ON THE GAUSS-BONNET FORMULA ON
ODD-DIMENSIONAL MANIFOLDS

A. REVENTÓS

1. Introduction. This work can be considered as a continuation of "A formula on some odd-dimensional Riemannian manifolds related to the Gauss-Bonnet formula" by S. Tanno [7].

Let B be a compact orientable Riemannian manifold of dimension $2n$. The Gauss-Bonnet-Chern formula says

$$(1) \qquad \int_B Q = 2^{2n}\pi^n n!\, \chi(B)$$

where Q denotes the Gauss-Bonnet form and $\chi(B)$ is the Euler-Poincaré characteristic. In this formula even dimensionality is essential. If B is odd-dimensional, then $\chi(B) = 0$. However, for odd-dimensional manifolds, we can think on the possibility of finding an analogous formula in which the right hand side is a certain expression containing the Betti numbers.

On trying to find a formula analogous to (1) we expect the non-vanishing fields on the manifold to play an important role.

On this line, S. Tanno [7] found the following result:

THEOREM. *Let (E^{2n+1}, g) be a compact Riemannian manifold having a unit Killing vector field ξ such that $R(X, \xi)Y = g(X, Y)\xi - g(Y, \xi)X$, for any X, Y vector fields on E^{2n+1} (i.e., a Sasakian manifold). Then, if the field ξ is regular (Palais [5]), we have*

$$(2) \qquad ((-1)^n/\ell(\xi)2^{2n}\pi^n n!)\int_E F(\Omega, \xi) = \sum_{r=0}^{n}(n+1-r)(-1)^r b_r(E),$$

where $b_i(E)$ is the i-th Betti number of E and $F(\Omega, \xi)$ is an expression which depends on the curvature Ω and the field ξ.

We want to obtain an expression like (2) for odd-dimensional Riemannian manifolds, which are not necessarily Sasakian.

However it seems convenient to continue assuming ξ to be a regular unit Killing vector field. Concretely we state the following problem.

Let E be a compact manifold of dimension $2n + 1$ with a regular field ξ on E. Let g be a Riemannian metric on E for which ξ is a unit Killing vector field. We want to find under these assumptions a formula

analogous to (2). The simplest example of a manifold satisfying our hypothesis and not Tanno's is the following:

Let B be a compact orientable Riemannian manifold of dimension $2n$. Let us suppose that $b_1(B)$ is even and $b_0(B) = 1$. We consider the product $B \times S^1$. A unit tangent field to S^1 defines a unit regular Killing vector field on $B \times S^1$, related to the product metric. So, $B \times S^1$ satisfies our hypothesis, but since $b_1(B \times S^1)$ is odd, on $B \times S^1$ there is not any Sasakian structure because it is well known that the first Betti number of a Sasakian manifold is zero or even.

The first result which we obtain is the following.

THEOREM 1. *Let* (E, ξ, g) *be a compact orientable Riemannian manifold of dimension* $2n + 1$, *with* ξ *a regular unit Killing vector field on* E. *Then*

$$(3) \qquad ((-1)^n/\angle(\xi)2^{2n}\pi^n n!) \int_E f(\Omega, \xi) = \sum_{r=0}^{n} (n + 1 - r)(-1)^r b_r(E)$$
$$+ \sum_{r=0}^{n-1} (-1)^r d_r ,$$

where $f(\Omega, \xi)$ *is a certain function of* Ω *and* ξ. $b_r(E)$ *is the* r-th *Betti number of* E *and* $d_r = \dim \operatorname{Ker} \{H^r(E/\xi, \mathbf{R}) \to H^{r+2}(E/\xi, \mathbf{R})\}$. *This map is the multiplication by the Euler class of the bundle* $E \to E/\xi$.

The left hand side of (3) depends on g and ξ. However the right hand side depends only on ξ (because d_r depends on ξ). The right hand side is concretely $\chi(E/\xi)$.

Nevertheless, it is possible that, in the most interesting cases, the conditions over the field ξ (unit Killing regular) determine $\chi(E/\xi)$. That is, if ξ and ξ' are two unit Killing regular vector fields on E, then $\chi(E/\xi) = \chi(E/\xi')$. In this case the right hand side of (3) could be calculated knowing only the Betti numbers of E.

In this work, we have tried to solve these problems in low dimensions (dim 3 and dim 5) and we have obtained some particular results for higher dimensions.

I am indebted to Professor J. Girbau who suggested this problem to me and helped in some particular cases.

I also wish to express my sincere gratitude to Professor S. Tanno for an invaluable review of my manuscript and his commentaries on Corollary 2.

2. **Preliminaries and notations.** To state our result in dimension 5 more easily we introduce the following definition.

DEFINITION. Let E be a compact connected manifold of dimension 5. We say that the Betti numbers $b_1(E)$ and $b_2(E)$ satisfy the R_1-condition if they satisfy any of the following conditions:

(1) $b_1(E) = 0$, $b_2(E)$ arbitrary

(2) $b_1(E) = 1$, $b_2(E) = 0$

(3) $b_1(E)$ even $\neq 2$, $b_2(E) = 1$

(4) $b_1(E) = 2$, $b_2(E) = 0$ or 1.

