Tôhoku Math. Journ.
31 (1979), 255-259.

LOCALLY SYMMETRIC EINSTEIN KAEHLER MANIFOLDS AND SPECTRAL GEOMETRY

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(Received July 27, 1978, revised October 19, 1978)

1. Introduction. Let (M, g) be a compact connected Riemannian manifold with the metric tensor g, and Δ be the Laplacian acting on differentiable functions of M, that is,

$$\Delta f = -\sum g^{ji}\nabla_j\nabla_i f ,$$

where ∇_j denotes the covariant differentiation $\nabla_{\partial/\partial x^j}$ with respect to the Riemannian connection. Let $\mathrm{Spec}(M, g) = \{0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots\}$ be the set of eigenvalues of Δ, where each eigenvalue is repeated as many times as its multiplicity. It is an interesting problem to investigate relations between $\mathrm{Spec}(M, g)$ and Riemannian structures.

A useful tool is a formula of Minakshisundaram:

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} \underset{t\to+0}{\sim} (4\pi t)^{-n/2}\sum_{i=0}^{\infty} a_i t^i ,$$

where $n = \dim M$.
Berger [1] has calculated the coefficients a_0, a_1 and a_2,

$$a_0 = \text{volume}\, M = \int_M dV$$

$$a_1 = (1/6)\int_M \tau dV$$

$$a_2 = (1/360)\int_M 5\tau^2 - 2|\rho|^2 + 2|R|^2 dV ,$$

where the notations τ, ρ, R denote the scalar curvature, the Ricci tensor and the curvature tensor, respectively. By difficult calculations, Sakai [4] derived a formula for a_3.

In this paper we prove the following result, making essential use of Sakai's formula.

THEOREM. *Let* (M, g, J) *and* (M', g', J') *be compact connected Einstein Kaehler manifolds with* $\dim_C M = n(\geqq 3)$ *which have nonzero scalar curvatures* τ, τ', *respectively. Assume that* $\mathrm{Spec}(M, g, J) = \mathrm{Spec}(M', g', J')$ (*which implies* $\dim_C M = \dim_C M'$) *and that* $c_1^{n-3}c_3[M] =$

$c'^{n-3}_1 c'_3[M']$. *Then* (M, g, J) *is locally symmetric if and only if* (M', g', J') *is locally symmetric.*

In this theorem $c_i \in H^{2i}(M, Z)$ is the i-th Chern class of M and $c_1^{n-3}c_3[M]$ is a Chern number of M. (Hirzebruch [3])

REMARK 1. The integer $c_1^{n-3}c_3[M]$ depends only on the complex structure of M.

REMARK 2. Let (M, g) and (M', g') be compact connected Einstein manifolds and assume that $\mathrm{Spec}\,(M, g) = \mathrm{Spec}\,(M', g')$. The following results are known.

(1) For $\dim M = 6$, assume that $\chi(M) = \chi(M')$. Then (M, g) is locally symmetric if and only if (M', g') is locally symmetric. (Sakai [4])
(2) For $\dim M \leqq 5$, (M, g) is locally symmetric if and only if (M', g') is locally symmetric. (Donnelly [2])

The author wishes to thank Professor K. Ogiue for his many valuable comments.

2. **Preliminaries.** Let (M, g, J) be an n-dimensional Kaehler manifold, and $e_1, e_2, \cdots, e_n, Je_1, Je_2, \cdots, Je_n$ be local orthonormal frames. We set

$$Z_\alpha = \{e_\alpha - \sqrt{-1}\,Je_\alpha\}/2$$
$$\bar{Z}_\alpha = \{e_\alpha + \sqrt{-1}\,Je_\alpha\}/2$$

and we denote dual frames by $\theta^1, \theta^2, \cdots, \theta^n, \bar{\theta}^1, \bar{\theta}^2, \cdots, \bar{\theta}^n$. With respect to these frames, local components of g are given by

