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ON TRANSFORMING THE CLASS OF BMO-MARTINGALES BY A CHANGE OF LAW

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1. Introduction. If Z is a positive uniformly integrable martingale such that $Z_0 = 1$, then we can define a change of the underlying probability measure dP by the formula $d\hat{P} = Z_\infty dP$. Our interest in this paper lies in investigating the transformation of BMO-martingales by this change of law. Let us denote by $B(P)$ (resp. $B(\hat{P})$) the space of BMO-martingales with respect to dP (resp. $d\hat{P}$). In the next section we shall deal only with discrete time martingales, and prove that $B(\hat{P})$ is isomorphic to $B(P)$ under a certain assumption. This equivalence corresponding to the continuous time case will be established in Section 4. Furthermore, in Section 3, we shall give a characterization of BMO-martingales.

2. The equivalence of $B(P)$ and $B(\hat{P})$; the discrete time case. Let (Ω, F, P) be a probability space, given a non-decreasing sequence (F_n) of sub σ-fields of F such that $\bigvee_{n=1}^\infty F_n = F$. We shall assume that F_0 contains all null sets. If $X = (X_n, F_n)$ is a martingale with difference sequence $x = (x_n)_{n\geqq 1}$, then the square function of X is $S(X) = (\sum_{n=1}^\infty x_n^2)^{1/2}$. Let $S_n(X) = (\sum_{k=1}^n x_k^2)^{1/2}$, $S_0(X) = X_0 = 0$ and if X converge a.s., let X_∞ denote its limit. The reader is assumed to be familiar with the martingale theory as is given in [2] and [3]. Throughout the paper, let us denote by C a positive constant and by C_p a positive constant depending only on the indexed parameter p, both letters are not necessarily the same in each occurrence. X is a BMO-martingale if

$$||X||_{B(P)} = \sup_n ||E[S(X)^2 - S_{n-1}(X)^2 | F_n]^{1/2}||_\infty < \infty .$$

The class of BMO-martingales depends on the underlying probability measure and so we shall denote it by $B(P)$. It is a real Banach space with norm $||\cdot||_{B(P)}$. The next lemma is fundamental in our investigation.

LEMMA 1. *The inequality*

$$(1) \qquad E[\exp\{S(X)^2 - S_{n-1}(X)^2\} | F_n] \leqq (1 - ||X||_{B(P)}^2)^{-1}$$

is valid for every martingale X such that $||X||_{B(P)} < 1$.

PROOF. This inequality is proved in [3], but for the reader's convenience we shall recall briefly the proof.

Let us set $A_j = S_{j+n-1}(X)^2 - S_{n-1}(X)^2$, which is F_{j+n-1}-measurable and $A_0 = 0 \leqq A_1 \leqq A_2 \leqq \cdots$. Then the left hand side of (1) is $E[\exp(A_\infty)|F_n]$ and without loss of generality we may assume that it is finite. By an elementary calculation we have

$$(2) \qquad E[\exp(A_\infty)|F_n] \leqq 1 + \sum_{j=1}^{\infty} E[b_j(A_\infty - A_{j-1})|F_n]$$

where $b_1 = \exp(A_1)$ and $b_j = \exp(A_j) - \exp(A_{j-1})$, $j \geqq 2$. But the right hand side of (2) is smaller than $1 + ||X||^2_{B(P)}E[\exp(A_\infty)|F_n]$, because $E[A_\infty - A_{j-1}|F_{j+n-1}] \leqq ||X||^2_{B(P)}$. Thus the lemma is proved.