And we say that they satisfy the R_2-condition if

(5) $b_1(E)$ odd $\neq 1$, and $b_2(E) < 12[(4b_1(E) - 4)/10] + 12\delta_{01} + 3 - 4b_1(E)$. ($\delta_{01} = 0$ if $((4b_1(E) - 4)/10) \in Z$, and $\delta_{01} = 1$ otherwise. The bracket is the Gauss symbol meaning the "integer part").

We give now a table which shows, for low Betti numbers, which ones satisfy the R_2-condition.

$b_1(E)$	$b_2(E)$
3	0, 1, 2.
5	0, 1, 2, 3, 4, 5, 6.
7	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
.	.
.	.

In the proof of Theorem 3, we use essentially the classification of complex surfaces given by Kodaira [3]. Let us recall the notation, definitions and some results.

Let S be a compact complex manifold of complex dimension 2. We denote by c_1 and c_2 the first and second Chern classes of S. We represent any cohomology class $c \in H^4(S, Z)$ by the value $c(S)$ of c on the fundamental cycle of the surface S oriented in the natural way with respect to its complex structure. Thus we consider c_1^2 and c_2 as integers.

We define $h^{r,s} = \dim H^s(S, \Omega^r)$ where Ω^r is the sheaf over S of germs of holomorfic r-forms. We set $p_g = h^{2,0} = h^{0,2}$ and $q = h^{0,1}$ and we call p_g and q, respectively, the geometric genus and the irregularity of S.

Moreover, if ξ and η are closed 2-forms on S we define

$$(\xi, \eta) = \int_S \xi \wedge \eta \,.$$

We may consider $(,)$ as a non-singular symmetric bilinear form defined on $H^2(S, R)$. We define b^+ and b^- to be respectively the numbers of positive and negative eigen values of the symmetric bilinear form.

Kodaira obtains the following theorem, which we call

THEOREM A. *If b_1 is even, then $2q = b_1$, $2p_g = b^+ - 1$, $h^{1,0} = q$. If*

b_1 *is odd, then* $2q = b_1 + 1$, $2p_g = b^+$, $h^{1,0} = q - 1$. *Moreover* $c_1^2 + 8q + b^- = 10p_g + 8$ *if* b_1 *is odd, and* $c_1^2 + 8q + b^- = 10p_g + 9$ *if* b_1 *is even.*

We use also the following theorem (Grauert):

"Any surface is obtained from a surface containing no exceptional curve by means of a finite number of quadratic transformations".

Let us recall that an exceptional curve (of the first kind) on S is, by definition, a non-singular rational curve C with $(C^2) = -1$.

Kodaira gives a proof of this theorem by showing that an exceptional curve (of the first kind) C on S is contractible, i.e., there exists $f: S \to S'$ with $f(C) = p$, $S - C$ isomorphe to $S' - p$, and f is a monoidal transformation.

We have $H_2(S, \mathbf{Z}) = H_2(S', \mathbf{Z}) \oplus \mathbf{Z}$. Since $H_2(S, \mathbf{Z})$ is a finitely generated $\mathbf{Z}$-module, in the classification of surfaces, we always assume that the surfaces do not contain exceptional curves.

From the last relation we have $c_1^2(S) = c_1^2(S') - 1$ which for a finite number n_t of quadratic transformations is $c_1^2(S) = c_1^2(S') - n_t$.

We also use the following theorem of Kodaira, which we call

THEOREM B. *We have the following table for surfaces which are free from exceptional curves:*

b_1	P_{12}	P_2	K	c_1^2	structure
even	0	0			plane or ruled
0	1	1	$=0$	0	K3 surface
4	1	1	$=0$	0	complex torus
even	$+$		$\neq 0$	0	elliptic
even	$+$	$+$		$+$	algebraic
odd	$+$			0	elliptic
1	0	0			?

where $P_m = \dim H^0(S, \Omega^0(mK))$ *and* K *is a canonical divisor.*

Finally we recall the following result (Kodaira).

THEOREM C. *If on a complex surface we have* $p_g = 0$ *and* $c_1^2 > 0$, *then* $q = 0$.

3. Proof of Theorem 1. Let E be a compact orientable manifold of dimension $2n + 1$. Let ξ be a regular field on E and g a Riemannian metric on E such that ξ is a unit Killing vector field.

Since ξ is regular and the manifold E is compact the integral curves of ξ are homeomorphic to S^1.

This allows us to define an effective action without fixed points of S^1 on E (Boothby and Wang [1]) and to construct in this way the principal circle bundle $E \to E/\xi = B$, where B denotes the set of orbits, which is a differentiable manifold (Palais [5]).

