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ON THE MAJORANT PROPERTIES IN $L^p(G)$

To the memory of the late Professor Karel deLeeuw

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Abstract. We extend the Hardy-Littlewood duality theorem to any locally compact abelian group G, namely, if $L^q(G)$ $(2<q<\infty)$ has the upper majorant property, then $L^p(G)$ has the lower majorant property, $p^{-1}+q^{-1}=1$. This settles the question of exactly which $L^p(G)$ has the lower majorant property.

1. Introduction. Let G be a compact abelian group. For $f, g \in L^1(G)$, we say as in [5] that g is a majorant of f if and only if $|\hat{f}| \leqq \hat{g}$. Let $1 \leqq p \leqq \infty$. We say as in [2] that $L^p(G)$ has the upper majorant property (UMP) if and only if there is a constant A_p such that

$$\|f\|_p \leqq A_p \|g\|_p$$

whenever $f, g \in L^p(G)$ and g is a majorant of f. We say also as in [2] that $L^p(G)$ has the lower majorant property (LMP) if and only if there is a constant B_p such that every $f \in L^p(G)$ has a majorant $g \in L^p(G)$ for which

$$\|g\|_p \leqq B_p \|f\|_p .$$

The majorant problem is to determine for which p the space $L^p(G)$ has the UMP or the LMP. To exclude trivialities we assume throughout that G is infinite. The problem was initiated by Hardy and Littlewood [5] and solved partially by them for the torus group T. The problem in the general compact abelian case has now been completely solved, collectively by Boas [2], Bachelis [1], and Fournier [4]. (See also Shapiro [10].) The results can be summarized in the following theorems.

THEOREM A. *$L^p(G)$ has UMP if and only if p is an even integer or ∞; and when $L^p(G)$ has the UMP the constant is 1.*

THEOREM B. *$L^p(G)$ has the UMP if and only if $L^q(G)$ has the LMP, with the same constant, $(q^{-1}+p^{-1}=1)$.*

As an immediate consequence of these one also has

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THEOREM C. *$L^p(G)$ has the* LMP *if and only if $p = 1$ or $p = 2k/(2k-1)$, $k \in N$.*

It should perhaps be mentioned that the difficult parts of the above theorems are the "only if" part of Theorem A and the "only if" part of Theorem B. The latter we shall call, following [9], the Hardy-Littlewood duality theorem, it being, in the torus case, the main theorem in [5]. The much easier half (the "if" part) of Theorem B was also named in [9] the Boas duality theorem.

Now the compact abelian groups being rather special, it seems difficult to adapt methods of the previous authors directly to the noncompact locally compact situation. Indeed, an attempt was made (see [3]) to extend the Hardy-Littlewood duality theorem to the integer group Z, but the arguments were erroneous. Recently in [8] we have managed to prove completely exact analogues of the above three theorems in $L^p(Z)$. At the same time, Rains [9] has extended Theorem A, as well as the Boas duality theorem and therefore the "only if" part of Theorem C, to all locally compact abelian groups. His proof of Theorem A in the $L^p(Z)$ case is the same as ours. On the other hand, as he says on p. 53 of [9], he has found neither a generalization of the Hardy-Littlewood duality theorem nor a direct proof that $L^p(G)$ has the LMP when $p = 2k/(2k-1)$, $k \in N$.

The object of this paper is to prove a generalization of the Hardy-Littlewood duality theorem and thereby, together with Rains' results, give a complete solution of the majorant problem in any locally compact abelian group. Our method is essentially a simple modification of the proof in our previous paper.

When the group G is not compact, $f \in L^p(G)$ need not always have an ordinary Fourier transform, so a few words must be said about the definitions of UMP and LMP in $L^p(G)$. One can proceed in either of the following ways as in [9]. Let

$$S(G) = L^1(G) \cap [L^1(\hat{G})]^\wedge .$$

One can define the UMP and LMP in $L^p(G)$ by taking the test functions f, g only from $S(G)$. The results of Rains mentioned previously have all been proved under this definition. Alternatively, one can define the concepts of majorants, UMP and LMP in the most general fashion, as follows. $S(G)$ is a Banach space under the norm

$$\|f\|_S = \|f\|_1 + \|\hat{f}\|_1 .$$

The Banach space dual of $S(G)$ is denoted by $S^*(G)$, which may be regarded as a space of distributions. The Fourier transform on $S^*(G)$

is defined in the usual dual manner. If $f \in S^*(G)$, we define $\hat{f} \in S^*(\hat{G})$ by

$$\langle \hat{f}, u \rangle = \langle f, \hat{u} \rangle , \quad u \in S(\hat{G}) .$$

For $f, g \in S^*(G)$, one then defines $|\hat{f}| \leqq \hat{g}$ by

$$|\langle \hat{f}, u \rangle| \leqq \langle \hat{g}, u \rangle , \quad u \in S(\hat{G}), u \geqq 0 .$$

Thus the concepts of the UMP and LMP are defined over all test functions in $L^p(G)$. However, as shown in [9], for $1 \leqq p < \infty$, the two definitions of UMP are equivalent (and with the same constants). This follows from a process of extension, since $S(G)$ is dense in $L^p(G)$, $1 \leqq p < \infty$. Whether the two kinds of definitions for LMP agree in general seems not to have been investigated. Rains has also proved the Boas duality theorem, but with different constants, when the LMP is taken in the distributional sense.

