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THE FIRST EIGENVALUE OF THE LAPLACIAN ON SPHERES

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1. Introduction. Let (S^2, h) be a 2-dimensional sphere with metric h, and let $\lambda_0 = 0 < \lambda_1 = \lambda_1(h) \leqq \lambda_2 \leqq \cdots$ be eigenvalues of the Laplacian Δ on (S^2, h) acting on smooth functions. J. Hersch [3] showed that

$$(*) \qquad 1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 \geqq (3/8\pi)\mathrm{Vol}(S^2, h)$$

holds, and in particular

$$(**) \qquad \lambda_1(h)\mathrm{Vol}(S^2, h) \leqq 8\pi ,$$

where $\mathrm{Vol}(S^2, h)$ denotes the volume of S^2 with respect to h. Equality in $(*)$ or $(**)$ holds if and only if h is a constant curvature metric.

M. Berger [1] showed that $(*)$ cannot be generalized for (S^m, h), $m \geqq 3$. With respect to $(**)$, M. Berger [1] posed a problem: Let M be a compact smooth manifold; then does there exist a constant $k(M)$ depending only on M such that the first eigenvalue $\lambda_1(h)$ of the Laplacian satisfies

$$(***) \qquad \lambda_1(h)\mathrm{Vol}(M, h)^{2/m} \leqq k(M)$$

for any Riemannian metric h?

H. Urakawa [5] showed the following: Let G be a compact connected Lie group with a non-trivial commutator subgroup; then there exists a family of left invariant Riemannian metrics $g(t)$ $(0 < t < \infty)$ on G such that

$$(****) \qquad \begin{cases} \lambda_1(g(t)) \to \infty & \text{as} \quad t \to \infty , \\ \lambda_1(g(t)) \to 0 & \text{as} \quad t \to 0 \end{cases}$$

and $\mathrm{Vol}(G, g(t)) = $ constant. In particular, since $SU(2)$ is diffeomorphic to S^3, there exists no constant $k(S^3)$ for a 3-dimensional sphere S^3 such that $(***)$ holds.

The purpose of this paper is to prove that for any odd dimensional sphere S^{2n+1} there exists no constant $k(S^{2n+1})$ such that $(***)$ holds. Namely we show the following.

THEOREM. *Any odd dimensional sphere S^{2n+1}, $n \geqq 1$, admits a family of Riemannian metrics $g(t)$ $(0 < t < \infty)$ such that the first* *eigenvalue* $\lambda_1(g(t))$ *of the Laplacian satisfies* (****) *and* $\mathrm{Vol}(S^{2n+1}, g(t))=$ *constant.*

2. Definition of $g(t)$. Let E^{m+1} be a Euclidean $(m+1)$-space and CE^{n+1} be a complex Euclidean $(n+1)$-space. Let (S^m, g) $(m=2n+1\geqq 3)$ be a unit sphere in E^{m+1} with the induced metric g and let ξ be a natural Sasakian structure on (S^m, g). That is, ξ is a unit Killing vector field with respect to g on S^m such that, for $x\in E^{m+1}\cap S^m$, two vectors x and ξ_x determine a holomorphic plane in E^{m+1} with respect to the complex structure of $CE^{n+1}=E^{m+1}$. Let η be the 1-form on S^m dual to ξ with respect to g. We define a one parameter family of Riemannian metrics $g(t)$ by

$$g(t) = t^{-1}g + (t^{m-1} - t^{-1})\eta\otimes\eta\ , \quad 0<t<\infty\ . \tag{2.1}$$

Easily we get

LEMMA 2.1. *Volume elements with respect to* $g(t)$ *and* $g(1)=g$ *are identical;* $dS^m(g(t)) = dS^m(g)$, *and* $\mathrm{Vol}(S^m, g(t)) = \mathrm{Vol}(S^m, g)$.

From now on by dS^m we denote both of the volume elements.