Let now Z be a positive uniformly integrable martingale with $Z_0 = 1$ and $Z_\infty > 0$ a.s. Throughout, we shall denote by $d\hat{P}$ the weighted probability measure $Z_\infty dP$ and by $\hat{E}[\cdot]$ the expectation over Ω with respect to $d\hat{P}$. It is clear that $\hat{P}(\Lambda) = \int_\Lambda Z_n dP$ for every $\Lambda \in F_n$, from which we have

$$(3) \qquad \hat{E}[U|F_n] = E[Z_\infty U|F_n]/Z_n \quad \text{a.s., under } dP \text{ and } d\hat{P}$$

for every $\hat{P}$-integrable random variable U. We shall often use this formula. Let X be a martingale such that every x_n is $\hat{P}$-integrable, and let us consider the process $\hat{X}$ defined by $\hat{X}_0 = 0, \hat{X}_n = \sum_{j=1}^n \hat{x}_j$ where $\hat{x}_j = x_j - \hat{E}[x_j|F_{j-1}]$, $j \geqq 1$. It is easy to see that $\hat{X}$ is a $\hat{P}$-martingale. $||\cdot||_{B(\hat{P})}$ denotes the BMO norm associated with $d\hat{P}$. Let W be the process defined by $W_n = Z_n^{-1}$. W is a $\hat{P}$-uniformly integrable martingale and $W_\infty d\hat{P} = dP$.

DEFINITION. Let $1 < p < \infty$. We say that Z satisfies (A_p) if the inequality

$$(4) \qquad Z_n E[Z_\infty^{-1/(p-1)}|F_n]^{p-1} \leqq C_p$$

is valid for every $n \geqq 1$.

For simplicity, let us say that (A_∞) holds if Z satisfies (A_p) for some $p > 1$. We shall denote by $(\hat{A}_p)$ the (A_p) condition associated with $d\hat{P}$. $(\hat{A}_\infty)$ is the (A_∞) condition with respect to $d\hat{P}$.

THEOREM 1. *Let $1 < p < \infty$. If Z satisfies (A_p), then the inequality*

$$(5) \qquad ||X||_{B(P)} \leqq C_p ||\hat{X}||_{B(\hat{P})}$$

is valid for every P-martingale X such that $x_n \in L_1(d\hat{P})$, $n \geqq 1$. Similarly, if W satisfies $(\hat{A}_p)$, then we have $||\hat{X}||_{B(\hat{P})} \leqq C_p ||X||_{B(P)}$.

PROOF. We show only (5): the proof of the latter half is similar, and is omitted.

If $\|\hat{X}\|_{B(\hat{P})} = 0$, then $\hat{X} = 0$ so that $\hat{x}_n = 0$ for all n. This implies that x_n is F_{n-1}-measurable. Thus $x_n = 0$ for all n. That is to say, $\|X\|_{B(P)} = 0$. So, we may assume that $0 < \|\hat{X}\|_{B(\hat{P})} < \infty$. As $|\hat{x}_n| \leqq \|\hat{X}\|_{B(\hat{P})}$ and $\hat{x}_n - E[\hat{x}_n|F_{n-1}] = x_n$, we get $|x_n| \leqq 2\|\hat{X}\|_{B(\hat{P})}$. Furthermore, a simple calculation shows that $E[x_j^2|F_n] \leqq E[\hat{x}_j^2|F_n]$ for $j \geqq n+1$. Thus we get

$$(6) \qquad E[S(X)^2 - S_{n-1}(X)^2|F_n] = x_n^2 + E[\sum_{j=n+1}^{\infty} x_j^2|F_n]$$
$$\leqq 4\|\hat{X}\|^2_{B(\hat{P})} + E[\sum_{j=n+1}^{\infty} \hat{x}_j^2|F_n] .$$