Let ω be the dual form of ξ by g, that is:

$$\omega(X) = g(\xi, X) \,.$$

Let us now introduce a Riemannian metric on B. For this we define

$$h(X, Y)(p) = g(X', Y')(q) - \omega \otimes \omega(X', Y')(q)$$

where X', Y' are fields on E, which are projected on X, Y respectively, and q is a point on the fibre of p.

It is easy to see that this definition is independent of the point q and of the fields X', Y' chosen. We also see that it is a Riemannian metric.

So, we have that (B, h) is a Riemannian manifold with $g = \pi^* h + \omega \otimes \omega$.

Moreover, since E is orientable, there is a non-vanishing $2n + 1$ form on E. This form, integrated over the fibres (the bundle is orientable), is a non-vanishing $2n$-form on B. So B is orientable.

Therefore on B we have:

$$(4) \qquad \int_B Q = 2^{2n} \pi^n n! \; \chi(B)$$

where $Q = (-1)^n \sum \varepsilon(i_1 \cdots i_{2n}) \Omega_{i_1}^{i_2} \wedge \cdots \wedge \Omega_{i_{2n-1}}^{i_{2n}}$, Ω_j^i denote the curvature forms of the metric h and $\chi(B)$ is the Euler-Poincaré characteristic of B.

Since $\omega(X) = g(\xi, X)$ and ξ is a Killing vector field we have that ω is a connection form on the fibre bundle and so we can write Ω_j^i as a function of the curvature on E with respect to the metric g (Kobayashi).

Let ψ_β^α, ω_j^i be the connection forms on E and B, with respect to g and h respectively. Let Ψ_β^α, Ω_j^i be the curvature forms of the connections defined by ψ_β^α and ω_j^i. Set $\Gamma = d\omega$.

Let U be a small open set in which $h = \sum_j \theta^j \otimes \theta^j$ where $\theta^1, \cdots, \theta^{2n}$ are 1-forms defined on U. $\Gamma = \pi^*(\sum_{ij} A_{ij} \theta^i \wedge \theta^j)$ with $A_{ij} + A_{ji} = 0$. Set $\varphi^0 = \omega$, $\varphi^i = \pi^*(\theta^i)$ so that $g = \sum_\alpha \varphi^\alpha \otimes \varphi^\alpha$.

Connections and curvatures are related as follows:

$$\psi_0^0 = 0$$
$$(5) \qquad \psi_0^i = -\psi_i^0 = -\sum_j A_{ij}\varphi^j$$
$$\psi_j^i = \pi^*(\omega_j^i) - A_{ij}\varphi^0$$

$$\Psi_0^0 = 0$$

$$(6) \qquad \Psi_0^i = -\Psi_i^0 = -\sum_{ks} A_{ik}A_{ks}\varphi^s \wedge \varphi^0 - \sum_{ks} A_{ik;s}\varphi^s \wedge \varphi^k$$

$$\Psi_j^i = \pi^*(\Omega_j^i) - \sum_{ks}(A_{ij}A_{ks} + A_{ik}A_{js})\varphi^k \wedge \varphi^s - \sum_k A_{ij;k}\varphi^k \wedge \varphi^0$$

where $\sum_k A_{ij;k}\theta^k = dA_{ij} - \sum_k A_{ik}\omega_j^k + \sum_k A_{kj}\omega_k^i$. It is easy to see that $A_{ij} = -(1/2)(g(\xi, [e_i, e_j])) = -g(e_i, \nabla_{e_j}\xi)$.

PROPOSITION 1. *In this hypothesis, all trajectories of ξ have the common length $\ell(\xi)$.*

PROOF. Since ξ is a unit vector field we have, by means of its 1-parameter group, a parametrization by arc-length. We set $\lambda(p) = \inf\{t \in R; t > 0 \text{ and } \varphi_t(p) = p\}$ the period of the orbit through p. It is constant on each orbit. Clearly $\lambda(p)$ coincides with the length of the orbit through p.

Let U be an open connected set, and $p \in U$. Let $V = \varphi_t((-\varepsilon, \varepsilon) \times V_0)$ be a neighborhood of p. $\varphi_0(p) = p$ with V_0 transversal with the 1-parameter group. For $x \in V$, by taking a smaller neighborhood if need be, we may suppose that there exists an orthogonal curve $\sigma: [0, 1] \to E$ with $\sigma(0) = p$ and $\sigma(1) = x'$ where x and x' lie on the same orbit.

Then, the transformed σ' of σ by $\varphi_{\lambda(p)}$ is another orthogonal curve (since ξ is a Killing field) with $\sigma(0) = \sigma'(0) = p$ and $\pi \circ \sigma' = \pi \circ \sigma$. Then by a straightforward application of the uniqueness of solutions of ordinary differential equations we have $\sigma = \sigma'$ and so the period of p coincides with the period of x'.

