{g,k} = g^{-1}(x(gk^{-1}))$ for some $x(g) \in A (g \in G)$. Hence, for each $h \in H$, $u_h \otimes 1$ induces an automorphism of $G \otimes A$. Put

$$\tilde{H} = \{\mathrm{Ad}\,(u_h \otimes 1);\ h \in H\}\,,$$

where $\mathrm{Ad}\,(u_h \otimes 1)(x) = (u_h \otimes 1)x(u_h \otimes 1)^*$ for all $x \in G \otimes A$. Then $\tilde{H}$ is an automorphism group of $G \otimes A$ satisfying the conditions (1), (2) and (3) in Theorem 3.

4. **Galois correspondence.** In this section, we shall consider a Galois correspondence between von Neumann subalgebras of a von Neumann algebra A and subgroups of an automorphism group of A. We shall

call a one-to-one correspondence between a class of von Neumann algebras and a class of subgroups a *Galois correspondence* if the correspondence is defined by the following way; each subgroup K in the class is corresponding the fixed point algebra A^K of A under K and each von Neumann subalgebra C in the class is corresponding the subgroup $G_C = \{g \in G;\ g(x) = x \text{ for all } x \in C\}$.

Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and ξ a cyclic and separating vector for A. Then the automorphism group $\mathrm{Aut}\,A$ of all automorphisms of A has a unitary representation $u_\alpha (\alpha \in \mathrm{Aut}\,A)$ such that $\alpha = \mathrm{Ad}\,u_\alpha$ $(\alpha \in \mathrm{Aut}\,A)$ by Araki [1] (also cf. [10]). This unitary representation u_α is called *canonical unitary representation* of $\mathrm{Aut}\,A$ with respect to ξ.

If a freely acting automorphism g of a von Neumann algebra A is induced by a unitary operator u, then $g(x') = ux'u^*(x' \in A')$ is a freely acting automorphism of A' (cf. [11] and [17]). Hence a freely acting automorphism group of A induced by a unitary group is also considered as a group of freely acting automorphisms of A'.

LEMMA 2. *Let A be a von Neumann algebra with a cyclic and separating vector ξ, G a group of freely acting automorphisms of A, B the fixed point algebra of A under G and H a group of automorphisms of A such that H is ergodic on the center of A and that $gh = hg$ for all $g \in G$ and $h \in H$. If B' is isomorphic to the crossed product $G \otimes A'$ of A' by G, by an isomorphism θ such that $\theta(A') = \pi(A')$ and $\theta(u_g) = \lambda(g)$ for all $g \in G$, then there exists a Galois correspondence between the class of all subgroups of G and the class of all globally H-invariant von Neumann subalgebras C of A containing B such that there exists a faithful normal expectation of B' onto C', where u is the canonical unitary representation of* $\mathrm{Aut}\,A$ *with respect to ξ.*

PROOF. Put $H_0 = \{\theta \circ \mathrm{Ad}\,u_h \circ \theta^{-1};\ h \in H\}$. Then H_0 is an automorphism group of $G \otimes A'$ and satisfies the conditions (1), (2) and (3) in Theorem 3 by replacing A by A'.

For a von Neumann subalgebra C of A containing B, C is globally invariant under H if and only if $\theta(C')$ is globally invariant under H_0. And there exists a faithful normal expectation of B' onto C' if and only if there exists a faithful normal expectation of $G \otimes A'$ onto $\theta(C')$. Hence, by Theorem 3, if C is a globally H-invariant von Neumann subalgebra of A containing B such that there exists a faithful normal expectation of B' onto C', then $\theta(C')$ corresponds with the subgroup $K = \{g \in G;\ \lambda(g) \in \theta(C')\}$. Now, we have that

$$\begin{aligned} K &= \{g \in G;\ \lambda(g) \in \theta(C')\} = \{g \in G;\ u_g \in C'\} \\ &= \{g \in G;\ g(x) = u_g x u_g^* = x \quad \text{for all} \quad x \in C\} \\ &= G_C\ . \end{aligned}$$

On the other hand, by Theorem 3 each subgroup K of G corresponds with $N(K) =$ the von Neumann algebra generated by $\pi(A')$ and $\{\lambda(g);\ g \in K\}$. Put $C = (\theta^{-1}(N(K)))'$. Then we have that

$$\begin{aligned} C &= (\theta^{-1}(N(K)))' = [A' \cup \{u_g;\ g \in K\}]' = A \cap \{u_g;\ g \in K\}' \\ &= A^K\ . \end{aligned}$$

Therefore, by Theorem 3, the proof completes.