Our problem is to prove the "only if" part of Theorem B and the "if" part of Theorem C, in any locally compact abelian group. Now it is trivial that $L^2(G)$ has the LMP, and the LMP for $L^1(G)$ can be established directly, exactly as in [5]. By Rains' analogue of Theorem A, $L^p(G)$ ($1 \leqq p < 2$) does not have the UMP. So $L^q(G)$ ($2 < q \leqq \infty$) does not have the LMP, by the Boas duality theorem. Thus what concerns us is whether $L^p(G)$ ($1 < p < 2$) has the LMP. Here functions have Fourier transforms in $L^q(\hat{G})$, by the Hausdorff-Young theorem, so that in our case the distributional definition of LMP is exactly the same as the definition set forth in the beginning paragraph of this paper.

We wish to thank Dr. M. A. Rains for sending us a copy of his thesis prior to its publication.

2. Generalization of the Hardy-Littlewood duality theorem. Our main theorem is

THEOREM 1. *Let G be a locally compact abelian group. If $L^q(G)$ ($2 < q < \infty$) has the upper majorant property, then $L^p(G)$ ($p^{-1} + q^{-1} = 1$) has the lower majorant property, and with the same constant.*

We first fix some notations. Generic elements in G will be denoted by $x, y, \cdots$, those in $\hat{G}$ by $\xi, \eta, \cdots$. $L^1(G)$ has an approximate identity $\{u_\alpha\}_{\alpha \in D}$ with the following properties [6, (33·12)]:

(i) $u_\alpha \in L^1(G) \cap C_0(G)$, $u_\alpha \geqq 0$;

(ii) $\int_G u_\alpha(x)dx = 1$;

(iii) $\hat{u}_\alpha \in C_c(\hat{G})$, $\hat{u}_\alpha \geqq 0$;

(iv) $\lim \hat{u}_\alpha(\xi) = 1$ uniformly on compact sets.

Note we shall often use without mentioning the fact that $u_\alpha(-x) =$

$u_\alpha(x)$, $\hat{u}_\alpha(-\xi) = \hat{u}_\alpha(\xi)$. Let $\{w_\alpha\}_{\alpha \in D}$ be an approximate identity for $L^1(\hat{G})$ with the same properties as above. Note that we can clearly arrange so that the index sets are the same.

We shall need the following lemma, which can be obtained by direct computation:

LEMMA 1. [5, Lemma 2; 7, (15·10)]. *Let E be any measure space. For $f, \phi \in L^p(E)$, $1 < p < \infty$, and real t,*

$$\left[\frac{d}{dt}\|f + t\phi\|_p^p\right]_{t=0} = p \operatorname{Re} \int_E |f(x)|^{p-1} \operatorname{sgn} \overline{f(x)} \phi(x) dx .$$

PROOF OF THEOREM 1. For $f \in L^p(G)$, $1 < p < 2$, $\widehat{u_\alpha * f} = \hat{u}_\alpha \hat{f} \in L^1(\hat{G})$, since $\hat{u}_\alpha \in C_c(\hat{G})$ and $\hat{f} \in L^q(\hat{G})$. Thus

$$w_\alpha * |\widehat{u_\alpha * f}|(\xi) \in L^1(\hat{G}) , \quad (w_\alpha * |\widehat{u_\alpha * f}|)\check{}(x) \in C_c(G) .$$

Hence the set

$$S_\alpha = \{a \in L^p(G) \mid \hat{a}(\xi) \geqq w_\alpha * |\widehat{u_\alpha * f}|(\xi), \text{ a.e.}\}$$

is not empty, because the inverse transform of $w_\alpha * |\hat{u}_\alpha \hat{f}|(\xi)$ belongs to S_α. S_α is closed in $L^p(G)$ by the Hausdorff-Young theorem, and is obviously convex. Since a closed convex set in the uniformly convex Banach space $L^p(G)$ has a unique element of minimum norm, we see that there exists a unique $g_\alpha \in S_\alpha$ such that

$$\|g_\alpha\|_p = \inf \{\|a\|_p \mid a \in S_\alpha\} .$$

Define

$$h_\alpha(x) = |g_\alpha(x)|^{p-1} \operatorname{sgn} \overline{g_\alpha(x)} ,$$

then

$$\|h_\alpha\|_q = \|g_\alpha\|_p^{p/q} .$$

LEMMA 2. *There is a positive Borel measure $\mu_{\alpha,\beta} \in M(\hat{G})$ such that $u_\beta * h_\alpha = \hat{\mu}_{\alpha,\beta}$.*