By (4) we have

$$(7) \qquad ((-1)^n/2^{2n}\pi^n n!) \ell(\xi)) \int_E \sum \varepsilon(i_1 \cdots i_{2n})\pi^*\Omega_{i_1}^{i_2}$$

$$\wedge \cdots \wedge \pi^*\Omega_{i_{2n-1}}^{i_{2n}} \wedge \varphi^0 = \chi(B) .$$

If we put $A(ij)$ instead of A_{ij} we have

$$(8) \qquad ((-1)^n/2^{2n}\pi^n n!) \ell(\xi)) \int_E \sum \varepsilon(i_1 \cdots i_{2n})(\Psi_{i_1}^{i_2}$$

$$+ \sum_{ks}(A(i_2 i_1)A(ks) + A(i_2 k)A(i_1 s))\varphi^k \wedge \varphi^s) \wedge \cdots \wedge (\Psi_{i_{2n-1}}^{i_{2n}}$$

$$+ \sum_{ks}(A(i_{2n}i_{2n-1})A(ks) + A(i_{2n}k)A(i_{2n-1}s))\varphi^k \wedge \varphi^s) \wedge \varphi^0$$

$$= \chi(B) .$$

Let us compute $\chi(B)$ in terms of the Betti numbers of E. The exact Gysin sequence of the bundle $\pi: E \to B$ is

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^\#} H^1(E) \xrightarrow{\int_{S^1}^\#} H^0(B) \xrightarrow{D_0} \cdots .$$

Splitting it

$$0 \to H^{p+2}(B)/\operatorname{Im} D_p \to H^{p+2}(E) \to \operatorname{Ker} D_{p+1} \to 0$$

and so, since they are vector spaces, we have:

$$H^{p+2}(E) = \operatorname{Ker} D_{p+1} \oplus \operatorname{Coker} D_p .$$

Then we get the following relations:

$$b_0(B) = b_0(E)$$
$$b_1(B) = b_1(E) - d_0$$
$$b_2(B) = b_2(E) + b_0(E) - d_0 - d_1$$
$$\vdots$$
$$b_n(B) = b_n(E) + b_{n-2}(E) + \cdots + b_{n-2k}(E) - d_0 - d_1 - \cdots - d_{n-1} ,$$

(with $n = 2k$ or $n = 2k + 1$, $d_i = \dim \operatorname{Ker} D_i$ and $D_i \colon H^i(B; R) \to H^{i+2}(B; R)$ the multiplication by the Euler class of the bundle).

Since

$$(9) \qquad \chi(B) = \sum_{r=0}^{n-1} 2b_r(B)(-1)^r + b_n(B)(-1)^n \quad (\dim B = 2n) ,$$

expressing each $b_r(B)$ as a function of $b_r(E)$ and d_r we have

$$(10) \qquad \chi(B) = \sum_{r=0}^{n} (n + 1 - r)(-1)^r b_r(E) + \sum_{r=0}^{n-1} (-1)^r d_r .$$

In this way we get formula (3), with $f(\Omega, \xi)$ given by (8). This $f(\Omega, \xi)$ is independent of the choice of ξ-frame fields.

4. Dimension 3.

THEOREM 2. *Let (E, g) be a compact connected orientable Riemannian manifold of dimension 3. Let ξ be a regular unit Killing vector field on E. Then*

$$(11) \qquad (1/2\pi \angle(\xi)) \int_E (K(\xi^\perp) + 3K(\xi))\eta = 2 - b_1(E)$$

if and only if $b_1(E)$ is even, and

$$(12) \qquad (1/2\pi \angle(\xi)) \int_E (K(\xi^\perp) + 3K(\xi))\eta = 3 - b_1(E)$$

if and only if $b_1(E)$ is odd, where $K(\xi^\perp)$ means the sectional curvature of the plane orthogonal to ξ, and $K(\xi)$ means the sectional curvature of any plane containing ξ (it is independent of the choice of the plane) and η denotes the volume form.

Moreover in the last case the bundle is trivial. (Then it is clear

that the right hand side is independent of ξ, in the sense that if ξ' is another regular unit Killing vector field on E, then $\chi(E/\xi) = \chi(E/\xi')$.

PROOF. $\chi(E/\xi) = 2 - b_1(E) + d_0$. E/ξ is a Riemann surface, so $b_1(E/\xi) \equiv \chi(E/\xi) \equiv 0 \pmod 2$. If $b_1(E)$ is even we have d_0 even and since $d_0 \leqq b_0(E/\xi) = 1$ (by connectedness) we have $d_0 = 0$. If $b_1(E)$ is odd we have d_0 odd and so $d_0 = 1$. By (8) we have

$$(13) \qquad (-1/4\pi\angle(\xi)) \int_E 2(\Psi_1^2 - 3A(12)^2\varphi^1 \wedge \varphi^2) \wedge \varphi^0$$

$$= (1/2\pi\angle(\xi)) \int_E (K(\xi^\perp) + 3K(\xi))\eta = \chi(B) .$$

Let us compute $A(12)^2$. Let X_1, X_2, ξ be an orthonormal frame. $R(\xi, X_1, \xi, X_1) = g(R(\xi, X_1)X_1; \xi) = \Psi_1^0(\xi, X_1) = A(12)A(21)\varphi^1 \wedge \varphi^0(\xi, X_1) = A(12)^2$.