Now let us set $a = \{2p\|\hat{X}\|_{B(\hat{P})}\}^{-1}$. The (A_p) condition implies that $E[(Z_n/Z_\infty)^{1/(p-1)}|F_n] \leqq C_p$ and by Lemma 1 we obtain $\hat{E}[\exp\{ap(S(\hat{X})^2 - S_n(\hat{X})^2)\}|F_{n+1}] \leqq 2$. Then, applying Hölder's inequality with exponents p and $p/(p-1)$, we can see that the second term on the right hand side of (6) is dominated by

$$\begin{aligned} &a^{-1}E[(Z_\infty/Z_n)^{1/p}\exp\{a(S(\hat{X})^2 - S_n(\hat{X})^2)\}(Z_n/Z_\infty)^{1/p}|F_n] \\ &\leqq a^{-1}\hat{E}[\exp\{ap(S(\hat{X})^2 - S_n(\hat{X})^2)\}|F_n]^{1/p}E[(Z_n/Z_\infty)^{1/(p-1)}|F_n]^{(p-1)/p} \\ &\leqq a^{-1}C_p\hat{E}[\hat{E}[\exp\{ap(S(\hat{X})^2 - S_n(\hat{X})^2)\}|F_{n+1}]|F_n]^{1/p} \\ &\leqq a^{-1}C_p = C_p\|\hat{X}\|^2_{B(\hat{P})}. \end{aligned}$$

This establishes our claim.

COROLLARY. *If Z and W satisfy (A_∞) and $(\hat{A}_\infty)$ respectively, then $B(\hat{P})$ is isomorphic to $B(P)$.*

PROOF. Clearly, $\phi: X \to \hat{X}$ is linear. It follows from Theorem 1 that it is an injective continuous mapping of $B(P)$ into $B(\hat{P})$. To see that it is surjective, let $X' \in B(\hat{P})$ and consider the process X given by $X_0 = 0$, $X_n = \sum_{j=1}^n x_j$, $n \geqq 1$ where $x_j = x'_j - E[x'_j|F_{j-1}]$ and $x'_j = X'_j - X'_{j-1}$. Obviously, X is a P-martingale and, as $\hat{E}[x'_j|F_{j-1}] = 0$, we get $\hat{x}_j = x_j - \hat{E}[x_j|F_{j-1}] = x'_j$. Namely, $X' = \hat{X}$, and by Theorem 1 we have $X \in B(P)$. It is clear that the inverse mapping of ϕ is continuous.

From (3) it follows immediately that W satisfies $(\hat{A}_p)$ if and only if $E[(Z_\infty/Z_n)^q|F_n] \leqq C_p$ where $p^{-1} + q^{-1} = 1$. Therefore, W satisfies $(\hat{A}_\infty)$ if and only if the "reverse Hölder's inequality"

$$(7) \qquad E[Z_\infty^{1+\delta}|F_n] \leqq C_\delta Z_n^{1+\delta}, \quad n \geqq 1$$

holds for some $\delta > 0$. It is proved in [1] that the inequality (7) holds in the special case where the underlying probability space is the d-

dimensional unit cube Q and the family of sub σ-fields is the sequence (F_n) of finite fields obtained by successive dyadic partitions of Q. Quite recently, C. Watari has pointed out that the reverse Hölder's inequality holds in the more general case where (F_n) is regular; namely, each F_n is atomic and there exists a constant $c > 0$ such that for any two atoms $A \in F_{n-1}$, $B \in E_n$ with $B \subset A$ we have $P(A)/P(B) \leqq c$. Therefore, in the regular case, from the (A_∞) condition it follows that $B(P)$ and $B(\hat{P})$ are isomorphic with the mapping ϕ.

We end this section with a simple remark. Let us consider the process M defined by $M_n = \sum_{j=1}^n m_j$ where $m_j = Z_j/Z_{j-1} - 1$. By an elementary calculation, $E[|m_j| \,|F_{j-1}] \leqq 2$, $E[m_j | F_{j-1}] = 0$ and so M is a martingale. By this definition we can easily verify that Z and M satisfy the relation $Z_n = 1 + \sum_{j=1}^n Z_{j-1} m_j$. If $X \in B(P)$, then from (3) it follows that $\hat{E}[x_j | F_{j-1}] = E[m_j x_j | F_{j-1}]$ for every $j \geqq 1$ so that we have $\hat{X}_n = X_n - \sum_{j=1}^n E[m_j x_j | F_{j-1}]$. In the next section we shall give a necessary and sufficient condition for the martingale M to be an element of $B(P)$.