$$g_{\alpha\bar{\beta}} = (1/2)\delta_{\alpha\beta}\ .$$

Then the fundamental 2-form Φ is given by

$$\Phi = (\sqrt{-1}/2)\sum \theta^\alpha \wedge \bar{\theta}^\alpha\ .$$

Let $\Omega^\alpha_\beta = \sum R^\alpha_{\beta\gamma\bar{\delta}}\theta^\gamma \wedge \bar{\theta}^\delta$ be the curvature form of M. Then the curvature tensor of M is the tensor field with local components $R^\alpha_{\beta\gamma\bar{\delta}}$, which will be denoted by R. The Ricci tensor ρ and the scalar curvature τ are given by

$$\rho = \sum \rho_{\alpha\bar{\beta}}\theta^\alpha \otimes \bar{\theta}^\beta + \bar{\rho}_{\alpha\bar{\beta}}\bar{\theta}^\alpha \otimes \theta^\beta$$
$$\tau = 4\sum \rho_{\alpha\bar{\alpha}}\ ,$$

where $\rho_{\alpha\bar{\beta}} = \sum R^\gamma_{\alpha\gamma\bar{\beta}}$.
If we define a closed $2k$-form γ_k by

$$\gamma_k = ((-1)^{k/2}/(2\pi)^k k!) \sum \delta^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_k} \Omega^{\beta_1}_{\alpha_1} \wedge \Omega^{\beta_2}_{\alpha_2} \cdots \wedge \Omega^{\beta_k}_{\alpha_k} ,$$

then the k-th Chern class c_k of M is represented by γ_k. By (,) we denote a local inner product in the space of p-forms. The inner product of θ^α and $\bar{\theta}^\beta$ is $2\delta_{\alpha\beta}$.

LEMMA 1.

$$\begin{aligned}(\Phi^3, \gamma_3) = (1/64\pi^3)\{&\tau^3 - 12\tau|\rho|^2 + 3\tau|R|^2 + 256 \sum \rho_{\alpha\bar{\beta}}\rho_{\beta\bar{\gamma}}\rho_{\gamma\bar{\alpha}} \\ &+ 384 \sum \rho_{\alpha\bar{\beta}}\rho_{\gamma\bar{\delta}}R^{\alpha}_{\beta\delta\bar{\gamma}} - 768 \sum \rho_{\alpha\bar{\beta}}R^{\gamma}_{\beta\delta\bar{\lambda}}R^{\alpha}_{\gamma\lambda\bar{\delta}} \\ &+ 128 \sum R^{\alpha}_{\gamma\delta\bar{\lambda}}R^{\beta}_{\alpha\lambda\bar{\mu}}R^{\gamma}_{\beta\mu\bar{\delta}} + 128 \sum R^{\alpha}_{\beta\delta\bar{\lambda}}R^{\beta}_{\gamma\lambda\bar{\mu}}R^{\gamma}_{\alpha\mu\bar{\delta}}\} ,\end{aligned}$$

where $|R|$ and $|\rho|$ denote the lengths of the curvature tensor and the Ricci tensor, respectively, so that

$$|R|^2 = 16 \sum R^{\alpha}_{\beta\gamma\bar{\delta}}R^{\beta}_{\alpha\delta\bar{\gamma}} , \quad |\rho|^2 = 8 \sum \rho_{\alpha\bar{\beta}}\rho_{\beta\bar{\alpha}} .$$

PROOF. By definition,

$$\begin{aligned}\gamma_3 = (-\sqrt{-1}/48\pi^3) \sum \{&\Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\gamma}_{\gamma} + \Omega^{\beta}_{\alpha} \wedge \Omega^{\gamma}_{\beta} \wedge \Omega^{\alpha}_{\gamma} + \Omega^{\gamma}_{\alpha} \wedge \Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\gamma} \\ &- \Omega^{\beta}_{\alpha} \wedge \Omega^{\alpha}_{\beta} \wedge \Omega^{\gamma}_{\gamma} - \Omega^{\gamma}_{\alpha} \wedge \Omega^{\beta}_{\beta} \wedge \Omega^{\alpha}_{\beta} - \Omega^{\alpha}_{\gamma} \wedge \Omega^{\gamma}_{\beta} \wedge \Omega^{\beta}_{\gamma}\}\end{aligned}$$

$$\Phi^3 = (-\sqrt{-1}/8) \sum \theta^\alpha \wedge \bar{\theta}^\alpha \wedge \theta^\beta \wedge \bar{\theta}^\beta \wedge \theta^\gamma \wedge \bar{\theta}^\gamma .$$

After calculations we get the result.