So $A(12)^2$ is the sectional curvature of the plane determinated by ξ and X_1. If $Z = \lambda X_1 + \mu X_2$ with $\lambda^2 + \mu^2 = 1$ it is easy to see that $R(\xi, Z, \xi, Z) = A(12)^2$. So we can say that $A(12)^2 = K(\xi) = $ the sectional curvature of any plane containing ξ.

Analogously we have $\Psi_1^2 \wedge \varphi^0 = -K(\xi^\perp)\varphi^0 \wedge \varphi^1 \wedge \varphi^2$.

So (11) and (12) are proved.

Tanno's theorem in dimension 3 is contained in (11) since for Sasakian manifolds we have $b_1(E)$ even and $K(\xi) = 1$.

5. Dimension 5. In the proof of Theorem 2 we use the fact that E/ξ is a Riemann surface, that is, an algebraic curve. If we want to do an analogous study in dimension 5, we would have to impose at least the condition that E/ξ is a complex manifold.

For this, it is sufficient to assume that E admits an almost contact normal structure from Morimoto's theorem [4].

THEOREM 3. *Let (E, g) be a connected compact Riemannian manifold of dimension 5 with a normal almost contact regular structure (φ, ξ, η), ξ unit Killing vector field. Then*

$$(14) \qquad (1/\angle(\xi)2^4\pi^2 2!) \int_E F(\Omega, \xi) = 3 - 2b_1(E) + b_2(E)$$

if the Betti numbers of E satisfy the R_1-condition. And

$$(15) \qquad (1/\angle(\xi)2^4\pi^2 2!) \int_E F(\Omega, \xi) = 5 - 3b_1(E) + b_2(E)$$

if the Betti numbers of E satisfy the R_2-condition.

It is obvious that the right hand side does not depend on the chosen

unit Killing vector field. In this formula $F(\Omega, \xi)$ is given by (8) where $A(ij) = g(e_i, \nabla_{e_j}\xi)$, φ^k dual of e_k, and $\xi, e_1, \cdots, e_{2n}$ an orthonormal frame.

We prove Theorem 3 studying separately the different cases of conditions R_1 and R_2.

THEOREM 3a. *Let E be a compact connected orientable Riemannian manifold of dimension 5, and ξ a regular unit Killing vector field on E such that E/ξ is a complex manifold. Then, if $b_1(E)$ and $b_2(E)$ satisfy the R_2-condition, we have $E = E/\xi \times S^1$ and $\chi(E/\xi) = 5 - 3b_1(E) + b_2(E)$.*

PROOF. If the fibration is not trivial we have $d_0 = 0$ and so $b_1(B) = b_1(E)$ odd $\neq 1$. By Grauert's theorem we have $B = Q \cdots Q(B')$, with B' free from exceptional curves (of the first kind). The geometric genus p_g and the irregularity q of B are invariant under quadratic transformations. Thus $b_1(B) = b_1(B') \neq 1$ and odd. So by Theorem B we have $c_1^2(B') = 0$. But $c_1^2(B) + n_t = c_1^2(B') = 0$ implies $c_1^2(B) \leq 0$. Moreover $c_1^2(B) + b^-(B) = c_1^2(B') + b^-(B') = b^-(B')$ implies $c_1^2(B) + b^-(B) \geq 0$. From $c_1^2(B) + b^-(B) = 10p_g(B) + 8 - 8q(B) \geq 0$ we have $p_g(B) \geq [(8q(B) - 8)/10] + \delta_{01}$.

Moreover, from the Gysin exact sequence, we have $b_2(E) = b_2(B) - b_0(E) + d_0 + d_1$. And from Theorem A, $b_2(E) = b^+(B) + b^-(B) - 1 + d_0 + d_1 = 2p_g(B) + b^-(B) - 1 + d_1 \geq 2p_g(B) + b^-(B) - 1$. Also from Theorem A we have:

$$b_2(E) \geq 12p_g(B) + 3 - 4b_1(E) , \quad \text{that is ,}$$
$$b_2(E) \geq 12[(4b_1(E) - 4)/10] + 12\delta_{01} + 3 - 4b_1(E) ,$$

but we have excluded this case.

Thus the fibration must be trivial and therefore $d_0 = 1$ and $d_1 = b_1(E) - 1$. So we have $\chi(B) = 5 - 3b_1(E) + b_2(E)$ independently of the unit Killing regular vector field.

THEOREM 3b. *Let E be a compact connected orientable Riemannian manifold of dimension 5, and ξ a regular unit Killing vector field on E such that E/ξ be a complex manifold. Then if $b_1(E)$ is even, $b_1(E) \neq 2$, and $b_2(E) = 0$ or $b_2(E) = 1$, we have $E \neq E/\xi \times S^1$ and $\chi(E/\xi) = 3 - 2b_1(E) + b_2(E)$.*

PROOF. If the fibration is trivial, we have $d_0 = 1$ and $b_1(B) = b_1(E) - 1$. Therefore, $b_1(B)$ is odd and $b_1(B) \neq 1$. As in the above case, we have $B = Q \cdots Q(B')$ with B' free from exceptional curves. Similarly we have $c_1^2(B') = 0$ and so $c_1^2(B) + b^-(B) = c_1^2(B') + b^-(B') \geq 0$. This implies $p_g(B) \geq [(4b_1(E) - 8)/10] + \delta_{01}$ and $b_2(E) \geq 2[(4b_1(E) - 8)/10] + 2\delta_{01}$ and this inequality is not true for $b_2(E) = 0$ or $b_2(E) = 1$.