3. A characterization of BMO-martingales.

Until now, in order to explain the basic structure of the transformation of martingales by a change of law, we dealt with the discrete time martingales. Now we are going to deal with the continuous time case. Let (F_t) be a non-decreasing right continuous family of sub σ-fields of F such that $\bigvee_{t \geqq 0} F_t = F$, and M_{loc} be the class of all locally square integrable martingales X such that $X_0 = 0$. As is well-known, for every $X \in M_{\mathrm{loc}}$ there is a unique predictable increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale. If $X, Y \in M_{\mathrm{loc}}$, then $\langle X, Y \rangle$ is the process defined by $\langle X, Y \rangle_t = (\langle X+Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t)/2$. On the other hand, any local martingale L can be split into the continuous part L^c, and the purely discontinuous part L^d, orthogonal to all continuous local martingales. Then one can define the increasing process $[L]$ for any local martingale L by $[L]_t = \langle L^c \rangle_t + \sum_{s \leqq t} (\Delta L_s)^2$ where $\Delta L_s = L_s - L_{s-}$. For two local martingales L and L' we set $[L, L'] = ([L + L'] - [L] - [L'])/2$ as above. It is well-known that, if $X, Y \in M_{\mathrm{loc}}$, then $[X, Y] - \langle X, Y \rangle$ is a local martingale. Let us denote by $||X||_{B(P)}$ the smallest positive constant c such that c^2 dominates a.s., $E[[X]_\infty - [X]_{T-} | F_T]$ for every stopping time T. We say that X is a BMO-martingale if $||X||_{B(P)} < \infty$. $B(P)$ denotes the class of all BMO-martingales as in Section 2.

LEMMA 2. *If* $||X||_{B(P)} < 1$, *then for every stopping time* T *we have*

$$E[\exp([X]_\infty - [X]_{T-}) | F_T] \leqq (1 - ||X||_{B(P)}^2)^{-1} \quad a.s. \tag{8}$$

We omit its proof, because it is the continuous parameter analog of Lemma 1 and is proved in [4].

Now let M be a fixed local martingale such that $M_0=0$, and Z be the local martingale defined by the formula $Z_t = \exp(M_t - \langle M^c\rangle_t) \prod_{s\leqq t}(1+\Delta M_s)\exp(-\Delta M_s)$. As is well-known nowadays, Z is a unique solution of the stochastic integral equation $Z_t = 1 + \int_0^t Z_{s-}dM_s$. Particularly, if $\Delta M_t > -1$ for every t, then Z is a positive local martingale and so it is a supermartingale. We always consider this case in the following. As is stated in Section 2, we say that the process Z satisfies (A_p) if the inequality $Z_T E[(1/Z_\infty)^{1/(p-1)} | F_T]^{p-1} \leqq C_p$ holds for every stopping time T, with a constant C_p.

In the next lemma we use a very simple inequality: $(1-x)^{-1} \leqq \exp(ex)$ for $0 \leqq x \leqq \rho$, where ρ is the root of the equation $1-x = \exp(-ex)$. It is easy to see that $\rho < 1$.

LEMMA 3. *If $M \in B(P)$ and $|\Delta M_t| \leqq \sqrt{\rho}$, then Z satisfies (A_∞).*