PROOF. For any $\phi \in L^p(G)$ with $\hat{\phi}(\xi) \geqq 0$, and for all $t \geqq 0$, $g_\alpha + t\phi \in S_\alpha$, and so

$$\|g_\alpha + t\phi\|_p \geqq \|g_\alpha\|_p ,$$

hence

$$\left[\frac{d}{dt}\|g_\alpha + t\phi\|_p^p\right]_{t=0} \geqq 0 .$$

By Lemma 1, we have

$$\operatorname{Re} \int h_\alpha(x)\phi(x)dx \geqq 0 ,$$

and since $\hat{g}_\alpha \geqq 0$, $\hat{\phi} \geqq 0$ entail $g_\alpha(-x) = \overline{g_\alpha(x)}$, $\phi(-x) = \overline{\phi(x)}$, we have in fact

$$\int h_\alpha(x)\phi(x)dx \geqq 0 .$$

Replacing ϕ by $u_\beta * \phi$, then

$$\int u_\beta * h_\alpha(x)\phi(x)dx \geqq 0 .$$

Now $\psi(x) = u_\beta * h_\alpha(x) \in C_0(G)$. We claim it is also positive definite, so that the conclusion of our lemma follows. For, fixing any $\xi \in \hat{G}$, consider $\phi(x) = \hat{w}_\delta(x)\xi(x) \in C_c(G)$. Since $\hat{\phi}(\eta) = w_\delta(\xi - \eta) \geqq 0$, we have

$$\int \hat{w}_\delta(x)\psi(x)\xi(x)dx \geqq 0 .$$

Hence for every $k \in C_c(G)$,

$$\iint \hat{w}_\delta(x)\psi(x)k(y)\overline{k(y-x)}dxdy = \int_G \hat{w}_\delta(x)\psi(x)dx \int_{\hat{G}} |\hat{k}(\xi)|^2 \xi(x)d\xi$$
$$= \int_{\hat{G}} |\hat{k}(\xi)|^2 d\xi \int_G \hat{w}_\delta(x)\psi(x)\xi(x)dx \geqq 0 .$$

But

$$\lim_\delta \iint \hat{w}_\delta(x)\psi(x)k(y)\overline{k(y-x)}dxdy = \iint \psi(x)k(y)\overline{k(y-x)}dxdy$$
$$= \iint \psi(x-y)k(x)\overline{k(y)}dxdy ,$$

so that

$$\iint \psi(x-y)k(x)\overline{k(y)}dxdy \geqq 0$$

for every $k \in C_c(G)$, which is sufficient.

LEMMA 3. $\|g_\alpha\|_p \leqq A_q \|f\|_p$.

PROOF. As a matter of fact, for all $t \geqq -1$,

$$\hat{g}_\alpha + t(\hat{g}_\alpha - w_\alpha * |\widehat{u_\alpha * f}|) \geqq w_\alpha * |\widehat{u_\alpha * f}| \quad \text{a.e.} ,$$

and since

$$\hat{g}_\alpha + w_\alpha * |\widehat{u_\alpha * f}| \in \hat{L}^p(G) ,$$

we have

$$||g_\alpha + t\{g_\alpha - (w_\alpha * |\hat{u}_\alpha \hat{f}|^\vee)^\wedge\}||_p \geqq ||g_\alpha||_p ,$$

so that

$$\left[\frac{d}{dt}||g_\alpha + t\{g_\alpha - (w_\alpha * |\hat{u}_\alpha \hat{f}|^\vee)^\wedge\}||_p^p\right]_{t=0} = 0 .$$

From Lemma 1, we obtain

$$\int h_\alpha(x) g_\alpha(x) dx = \int h_\alpha(x) \hat{w}_\alpha(-x) |\hat{u}_\alpha \hat{f}|^\wedge(-x) dx .$$

Note that $h_\alpha \in L^q(G)$ and $\hat{w}_\alpha |\hat{u}_\alpha \hat{f}|^\wedge \in C_c(G)$, so

$$\begin{aligned}\int h_\alpha(x)\hat{w}_\alpha(-x)|\hat{u}_\alpha\hat{f}|^\wedge(-x)dx &= \lim_\beta \int u_\beta * h_\alpha(x)\hat{w}_\alpha(-x)|\hat{u}_\alpha\hat{f}|^\wedge(-x)dx \\ &= \lim_\beta \int |\hat{u}_\alpha \hat{f}|(\xi) w_\alpha * \mu_{\alpha,\beta}(\xi) d\xi ,\end{aligned}$$

where the last equality follows from Parseval's relation. Now set

$$\tau_{\alpha,\beta}(\xi) = w_\alpha * \mu_{\alpha,\beta}(\xi) \operatorname{sgn} \overline{\hat{f}(\xi)} .$$