Thus the fibration must be non-trivial. In this case, $d_0 = 0$ and

$b_1(B) = b_1(E)$ even different from 2. Since $b_2(B) = b_2(E) + 1 - d_1 = 2p_g(B) + 1 + b^-(B)$, if $b_2(E) = 0$ we have $1 - d_1 = 2p_g(B) + 1 + b^-(B)$ and this implies $\chi(E/\xi) = 3 - 2b_1(E)$. This proves the theorem for $b_2(E) = 0$.

If $b_2(E) = 1$, let us see that $c_1^2(B') < 0$: Otherwise we have $c_1^2(B) + b^-(B) \geqq 0$, and then $p_g(B) \geqq [(4b_1(E) - 9)/10] + \delta_{01}$ by Theorem A. Also $b_2(E) = b_2(B) - 1 + d_1 \geqq 2p_g(B)$, and so $b_2(E) \geqq 2[(4b_1(E) - 9)/10] + 2\delta_{01}$, and this inequality is not true for $b_2(E) = 1$. Thus $c_1^2(B') < 0$. Then, since $b_1(B')$ is even we are in the first five cases in the table of Theorem B, and since $c_1^2(B') < 0$, we are in the first case.

Since $c_1^2(P_2(C)) = 9$, B' is a ruled surface, that is, $B' = P_1(C) \times S$, where S is a curve of genus p ($\dim_C S = 1$). Therefore $b_2(B') = b_2(P_1(C)) + b_1(P_1(C)) \cdot b_1(S) + b_2(S) = 1 + 1 = 2$, since $b_1(S) = 2p$, $b_0(S) = 1$, $b_2(S) = 1$ and $b_1(P_1(C)) = 0$, $b_0(P_1(C)) = 1$, $b_2(P_1(C)) = 1$.

Moreover $b_2(B') = 2$ implies $b^-(B) = 2 - b^+(B') = 2 - 2p_g(B') - 1$ and this implies $p_g(B') = p_g(B) = 0$ and $b^-(B') = 1$. Therefore $b^-(B) = b^-(B') + n_t \geqq 1$. From $b_2(E) + 1 - d_1 = 2p_g(B) + 1 + b^-(B)$ we have $d_1 = 0$ and $\chi(E/\xi) = 3 - 2b_1(E) + 1$. This proves the theorem.

In both cases we have proved that the surface E/ξ is free from exceptional curves.

Let us study separately the cases 1, 2, and 3 of the condition R_1.

PROPOSITION 2. *Let (E, ξ) be as in the above theorem. Then if $b_1(E) = 2$ and $b_2(E) = 0$ we have $\chi(E/\xi) = -1$ and the fibration is not trivial.*

PROOF. In this case $b_2(B) = 1 - (d_0 + d_1)$. $d_0 = 0$ implies $b_1(B) = b_1(E) = 2$ and so $b_2(B) = 2p_g(B) + 1 + b^-(B)$. This implies $d_1 = 0$ and $\chi(E/\xi) = -1$. $d_0 = 1$ implies $b_1(B) = d_1 = b_1(E) - d_0 = 1$ and this implies $b_2(B) = 1 - (d_0 + d_1) = 1 - 2$ and this is impossible.

PROPOSITION 3. *Let (E, ξ) be as in the above proposition. Then $b_1(E) = 0$ implies $\chi(E/\xi) = 3 + b_2(E)$ and the fibration is not trivial.*

PROOF. $b_1(B) = b_1(E) - d_0 = -d_0$ implies $d_0 = b_1(B) = 0$. Since $d_1 \leqq b_1(B)$ we have $d_1 = 0$.

PROPOSITION 4. *Let (E, ξ) be as in the above proposition. If $b_1(E) = 2$ and $b_2(E) = 1$, we have $\chi(E/\xi) = 0$.*

PROOF. $b_2(B) = 1 + 1 - (d_0 + d_1)$. (i) $d_0 = 0$ implies $b_1(B) = 2$ and $b_2(B) = 2 - d_1 = 2p_g(B) + 1 + b^-(B)$ and this implies $d_1 = 0$ or 1.