PROOF. Let T be any stopping time, and let us take $p > 2$ such that $e\|M\|^2_{B(P)}/(p-2) < 1$. Then $E[\exp\{e([M]_\infty - [M]_{T-})/(p-2)\} | F_T] \leqq \{1 - e\|M\|^2_{B(P)}/(p-2)\}^{-1}$ by Lemma 2. As $(\Delta M)^2 \leqq \rho < 1$, Z is a positive local martingale and $\{1-(\Delta M_t)^2\}^{-1} \leqq \exp\{e(\Delta M_t)^2\}$ for every t. Thus we have

$$\begin{aligned} Z_T/Z_\infty &= \exp\{-(M_\infty - M_T) + (\langle M^c\rangle_\infty - \langle M^c\rangle_T)/2\} \prod_{T<t}(1+\Delta M_t)^{-1}\exp(\Delta M_t) \\ &= \exp\{-(M_\infty - M_T) - (\langle M^c\rangle_\infty - \langle M^c\rangle_T)/2\} \prod_{T<t}(1-\Delta M_t)\exp(\Delta M_t) \\ &\qquad \times \exp(\langle M^c\rangle_\infty - \langle M^c\rangle_T) \prod_{T<t}(1-(\Delta M_t)^2)^{-1} \\ &\leqq \exp\{-(M_\infty - M_T) - (\langle M^c\rangle_\infty - \langle M^c\rangle_T)/2\} \prod_{T<t}(1-\Delta M_t)\exp(\Delta M_t) \\ &\qquad \times \exp\{e([M]_\infty - [M]_{T-})\}\,. \end{aligned}$$

By using Hölder's inequality with exponents $p-1$ and $(p-1)/(p-2)$ we get

$$\begin{aligned} E[(Z_T/Z_\infty)^{1/(p-1)} | F_T] \leqq\; & E[\exp\{-(M_\infty - M_T) - (\langle M^c\rangle_\infty - \langle M^c\rangle_T)/2\} \\ & \times \textstyle\prod_{T<t}(1-\Delta M_t)\exp(\Delta M_t) | F_T]^{1/(p-1)} \\ & \times E[\exp\{e([M_\infty] - [M]_{T-})/(p-2)\} | F_T]^{(p-2)/(p-1)}\,. \end{aligned}$$

By the supermartingale inequality the first term on the right hand side is smaller than 1, and the second term is bounded by $\{1 - e\|M\|^2_{B(P)}/(p-2)\}^{-(p-2)/(p-1)}$. This completes the proof.

LEMMA 4. *If $-1 < \Delta M_t \leqq C$ for every t and Z satisfies (A_∞), then M is a* BMO-*martingale.*

PROOF. Let T_n be stopping times, increasing to ∞ a.s., such that for each n the process $M^{T_n} = (M_{t\wedge T_n})$ is a uniformly integrable martingale, and let us assume that Z satisfies (A_{p-1}) for some $p > 2$. Then for each n the process $Z^{T_n} = (Z_{t\wedge T_n})$ satisfies (A_p). To see this, let S be any stopping time, and we now apply Hölder's inequality with exponents $p-1$ and $(p-1)/(p-2)$:

$$\begin{aligned} E[(Z_{S\wedge T_n}/Z_{T_n})^{1/(p-1)} \mid F_{S\wedge T_n}] &= E[(Z_{S\wedge T_n}/Z_\infty)^{1/(p-1)}(Z_\infty/Z_{T_n})^{1/(p-1)} \mid F_{S\wedge T_n}] \\ &\leqq E[(Z_{S\wedge T_n}/Z_\infty)^{1/(p-2)} \mid F_{S\wedge T_n}]^{(p-2)/(p-1)} E[Z_\infty/Z_{T_n} \mid F_{S\wedge T_n}]^{1/(p-1)} . \end{aligned}$$

But, Z being a positive local martingale, the second term on the right hand side is bounded by 1, and from the definition of the (A_{p-1}) condition it follows that the first term is also dominated by some constant C_p. This implies that Z^{T_n} satisfies (A_p).