By (g^{jk}) we denote the inverse matrix of (g_{ij}) in a local coordinate neighborhood (U, x^i). By ∇ and Δ we denote the Riemannian connection and the Laplacian with respect to g. We also write ${}^{(t)}g$ instead of $g(t)$. $({}^{(t)}g^{jk})$, ${}^{(t)}\nabla$ and ${}^{(t)}\Delta$ etc. denote ones with respect to ${}^{(t)}g$. The relation between $({}^{(t)}g^{jk})$ and (g^{jk}) is given by (cf. S. Tanno [4], p. 702)

$${}^{(t)}g^{jk} = tg^{jk} - t(1-t^{-m})\xi^j\xi^k\ . \tag{2.2}$$

The difference $W^i_{jk} = {}^{(t)}\Gamma^i_{jk} - \Gamma^i_{jk}$ of the Christoffel's symbols is given by ([4], p. 702)

$$W^i_{jk} = (1-t^m)\ (\phi^i_j\eta_k + \eta_j\phi^i_k)\ ,$$

where $\phi^i_j = -\nabla_j\xi^i$. Note that $\phi\xi=0$ and hence,

$$g^{jk}W^i_{jk} = 0\ , \quad \xi^j\xi^k W^i_{jk} = 0\ .$$

Let f be a function on S^m and put $df = (f_i) = (\partial f/\partial x^i)$. Then

$${}^{(t)}\Delta f = {}^{(t)}g^{jk}\ {}^{(t)}\nabla_j f_k\ .$$

By (2.2) and ${}^{(t)}\nabla_j f_k = \nabla_j f_k - W^r_{jk}f_r$, we get

$${}^{(t)}\Delta f = t\Delta f - t(1-t^{-m})L_\xi L_\xi f\ , \tag{2.3}$$

where L_ξ denotes the Lie derivation by ξ and

$$L_\xi L_\xi f = \nabla_j f_k \xi^j\xi^k\ .$$

3. Eigenfunctions on (S^m, g). Contrary to the case of the introduction, we denote by λ_k the k-th eigenvalue with multiplicity $\mu(\lambda_k)$. Then, (cf. for example, [2])

$$\mathrm{Spec}(S^m, g) = \{\lambda_k = k(m + k - 1);\ k = 0, 1, 2, \cdots\}\ ,$$
$$\mu(\lambda_k) = \binom{m+k}{k} - \binom{m+k-2}{k-2}\ ;\quad k \geqq 2\ ,$$

$\mu(\lambda_0) = 1$ and $\mu(\lambda_1) = m + 1$. Let $\{\varphi_{k,v}\}$ be a complete basis of the space of smooth functions on S^m, $m = 2n + 1$, such that

$$\Delta\varphi_{k,v} + \lambda_k\varphi_{k,v} = 0\ ,\quad v = 1, 2, \cdots, \mu(\lambda_k)$$
$$\langle\varphi_{k,v}, \varphi_{j,r}\rangle = \delta_{kj}\delta_{vr}\ ,$$

where $\langle f_1, f_2\rangle = \int f_1 f_2 dS^m$ for functions f_1 and f_2.

By $V(\lambda_k)$ we denote the eigenspace corresponding to the eigenvalue λ_k.

With respect to the complex projective space (CP^n, g_0) with the Fubini-Study metric g_0 of constant holomorphic sectional curvature 4, it is known that

$$\mathrm{Spec}(CP^n, g_0) = \{\kappa_q = 4q(n + q);\ q = 0, 1, 2, \cdots\}\ ,$$
$$\mu(\kappa_q) = \binom{n+q}{q}^2 - \binom{n+q-1}{q-1}^2,\quad q \geqq 1\ .$$