LEMMA 2. *Let (M, g, J) be an n-dimensional ($n \geqq 3$) compact connected Einstein Kaehler manifold with a nonzero scalar curvature τ. Then*

$$\int_M (\Phi^3, \gamma_3) dV = (4n\pi/\tau)^{n-3}(3!/(n-3)!)c_1^{n-3}c_3[M] .$$

PROOF. c_1 is represented by

$$\gamma_1 = (\sqrt{-1}/2\pi) \sum \Omega^{\alpha}_{\alpha} = (\sqrt{-1}/2\pi) \sum \rho_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta .$$

Since M is an Einstein manifold, we have $\rho_{\alpha\bar{\beta}} = (\tau/2n)g_{\alpha\bar{\beta}} = (\tau/4n)\delta_{\alpha\beta}$ and hence

$$\gamma_1 = (\sqrt{-1}\tau/8n\pi) \sum \theta^\alpha \wedge \bar{\theta}^\alpha = (\tau/4n\pi)\Phi .$$

Therefore

$$\begin{aligned}(\Phi^3, \gamma_3)dV &= \gamma_3 \wedge * \Phi^3 = \gamma_3 \wedge (3!/(n-3)!)\Phi^{n-3} \\ &= (3!/(n-3)!)(4n\pi/\tau)^{n-3}\gamma_3 \wedge \gamma_1^{n-3} .\end{aligned}$$

Thus we have

$$\begin{aligned}\int_M (\Phi^3, \gamma_3)dV &= (3!/(n-3)!)(4n\pi/\tau)^{n-3}\int_M \gamma_3 \wedge \gamma_1^{n-3} \\ &= (3!/(n-3)!)(4n\pi/\tau)^{n-3}c_1^{n-3}c_3[M] .\end{aligned}$$

3. Proof of Theorem. Let R, ρ, τ (resp. R', ρ', τ') be the curvature tensor, the Ricci tensor and the scalar curvature of M (resp. M'), respectively. It follows directly from the formulas for the coefficients a_0, a_1 and a_2, that $\tau = \tau'$ and that

$$\int_M |R|^2 dV = \int_{M'} |R'|^2 dV' .$$

Moreover since scalar curvatures are constant, we have

$$(1) \qquad \int_M \tau^3 dV = \int_{M'} \tau'^3 dV'$$

and

$$(2) \qquad \int_M \tau |R|^2 dV = \int_{M'} \tau' |R'|^2 dV' .$$

Next we notice the following:

$$\sum R^*_{ijkl} R^*_{ijuv} R^*_{kluv} = 64 \sum R^{\alpha}_{\beta\delta\bar{\lambda}} R^{\beta}_{\gamma\lambda\bar{\mu}} R^{\gamma}_{\alpha\mu\bar{\delta}} ,$$
$$\sum R^*_{ijkl} R^*_{iukv} R^*_{julv} = 16 \sum (R^{\alpha}_{\beta\delta\bar{\lambda}} R^{\beta}_{\gamma\lambda\bar{\mu}} R^{\gamma}_{\alpha\mu\bar{\delta}} - R^{\alpha}_{\gamma\delta\bar{\lambda}} R^{\beta}_{\alpha\lambda\bar{\mu}} R^{\gamma}_{\beta\mu\bar{\delta}}) ,$$

where R^*_{ijkl} denote the components of R with respect to the real local orthonormal frames. Since M and M' are Einstein, our assumption $a_3 = a'_3$, together with Sakai's formula in [4], implies