LEMMA 3. *Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, G a group of freely acting automorphisms of A with a unitary representation $u_g (g \in G)$ into $\mathfrak{H}$ such that $g = \mathrm{Ad}\, u_g$ for all $g \in G$ and B the fixed point algebra of A under G. Then a necessary and sufficient condition that there exists a faithful normal expectation of B' onto A' is that there exists an isomorphism θ of B' onto $G \otimes A'$ satisfying the following conditions*:

(1) $\theta(A') = \pi(A')$

and

(2) $\theta(u_g) = \lambda(g)$ *for all* $g \in G$.

PROOF. Suppose that there exists a faithful normal expectation of B' onto A'. The commutant B' of B is generated by A' and $\{u_g;\ g \in G\}$. So, by [4, Corollary 5], there exists an isomorphism θ of B' onto $G \otimes A'$ with desired properties.

Conversely, let θ be an isomorphism of B' onto $G \otimes A'$ with properties (1) and (2) and e a faithful normal expectation of $G \otimes A'$ onto $\pi(A')$. Then $e_0 = \theta^{-1} \circ e \circ \theta$ is a faithful normal expectation of B' onto A'.

LEMMA 4. *Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and B a von Neumann subalgebra of A such that there exists a faithful normal expectation of B' onto A'. If θ is an isomorphism of A onto a von Neumann algebra C, then there exists a faithful normal expectation of $\theta(B)'$ onto C'.*

PROOF. By [8, Theorem 3, §4, Chap. 1], the isomorphism θ is decomposed into $\theta_3 \circ \theta_2 \circ \theta_1$, where θ_1 is an ampliation, θ_2 is an induction and θ_3 is a spatial isomorphism. Hence, it is sufficient to consider each $\theta_i (i = 1, 2, 3)$. Let e be a faithful normal expectation of B' onto A'. Suppose θ is an ampliation, i.e., $\theta(x) = x \otimes 1$ for all $x \in A$, where 1 is an identity operator on a Hilbert space $\mathfrak{K}$. Then we have that $\theta(B)' =$

$B'\otimes L(\mathfrak{K})$ and $\theta(A)'=A'\otimes L(\mathfrak{K})$. Hence, the tensor product $e\otimes i$ of e and identity map i on $L(\mathfrak{K})$ is a faithful normal expectation of $\theta(B)'$ onto $\theta(A)'$(cf. [20]). Next, we suppose that θ is an induction, i.e., $\theta(x) = x_f$ for all $x \in A$, where f is a projection in A'. Put $e_0(x_f) = e(x)_f$ for all $x \in B'$. Then e_0 is a faithful normal expectation of $(\theta(B))' = B'_f$ onto $(\theta(A))' = A'_f$. At last, we suppose that θ is a spatial isomorphism induced by an isomorphism u of the Hilbert space $\mathfrak{H}$ onto a Hilbert space $\mathfrak{K}$. Put $e_0(x) = ue(u^*xu)u^*$ for all $x \in \theta(B)'$. Then e_0 is a faithful normal expectation of $\theta(B)'$ onto $\theta(A)'$.

THEOREM 6. *Let A be a von Neumann algebra, G a group of freely acting automorphisms of A, B the fixed point algebra of A under G and H a group of automorphisms of A such that H is ergodic on the center of A and that $gh = hg$ for all $g \in G$ and $h \in H$. If there exists a faithful normal expectation of B' onto A', then there exists a Galois correspondence between the class of all subgroups of G and the class of all globally H-invariant von Neumann subalgebras C of A containing B such that there exists a faithful normal expectation of B' onto C'.*

PROOF. Since underlying space is separable, there exists a faithful normal state ϕ on A. Let $(\pi_\phi, \mathfrak{H}_\phi, \xi_\phi)$ be the representation of A with respect to ϕ. Then π_ϕ is an isomorphism of A onto the von Neumann algebra $\pi_\phi(A)$ and ξ_ϕ is a cyclic and separating vector for $\pi_\phi(A)$. Put

$$\pi_\phi G\pi_\phi^{-1} = \{\pi_\phi\circ g\circ\pi_\phi^{-1};\ g \in G\}\ ,$$

and

$$\pi_\phi H\pi_\phi^{-1} = \{\pi_\phi\circ h\circ\pi_\phi^{-1};\ h \in H\}\ .$$

Then we have that

$$\begin{aligned}\{y \in \pi_\phi(A);\ g_0(y) = y \quad \text{for all} \quad g_0 \in \pi_\phi G\pi_\phi^{-1}\}\\ = \{\pi_\phi(x) \in \pi_\phi(A);\ \pi_\phi(g(x)) = \pi_\phi(x) \quad \text{for all} \quad g \in G\}\\ = \{\pi_\phi(x) \in \pi_\phi(A);\ g(x) = x \quad \text{for all} \quad g \in G\}\\ = \pi_\phi(B)\ .\end{aligned}$$