Let $W(\kappa_q)$ denote the subspace of $V(\lambda_{2q})$ which is invariant by $\exp s\xi$, that is, each element of $W(\kappa_q)$ is a lift of an eigenfunction corresponding to the q-th eigenvalue $\kappa_q = \lambda_{2q}$ of the Laplacian on CP^n, by the Hopf fibration;

$$\pi\colon (S^{2n+1}, g) \to (CP^n, g_0) = (S^{2n+1}/\xi, g_0)\ .$$

Let $(x^\alpha, y^\alpha; \alpha = 1, \cdots, n + 1)$ be coordinates in $E^{m+1} = CE^{n+1}$. For a point $x = (x_0^\alpha, y_0^\alpha)$ of S^m, Jx is given by

$$Jx = (y_0^\alpha, -x_0^\alpha)\ ,$$

where J is the complex structure of CE^{n+1}. Then the trajectory $l = \{l(s)\}$ of ξ passing through the point x is a great circle of S^m and is given by

$$l(s) = (x_0^\alpha \cos s + y_0^\alpha \sin s,\ y_0^\alpha \cos s - x_0^\alpha \sin s)\ .$$

Let f be a function in $V(\lambda_k)$. Since f is the restriction $F|S^m$ of a harmonic homogeneous polynomial F of degree k in E^{m+1}, writing down F and substituting $l(s)$, we see that $f(s) = F(s) = F(l(s))$ is of the form;

$$f(s) = \sum_{\nu=0}^{k} Q_\nu \cos^\nu s \sin^{k-\nu} s \ , \tag{3.1}$$

where Q_ν are constants depending on l.

Now operating L_ξ to $\Delta f + \lambda_k f = 0$ and noticing that L_ξ and Δ commute, we see that L_ξ is a linear transformation of $V(\lambda_k)$. By Green's theorem we get

$$\langle L_\xi f, h \rangle = \int \xi^i f_i h dS^m = -\int \xi^i f h_i dS^m \ ,$$

and hence, $\langle L_\xi f, h \rangle + \langle f, L_\xi h \rangle = 0$ holds for any C^1-functions f and h. Therefore L_ξ is a skew-symmetric linear transformation of $V(\lambda_k)$.

LEMMA 3.1. *For each eigenvalue λ_k of Δ, $V(\lambda_k)$ has the orthogonal decomposition [here we do not care if some $V_\theta(\lambda_k)$ is trivial or not]:*

$$V(\lambda_k) = V_k(\lambda_k) + V_{k-2}(\lambda_k) + \cdots + V_{k-2[k/2]}(\lambda_k) \ , \tag{3.2}$$

where $[k/2]$ is the integral part of $k/2$, and for $\varphi \in V_\theta(\lambda_k)$, $\theta = k - 2p$,

$$L_\xi L_\xi \varphi + (k-2p)^2 \varphi = 0 \ , \quad 0 \leqq p \leqq [k/2] \ . \tag{3.3}$$

PROOF. Since L_ξ is a skew-symmetric transformation of $V(\lambda_k)$, each non-zero eigenvalue of L_ξ is purely imaginary. Hence, each eigenvalue of $L_\xi L_\xi$ is real and non-positive. Let f be an eigenfunction of $L_\xi L_\xi$;

$$L_\xi L_\xi f + \theta^2 f = 0 \ , \qquad \theta \geqq 0 \ . \tag{3.4}$$

Solving (3.4) on $l = \{l(s)\}$, we get

$$f(s) = b \sin(\theta s + c) \ , \tag{3.5}$$

where b and c are constants depending on l. (3.1) and (3.5) imply that θ is of the following form;

$$\theta = k, k-2, \cdots, k - 2[k/2]$$

according as the expression of (3.1) reduces to the lower degree. Here, $\theta = k - 2p$ means that the degree of the reduced expression of (3.1) is equal to $k - 2p$ for some l and $\leqq k - 2p$ for any l. Denoting by $V_\theta(\lambda_k)$ the eigenspace of $L_\xi L_\xi$ corresponding to $-\theta^2 = -(k-2p)^2$, we have the decomposition (3.2). q.e.d.