We are now going to prove that the local martingale M belongs to $B(P)$. For that, let $\kappa = \inf_{-1<x\leqq C} x^{-2} \log 2^{-1}\{1 + e^x/(1+x)\}$. Then $0 < \kappa < 1/2$ and $\exp(\kappa x^2) \leqq e^x/(1+x)$ for $-1 < x \leqq C$, from which the inequality $\exp\{\kappa(\Delta M_t)^2\} \leqq \exp(\Delta M_t)/(1+\Delta M_t)$ follows at once. Thus we have

$$Z_{S\wedge T_n}/Z_{T_n} \geqq \exp\{-(M_{T_n} - M_{S\wedge T_n}) + \kappa([M]_{T_n} - [M]_{S\wedge T_n})\} .$$

Then, applying Jensen's inequality we get $E[[M]_{T_n} - [M]_{S\wedge T_n} \mid F_{S\wedge T_n}] \leqq C_p$, $n \geqq 1$. The constant C_p does not depend on (T_n), so that, letting $n \to \infty$, we obtain $E[[M]_\infty - [M]_S \mid F_S] \leqq C_p$. Consequently, if $-1 < \Delta M_t \leqq C$, then $E[[M]_\infty - [M]_{S-} \mid F_S] \leqq C_p$. Thus the lemma is proved.

Now, let $Z^{(a)}$ denote the process given by the formula

$$Z_t^{(a)} = \exp(aM_t - a^2\langle M^c\rangle_t/2) \prod_{s\leqq t}(1 + a\Delta M_s)\exp(-a\Delta M_s) , \quad a > 0 .$$

Of course, it is also a local martingale. Lemmas 3 and 4 combined have the following result.

THEOREM 2. *M belongs to $B(P)$ if and only if for some $a>0$* (i) $-1 < a\Delta M_t \leqq C_a$ *and* (ii) $Z^{(a)}$ *satisfies* (A_∞).

PROOF. If $M \in B(P)$, then $|\Delta M_t| \leqq ||M||_{B(P)}$ for every t. Let us take $a > 0$ such that $-1 < a\Delta M_t$ and $a^2(\Delta M_t)^2 \leqq \rho$, where ρ is the same constant as in Lemma 3. Then Lemma 3 implies (ii). The converse follows at once from Lemma 4.

It should be noted that, if M is continuous, then a may be taken to be 1. This is proved in [4].

4. The equivalence of $B(P)$ and $B(\hat{P})$; the continuous time case. In this section we assume that the process M is a locally square integrable martingale such that $\Delta M_t > -1$ for every t. In addition, let us assume that Z is a uniformly integrable martingale and $Z_\infty > 0$. $d\hat{P}$ denotes always the weighted probability measure $Z_\infty dP$. Recall that $\hat{P}(A) = \int_A Z_t dP$ for every $A \in F_t$. Any local martingale with respect to dP is a local martingale under $d\hat{P}$? In general, the answer is negative. But, in 1960, it was proved by I. V. Girsanov that under the absolutely continuous change in probability measure a Brownian motion is transformed into the sum of a Brownian motion and a second process with sample functions which are absolutely continuous with respect to the Lebesgue measure. J. H. Van Schuppen and E. Wong [6] gave a natural generalization of this result as is stated in Lemma 5.

A semi-martingale is a process Y of the form $Y_t = Y_0 + L_t + A_t$ where L is a local martingale and the sample functions of A have bounded variation on every finite interval. As the continuous part L^c of L is independent of the decomposition, one can define another increasing process $[Y]_t = \langle L^c \rangle_t + \sum_{s \leq t} (\Delta Y_s)^2$ for a semi-martingale Y.

LEMMA 5. *For any* $X \in M_{\text{loc}}$, $\hat{X} = X - \langle X, M \rangle$ *is a local martingale with respect to* $d\hat{P}$. *Particularly, if* $\langle X, M \rangle$ *is continuous, then we have* $[\hat{X}] = [X]$ *under either probability measure.*

PROOF. An application of the change of variables formula shows that $\hat{X}Z$ is a local martingale. This means that $\hat{X}$ is a local martingale with respect to $d\hat{P}$. And, $\hat{X}$ being a semi-martingale under dP, from the definition of $[\hat{X}]$ we have $[\hat{X}]_t = \langle X^c \rangle_t + \sum_{s \leq t} (\Delta X_s - \Delta \langle X, M \rangle_s)^2$. Furthermore, if $\langle X, M \rangle$ is continuous, then the right hand side is $\langle X^c \rangle_t + \sum_{s \leq t} (\Delta X_s)^2$. Thus $[\hat{X}] = [X]$. The same conclusion follows under $d\hat{P}$. For details, see [6].