Noting that $\tau_{\alpha,\beta} \in L^1(\hat{G}) \cap L^\infty(\hat{G}) \subset L^p(\hat{G})$, we have

$$\begin{aligned}\int |\hat{u}_\alpha \hat{f}|(\xi) w_\alpha * \mu_{\alpha,\beta}(\xi) d\xi &= \int \widehat{u_\alpha * f}(\xi) \tau_{\alpha,\beta}(\xi) d\xi = \int u_\alpha * f(x) \hat{\tau}_{\alpha,\beta}(x) dx \\ &\leqq ||u_\alpha * f||_p ||\hat{\tau}_{\alpha,\beta}||_q \leqq ||f||_p ||\hat{\tau}_{\alpha,\beta}||_q .\end{aligned}$$

We now apply the upper majorant property for $L^q(G)$ (with constant A_q) to the pair

$$\hat{\tau}_{\alpha,\beta} \quad \text{and} \quad \widehat{w_\alpha * \mu_{\alpha,\beta}} .$$

Recall that $\tau_{\alpha,\beta} \in L^2(\hat{G})$ also, so that the distributional definition of the Fourier transform of $\hat{\tau}_{\alpha,\beta} \in L^q(G)$ agrees with $\tau_{\alpha,\beta}{}^\vee$. Thus the definition of $\tau_{\alpha,\beta}$ shows that

$$\widehat{w_\alpha * \mu_{\alpha,\beta}} \quad \text{is a majorant of } \hat{\tau}_{\alpha,\beta}$$

and so by hypothesis

$$\begin{aligned}||\hat{\tau}_{\alpha,\beta}||_q &\leqq A_q ||\hat{w}_\alpha \hat{\mu}_{\alpha,\beta}||_q \leqq A_q ||\hat{\mu}_{\alpha,\beta}||_q = A_q ||u_\beta * h_\alpha||_q \leqq A_q ||h_\alpha||_q \\ &= A_\alpha ||g_\alpha||_p^{p/q} .\end{aligned}$$

Combining the above steps, we have

$$\|g_\alpha\|_p^p = \int h_\alpha(x) g_\alpha(x) dx = \lim_\beta \int |\hat{u}_\alpha \hat{f}|(\xi) w_\alpha * \mu_{\alpha,\beta}(\xi) d\xi$$
$$\leqq \|f\|_p \|\hat{\tau}_{\alpha,\beta}\|_q \leqq A_q \|f\|_p \|g_\alpha\|_p^{p/q} ,$$

which is what we need.

CONCLUSION OF THE PROOF. By Lemma 3,

$$\|g_\alpha\|_p \leqq A_q \|f\|_p .$$

Hence there exists a subnet of $\{g_\alpha\}$, also denoted by $\{g_\alpha\}$ for notational simplicity, which converges weakly to some $g \in L^p(G)$ and

$$\|g\|_p \leqq A_q \|f\|_p .$$

Thus, it remains to show that

$$|\hat{f}(\xi)| \leqq \hat{g}(\xi) \quad \text{a.e.}$$

For any $h \in L^p(\hat{G})$, $h(\xi) \geqq 0$ a.e., we have

$$\hat{g}_\alpha * h(\xi) = \int g_\alpha(x) \overline{\check{h}(x) \xi(x)} dx ,$$

which converges to

$$\int g(x) \overline{\check{h}(x) \xi(x)} dx = \hat{g} * h(\xi) .$$

But, from the definition of g_α,

$$\hat{g}_\alpha * h(\xi) \geqq h * w_\alpha * |\widehat{u_\alpha * f}|(\xi) \quad \text{a.e.}$$

with the right side converging to $h * |\hat{f}|(\xi)$. Hence

$$h * |\hat{f}|(\xi) \leqq h * \hat{g}(\xi) \quad \text{a.e.}$$

Now replace h by w_α. Since the corresponding nets converge in $L^q(\hat{G})$, so do a sequence thereof, and passing to an appropriate subsequence, we see that

$$|\hat{f}(\xi)| \leqq \hat{g}(\xi) \quad \text{a.e.}$$

as asserted.

From Theorem 1 and Rains' Theorem A as well as the Boas duality part of Theorem B, we immediately have

THEOREM 2. *For any locally compact abelian group G, $L^p(G)$ has the lower majorant property if and only if $p = 1$ or $p = 2k/(2k-1)$, $k \in N$.*

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