REMARK 1. We show that $V_k(\lambda_k) \neq \{0\}$. Let F be a harmonic homogeneous polynomial of degree k in $E^{m+1}(x^\alpha, y^\alpha)$ such that

$$F = F(x^1, x^2) = (x^1)^k + a_1 (x^1)^{k-1}(x^2) + \cdots + a_k (x^2)^k \ .$$

Take the trajectory l of ξ passing through the point $(1, 0, \cdots, 0)$. Then l lies in the (x^1, y^1)-plane (i.e., $x^2 = 0$) and $F(s) = F(l(s))$ is of degree k.

REMARK 2. We show that $V_1(\lambda_k) \neq \{0\}$ (in this case k = odd) and $V_2(\lambda_k) \neq 0$ (in this case k = even) for $m = 2n + 1 \geqq 5$. We extend ξ to a vector field ${}^\#\xi$ on E^{m+1} by

$$ {}^\#\xi = y^\alpha(\partial/\partial x^\alpha) - x^\alpha(\partial/\partial y^\alpha) . $$

Then $L_{({}^\#\xi)}F = 0$ if and only if $L_\xi(F|S^m) = 0$ for any homogeneous polynomial F in E^{m+1}. For each $2q$, let F be a (non-trivial) harmonic homogeneous polynomial of degree $2q$ in E^{m+1} such that

$$ F = F(x^1, \cdots, x^n, y^1, \cdots, y^n) , $$
$$ L_{({}^\#\xi)}F = 0 . $$

Existence of such an F is seen by considering the Hopf fibration $\pi: S^{m-2} \to CP^{n-1}$. We put

$$ F^* = x^{n+1}F , \quad {}^*F = x^{n+1}y^{n+1}F . $$

Since

$$ \partial F/\partial x^{n+1} = \partial F/\partial y^{n+1} = 0 , $$

we see that F^* is a harmonic homogeneous polynomial of degree $2q + 1$, and ${}^*F$ is a harmonic homogeneous polynomial of degree $2q + 2$ in E^{m+1}. By $L_{({}^\#\xi)}F = 0$, we see that $F^*(s)$ is of degree 1, and ${}^*F(s)$ is of degree 2. Thus, for $k = 2q + 1$, $V_1(\lambda_k) \neq \{0\}$, and for $k = 2q + 2$, $V_2(\lambda_k) \neq \{0\}$.

REMARK 3. For $m = 3$ we show that $V_1(\lambda_3) \neq \{0\}$ and $V_2(\lambda_4) \neq \{0\}$. First we notice that $L_{({}^\#\xi)}F = 0$, where

$$ F = a[(x^1)^2 + (y^1)^2] + b[(x^2)^2 + (y^2)^2] . $$

Next we verify that, if $b = -2a$, x^1F is a harmonic homogeneous polynomial of degree 3 in E^4, and $(x^1F)(s)$ is of degree 1. Similarly, if $b = -a$, x^1x^2F is a harmonic homogeneous polynomial of degree 4 in E^4, and $(x^1x^2F)(s)$ is of degree 2.

REMARK 4. For $k = 2q$, $V_0(\lambda_k) = W(\kappa_q) \neq \{0\}$. So, by above remarks we see that in the decompositions;

$$ \begin{aligned} V(\lambda_1) &= V_1(\lambda_1) , \\ V(\lambda_2) &= V_2(\lambda_2) + V_0(\lambda_2) , \\ V(\lambda_3) &= V_3(\lambda_3) + V_1(\lambda_3) , \\ V(\lambda_4) &= V_4(\lambda_4) + V_2(\lambda_4) + V_0(\lambda_4) , \end{aligned} $$

all subspaces are non-trivial.

4. **Eigenfunctions on** $(S^m, g(t))$. Let $\{\varphi_{k,v}\}$ be a complete orthonormal

base stated in Section 3. By Lemma 3.1 we can assume that each $\varphi_{k,v}$ is contained in some $V_\theta(\lambda_k)$ in (3.2).