Thus, $\pi_\phi(B)$ is the fixed point algebra of $\pi_\phi(A)$ under $\pi_\phi G\pi_\phi^{-1}$. By Lemma 4, there exists a faithful normal expectation of $\pi_\phi(B)'$ onto $\pi_\phi(A)'$. Hence, by Lemma 3, $\pi_\phi(A)$, $\pi_\phi G\pi_\phi^{-1}$, $\pi_\phi H\pi_\phi^{-1}$, $\pi_\phi(B)$ and the canonical unitary representation u_g of $\pi_\phi G\pi_\phi^{-1}$ with respect to ξ_ϕ satisfy the conditions of Lemma 2. Therefore, there exists a Galois correspondence between the class of all subgroups of $\pi_\phi G\pi_\phi^{-1}$ and the class of all globally $\pi_\phi H\pi_\phi^{-1}$-invariant von Neumann subalgebras D of $\pi_\phi(A)$ containing $\pi_\phi(B)$ such that there exists a faithful normal expectation of $\pi_\phi(B)'$ onto D'.

On the other hand, by Lemma 4 and the definition of $\pi_\phi H \pi_\phi^{-1}$, a von Neumann subalgebra C of A containing B is globally H-invariant and has a faithful normal expectation of B' onto C' if and only if the isomorphic von Neumann subalgebra $\pi_\phi(C)$ of $\pi_\phi(A)$ is globally $\pi_\phi H \pi_\phi^{-1}$-invariant and has a faithful normal expectation of $\pi_\phi(B)'$ onto $\pi_\phi(C)'$. A von Neumann subalgebra C of A containing B is the fixed point algebra of A under a subgroup K of G if and only if $\pi_\phi(C)$ is the fixed point algebra of $\pi_\phi(A)$ under $\pi_\phi K \pi_\phi^{-1}$.

Therefore, we have a Galois correspondence between the class of all subgroups of G and the class of all globally H-invariant von Neumann subalgebras C of A containing B such that there exists a faithful normal expectation of B' onto C'.

REMARK 2. Let G be a finite group of freely acting automorphisms of a von Neumann algebra A. Then, by [18, Lemma 6] (also, cf. [11] and [12]), there exists a faithful normal representation of A with a unitary representation $u_g (g \in G)$ such that $g = \mathrm{Ad}\, u_g$ for all $g \in G$ and that there exists an isomorphism θ of B' onto $G \otimes A'$ satisfying $\theta(u_g) = \lambda(g)$ for all $g \in G$ and $\theta(A') = \pi(A')$. Therefore, if there exists a group H of automorphisms of A with the same properties as in Theorem 6, then there exists the same Galois correspondence as in Theorem 6.

Considering the identity automorphism group as in Corollary 4, we have the following:

COROLLARY 7. *Let A be a factor, G a group of outer automorphisms of A and B the fixed point algebra of A under G. If there exists a faithful normal expectation of B' onto A', then there exists a Galois correspondence between the class of all subgroups of G and the class of all von Neumann subalgebras C of A containing B such that there exists a faithful normal expectation of B' onto C'.*

Using Proposition 5 and Lemma 3, we have the following Henle's type proposition. The proposition is another form of Theorem 6 in the case where there does not exist a group H of automorphisms of the von Neumann algebra with the properties in Theorem 6.

PROPOSITION 8. *Let A be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, G a group of freely acting automorphisms of A with a unitary representation $u_g (g \in G)$ on $\mathfrak{H}$ such that $g = \mathrm{Ad}\, u_g$ for all $g \in G$, and B the fixed point algebra of A under G. If there exists a faithful normal expectation of B' onto A', then there exists a Galois correspondence between the class of all subgroups of G and the class of all von Neumann* *subalgebras C of A containing B such that there exists a faithful normal expectation e of B' onto C' with $e(u_g) = 0$ for all $u_g \notin C'$.*

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