Let us see that $d_1 \neq 1$. Assume $b_2(B) = 1$, then $p_g(B) = b^-(B) = 0$. But $b^-(B) = 0$ implies that B is free from exceptional curves (cf. proof

of Theorem 3a). By Theorem A, we have $c_1^2(B) = 10p_g + 9 - b^- - 8q = 1 > 0$ and $q = 1$. But by Theorem C, we have $q = 0$. This is a contradiction. Thus $d_1 = 0$. (ii) If $d_0 = 1$, then $d_1 = b_1(B) = b_1(E) - d_0 = 1$. In both cases $\chi(E/\xi) = 3 - 2b_1(E) + b_2(E) + d_0 - d_1 = 3 - 2b_1(E) + b_2(E) = 0$.

PROPOSITION 5. *Let (E, ξ) be as in the above proposition. If $b_1(E) = 1$ and $b_2(E) = 0$, we have $\chi(E/\xi) = 1$ and the fibration is not trivial.*

PROOF. $b_2(B) = 1 - (d_0 + d_1)$. If $d_0 = 0$, $b_1(B) = b_1(E) = 1$. Therefore $b_2(B) = 2p_g(B) + b^-(B) = b_2(E) + b_0(E) - d_0 - d_1 = 1 - d_1$. This implies $p_g(B) = 0$ and $b^-(B) + d_1 = 1$. If $d_1 = 1$, $b^-(B) = 0$ implies $b_2(B) = 0$, but this is impossible, since $\chi_E \in H^2(B; R)$ and χ_E is not zero because $d_0 = 0$. If $d_0 = 1$, $d_1 = b_1(B) = b_1(E) - 1 = 0$, but then $b_2(B) = 0 = 2p_g(B) + 1 + b^-(B)$ and this is impossible. The last equality follows from Theorem A. Therefore, we have $d_0 = 0$ and $d_1 = 0$. So $\chi(E/\xi) = 1$ and the fibration is not trivial.

Obviously, Theorem 3 follows from Theorems 3a and 3b, and Propositions 2, 3, 4 and 5. The integrand is obtained from (8) for $n = 2$.

We have obtained, too, the following theorem:

THEOREM 4. *Let E be a compact connected manifold of dimension 5, with $b_1(E)$ even, different from zero and two, and $b_2(E) = 0$, then E cannot admit any almost contact normal regular structure.*

PROOF. In the proof of Theorem 3b, the existence of a field ξ with E/ξ being a complex manifold, implies the fibration being non-trivial and $p_g(B) = b^-(B) = 0$. From this last equality we see that B is free from exceptional curves.

Then, from Theorem A we would have $c_1^2(B) = 9 - 8q < 0$, and using Theorem B, B would be a ruled surface and so $b_2(B) = 2 = b^+(B) + b^-(B) = 2p_g(B) + 1 + b^-(B)$, and this is impossible.

6. Higher dimensions. In higher dimensions we have obtained the following result:

THEOREM 5. *Let E be a compact connected Riemannian manifold of dimension $n = 3 + 4m$, $m > 0$, which admits a regular unit Killing vector field ξ. If $b_2(E) = b_4(E) = \cdots = b_{2m}(E) = 0$, then*

$$(16) \qquad \chi(E/\xi) = \sum_{r=0}^{2m+1} (n + 1 - r)(-1)^r b_r(E) + 1$$

if and only if $b_1(E) + b_3(E) + \cdots + b_{2m+1}(E)$ is odd. And

(17) $$\chi(E/\xi) = \sum_{r=0}^{2m+1} (n + 1 - r)(-1)^r b_r(E)$$

if and only if $b_1(E) + b_3(E) + \cdots + b_{2m+1}(E)$ is even.
  If $b_1(E) = \cdots = b_{2m+1}(E) = 0$. Then

(18) $$\chi(E/\xi) = \sum_{r=0}^{2m+1} (n + 1 - r)(-1)^r b_r(E) .$$

(Therefore, $\chi(E/\xi)$ is independent of the unit regular Killing vector field ξ.)

PROOF. By Gysin exact sequence and the Poincaré duality we have the following relations:

$b_1(B) = b_1(E) - d_0 = b_{4m+1}(B) = b_{4m+1}(E) + \cdots + b_1(E) - d_0 - \cdots - d_{4m}$.

$b_2(B) = b_2(E) + 1 - d_0 - d_1 = b_{4m}(E) + \cdots + b_0(E) - d_0 - \cdots - d_{4m-1}$.

  Thus

(19) $$b_{4m+1}(E) + \cdots + b_3(E) = d_1 + \cdots + d_{4m}$$

and

(20) $$b_{4m}(E) + \cdots + b_4(E) = d_2 + \cdots + d_{4m-1} .$$

  But $b_{4m}(E) = b_3(E), \cdots,$ and substracting (20) from (19) we have $b_{4m+1}(E) = b_2(E) = d_1 + d_{4m}$. Analogously $b_4(E) = d_3 + d_{4m-2}$.
  Therefore, it follows from hypothesis that $d_1 = d_3 = \cdots = d_{2m-1} = 0$. Moreover, if ξ' is another regular unit Killing vector field with $B' = E/\xi'$, we have

(21) $b_{2m+1}(B) - b_{2m+1}(B') = d_0' - d_0 + d_2' - d_2 + \cdots + d_{2m}' - d_{2m}$

(22) $b_{2m+1}(B) - b_{2m+1}(B') \equiv 0 \pmod 2$.