$$(3) \qquad \int_M \{\tau^3(5/9 - 1/3n - 4/63n^2) + \tau|R|^2(2/3 + 68/105n) + (3/5)|\nabla R|^2 + (14336/315) \sum R^{\alpha}_{\gamma\delta\bar{\lambda}} R^{\beta}_{\alpha\lambda\bar{\mu}} R^{\gamma}_{\beta\mu\bar{\delta}} - (20992/315) \sum R^{\alpha}_{\beta\delta\bar{\lambda}} R^{\beta}_{\gamma\lambda\bar{\mu}} R^{\gamma}_{\alpha\mu\bar{\delta}}\} dV$$
$$= \int_{M'} \{\tau'^3(5/9 - 1/3n - 4/63n^2) + \tau'|R'|^2(2/3 + 68/105n) + (3/5)|\nabla R'|^2 + (14336/315) \sum R'^{\alpha}_{\gamma\delta\bar{\lambda}} R'^{\beta}_{\alpha\lambda\bar{\mu}} R'^{\gamma}_{\beta\mu\bar{\delta}} - (20992/315) \sum R'^{\alpha}_{\beta\delta\bar{\lambda}} R'^{\beta}_{\gamma\lambda\bar{\mu}} R'^{\gamma}_{\alpha\mu\bar{\delta}}\} dV' .$$

Since $c_1^{n-3}c_3[M] = c_1'^{n-3}c'_3[M']$ and M, M' have the same nonzero constant scalar curvatures, Lemma 2 implies

$$\int_M (\Phi^3, \gamma_3) dV = \int_{M'} (\Phi'^3, \gamma'_3) dV' .$$

Then, using Lemma 1 in the Einstein case, we get the following equation.

$$(4) \qquad \int_M \{\tau^3(1 - 6/n + 10/n^2) + (3 - 12/n)\tau|R|^2 + 128 \sum R^{\alpha}_{\gamma\delta\bar{\lambda}} R^{\beta}_{\alpha\lambda\bar{\mu}} R^{\gamma}_{\beta\mu\bar{\delta}} + 128 \sum R^{\alpha}_{\beta\delta\bar{\lambda}} R^{\beta}_{\gamma\lambda\bar{\mu}} R^{\gamma}_{\alpha\mu\bar{\delta}}\} dV$$

$$= \int_{M'} \{\tau'^3(1 - 6/n + 10/n^2) + (3 - 12/n)\tau'|R'|^2 + 128 \sum R'^{\alpha}_{\gamma\delta\bar{\lambda}} R'^{\beta}_{\alpha\lambda\bar{\mu}} R'^{\gamma}_{\beta\mu\bar{\delta}} + 128 \sum R'^{\alpha}_{\beta\delta\bar{\lambda}} R'^{\beta}_{\gamma\lambda\bar{\mu}} R'^{\gamma}_{\alpha\mu\bar{\delta}}\} dV' .$$

By computing $\Delta |R|^2$ and applying Green's theorem, we obtain

$$(5) \qquad \int_M \{(\tau/4n)|R|^2 + (1/4)|\nabla R|^2 + 16 \sum R^{\alpha}_{\gamma\delta\bar{\lambda}} R^{\beta}_{\alpha\lambda\bar{\mu}} R^{\gamma}_{\beta\mu\bar{\delta}} - 32 \sum R^{\alpha}_{\beta\delta\bar{\lambda}} R^{\beta}_{\gamma\lambda\bar{\mu}} R^{\gamma}_{\alpha\mu\bar{\delta}}\} dV = 0 .$$

Similarly we have

$$(6) \qquad \int_{M'} \{(\tau'/4n)|R'|^2 + (1/4)|\nabla R'|^2 + 16 \sum R'^{\alpha}_{\gamma\delta\bar{\lambda}} R'^{\beta}_{\alpha\lambda\bar{\mu}} R'^{\gamma}_{\beta\mu\bar{\delta}} - 32 \sum R'^{\alpha}_{\beta\delta\bar{\lambda}} R'^{\beta}_{\gamma\lambda\bar{\mu}} R'^{\gamma}_{\alpha\mu\bar{\delta}}\} dV' = 0 .$$

By (1), (2), (3), (4), (5) and (6) we have

$$\int_M |\nabla R|^2 dV = \int_{M'} |\nabla R'|^2 dV' ,$$

from which Theorem follows.

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