LEMMA 4.1. *Each eigenfunction $\varphi_{k,v}$ of Δ corresponding to λ_k is also an eigenfunction of ${}^{(t)}\Delta$ corresponding to*

$$t\lambda_k - t(1 - t^{-m})(k - 2p)^2 , \quad 0 \leqq p \leqq [k/2] \tag{4.1}$$

according as $\varphi_{k,v} \in V_{k-2p}(\lambda_k)$.

In particular, each eigenvalue of ${}^{(t)}\Delta$ takes the above form.

PROOF. (4.1) follows from (2.3) and (3.3). Since $\{\varphi_{k,v}\}$ is also a complete orthonormal base of the space of smooth functions on S^m with respect to $g(t)$, and since each $\varphi_{k,v}$ is an eigenfunction of ${}^{(t)}\Delta$, Spec(S^m, $g(t)$) is given by eigenvalues for $\{\varphi_{k,v}\}$ (cf. [2], Lemma A.II. 1, p. 143).

PROPOSITION 4.2. *The first eigenvalue of $(S^m, g(t))$, $m = 2n + 1$, is given by*

$$\lambda_1(g(t)) = \begin{cases} (2n + t^{-m})t & \text{for} \quad t^{-m} \leqq m + 3 \\ 4(n + 1)t & \text{for} \quad t^{-m} \geqq m + 3 . \end{cases}$$

In particular,

$$2nt < \lambda_1(g(t)) \leqq 4(n + 1)t , \quad 0 < t < \infty .$$

PROOF. Since $\lambda_k = k(m + k - 1)$, by (4.1) the first (non-zero) eigenvalue can be found among

(i) $tk(m + k - 1) - t(1 - t^{-m})k^2 \qquad k \geqq 1$,
(ii) $tk(m + k - 1) - t(1 - t^{-m}) \qquad k = \text{odd} \geqq 1$,
(iii) $tk(m + k - 1) \qquad k = \text{even} \geqq 2$.

The minimum for (i), (ii) is given by $t(m - 1 + t^{-m})$, and the minimum for (iii) is given by $2t(m + 1)$. q.e.d.

5. **Remarks.** (a). In Proposition 4.2 the multiplicity of $(2n + t^{-m})t$ for $t^{-m} < m + 3$ is $\mu(\lambda_1) = m + 1$. The multiplicity of $4(n + 1)t$ for $t^{-m} > m + 3$ is

$$\dim V_0(\lambda_2) = \mu(\kappa_1) = \begin{pmatrix} n + 1 \\ 1 \end{pmatrix}^2 - 1 = n(n + 2) .$$

The multiplicity of $4(n + 1)t$ for $t^{-m} = m + 3$ is equal to the sum of the above two; $n^2 + 4n + 2$. Thus,

There exists a Riemannian metric on S^m $(m = 2n + 1)$ such that the first eigenvalue has multiplicity $(m^2 + 6m + 1)/4$.

There is a natural problem: What is the maximum of multiplicity of the first eigenvalue of the Laplacian (for fixed dimension m of compact manifolds)?

(b). M. Berger [1] showed the existence of a Riemannian metric h on S^m, $m \geqq 3$, such that, for the first $m+1$ eigenvalues $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{m+1}$,

$$\sum_{j=1}^{m+1} \frac{1}{\lambda_j} < \frac{m+1}{m} \cdot \frac{\mathrm{Vol}(S^m, h)^{2/m}}{\mathrm{Vol}(S^m, g)^{2/m}} \tag{5.1}$$

holds, where g is a constant curvature metric. This is a counter-example to the natural generalization of $(*)$ in the introduction.

For each odd dimensional sphere S^{2n+1}, as a simple example of such a Riemannian metric h we may put $h = g(t)$ given by (2.1) where t is sufficiently near 1. In fact, $\lambda_1(g(t)) = (2n + t^{-m})t$ has multiplicity $m+1$, and

$$(2n + t^{-m})t > m .$$

Thus, (5.1) holds for any $g(t)$; $t^{-m} < m + 3$, $t \neq 1$.

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