  We also have

(23) $$b_{2m}(B) = 1 - (d_0 + \cdots + d_{2m-1}) .$$

  By (21) and (22) we have

(24) $$d_0' + d_2' + \cdots + d_{2m}' = 2k + d_0 + d_2 \cdots + d_{2m} .$$

  Then if $d_0 = 1$, by (23) we have $d_2 = \cdots = d_{2m} = 0$ and so $d_0 + \cdots + d_{2m} = 1$. If $d_0 = 0$, by (23) we have $b_{2m}(B) = 0$ or 1, $d_{2m} = 0$ or 1 and $d_2 + \cdots + d_{2m-2} = 0$ or 1. If $d_2 + \cdots + d_{2m-2} = 0$, we have $d_0 + \cdots + d_{2m} = 0$ or 1. If $d_2 + \cdots + d_{2m-2} = 1$, we have $b_{2m}(B) = 1 - 1 = 0$ and so $d_{2m} = 0$, thus $d_0 + \cdots + d_{2m} = 0$ or 1.
  Analogously, $d_0' + \cdots + d_{2m}' = 0$ or 1. By (24) we have $d_0' + \cdots + d_{2m}' = d_0 + \cdots + d_{2m}$. Therefore, $\chi(E/\xi) = \chi(E/\xi') = \sum_{r=0}^{n} (n + 1 - r)(-1)^r b_r(E) +$

$d_0 + \cdots + d_{2m}$. Since $b_{2m+1}(B) = b_{2m+1}(E) + \cdots + b_1(E) - (d_0 + \cdots + d_{2m})$ is even, we have proved (17) and (18).

If $b_1(E) = \cdots = b_{2m+1}(E) = 0$, it follows that $b_{2m+1}(B) = -(d_0 + \cdots + d_{2m})$ and so:

$$\chi(E/\xi) = \sum_{r=0}^{n} (n + 1 - r)(-1)^r b_r(E) .$$

This proves Theorem 5.

COROLLARY 1. *Let (E, ξ) be a compact connected Sasakian manifold of dimension 7, with ξ regular, whose curvature is strictly positive and ξ' another unit Killing regular vector field on E. Then $\chi(E/\xi) = \chi(E/\xi')$.*

PROOF. For this manifold $b_2(E) = 0$. (Tanno [6]).

COROLLARY 2. $S^{2p+1} \times S^{2q}$ $(p + q = 2m + 1, p \leqq m < q)$ *does not admit a regular Sasakian structure.*

PROOF. The Betti numbers of $S^{2p+1} \times S^{2q}$ $(p + q = 2m + 1, p \leqq m < q)$ satisfy $b_2 = \cdots = b_{2m} = 0$ and $b_1 + \cdots + b_{2m+1}$ odd. So if ξ is a regular unit Killing vector field on $S^{2p+1} \times S^{2q}$ we have

$$\chi(E/\xi) = \sum_{r=0}^{2m+1} (n + 1 - r)(-1)^r b_r(E) + 1$$

and therefore ξ is not Sasakian.

Nevertheless $S^2 \times S^3$ $(S^6 \times S^7)$ admits a Sasakian regular structure.

7. Remarks. (i) Let (E, g) be as in Theorem 2. If (E, g) is of constant curvature k, we have $k \geqq 0$ and:

If $k = 0$, then E is homeomorphic to $S^1 \times S^1 \times S^1$.

If $k > 0$, then E is a fibre bundle on S^2.

If (E, g) is not of constant curvature but $K(\xi) = k \neq 0$, then (E, g', ξ') with $g' = kg$ and $\xi' = \xi/\sqrt{k}$ is a Sasakian manifold.

(ii) Let (E, g) be as in Theorem 3. We want to know in which cases the integrand of (14) (or (15)) is positive. We know that the integrand of the Gauss-Bonnet formula in dimension 4 can be written as follows (Chern, Abh. Math. Sem. Univ. Hamburg, 20 (1955), p. 124):

$$(25) \qquad (K_{1212}K_{3434} + K_{1234}^2 + K_{1313}K_{4242}$$
$$+ K_{1342}^2 + K_{1414}K_{2323} + K_{1423}^2)\theta^1 \wedge \cdots \wedge \theta^4 ,$$

where

$$\Omega_j^i = (1/2) \sum_{k,r} K_{ijkr} \theta^k \wedge \theta^r .$$

If

$$\Psi^\alpha_\beta = (1/2) \sum_{\lambda,\mu} R_{\alpha\beta\lambda\mu} \varphi^\lambda \wedge \varphi^\mu , \quad \text{by (6)}$$

we have

(26) $R_{ijij} = K_{ijij} - 3A(ij)^2 .$

Then, by (25) and (26), we have the following result: "If the sectional curvature is always positive, the integrand is positive. If the sectional curvature along the planes orthogonals to ξ is always positive, the integrand is positive".

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