As in the discrete time case, W is the process defined by $W_t = Z_t^{-1}$ and $(\hat{A}_p)$ is the (A_p) condition associated with $d\hat{P}$. The $(\hat{A}_\infty)$ condition means that $(\hat{A}_p)$ holds for some $p > 1$.

THEOREM 3. *Assume that* (F_t) *has no times of discontinuity and that* W *is a* $\hat{P}$*-locally square integrable martingale. If* Z *and* W *satisfy* (A_∞) *and* $(\hat{A}_\infty)$ *respectively, then* $B(\hat{P})$ *is isomorphic to* $B(P)$.

PROOF. First we show that, if Z satisfies (A_p), then the inequality

$$(9) \qquad \|X\|_{B(P)} \leqq C_p \|\hat{X}\|_{B(\hat{P})}$$

is valid for all $X \in M_{\text{loc}}$. $\|\hat{X}\|_{B(\hat{P})} = 0$ implies $X = 0$ so that we may assume $0 < \|\hat{X}\|_{B(\hat{P})} < \infty$. Now let T be any stopping time, and let $a = \{2p\|\hat{X}\|^2_{B(\hat{P})}\}^{-1}$. Then $\hat{E}[\exp\{ap([X]_\infty - [X]_{T-})\} | F_T] \leqq 2$ by Lemma 2, and from (A_p) it follows that $E[(Z_T/Z_\infty)^{1/(p-1)}|F_T] \leqq C_p$. If (F_t) has no times of discontinuity, then $\langle X, M\rangle$ is continuous for every $X \in M_{\text{loc}}$ so that by Lemma 5 we have $[\hat{X}] = [X]$ under dP and $d\hat{P}$. As in the proof of Theorem 1, an application of Hölder's inequality shows that

$$E[[X]_\infty - [X]_{T-}|F_T] \\ \leqq a^{-1}E[(Z_T/Z_\infty)^{1/(p-1)}|F_T]^{(p-1)/p}\hat{E}[\exp\{ap([\hat{X}]_\infty - [\hat{X}]_{T-})\}|F_T]^{1/p} .$$

Hence the right hand side is smaller than $C_p||\hat{X}||^2_{B(\hat{P})}$ and (9) is proved. Similarly, we can see that, if W satisfies (A_p), then the inequality

$$||\hat{X}||_{B(\hat{P})} \leqq C_p||X||_{B(P)} \tag{10}$$

is valid for all $X \in M_{\text{loc}}$. Therefore, ϕ: $X \to \hat{X} = X - \langle X, M\rangle$ defines an injective continuous linear mapping of $B(P)$ into $B(\hat{P})$. So, to prove the theorem, it suffices to verify that ϕ is surjective. For that, consider the process M' defined by $M'_t = \int_0^t Z_{s-}dW_s$, which is a locally square integrable martingale under $d\hat{P}$. Since W satisfies the equation $W_t = 1 + \int_0^t W_{s-}dM'_s$, we have $W_t = \exp(M'_t - \langle M'^c\rangle_t/2)\prod_{s\leq t}(1 + \Delta M'_s)\exp(-\Delta M'_s)$. Furthermore, $\langle X', M'\rangle$ is also continuous for every $\hat{P}$-locally square integrable martingale X', because the family (F_t) has no times of discontinuity. Let now $X' \in B(\hat{P})$. Then it follows from (9) that $X = X' - \langle X', M'\rangle$ belongs to $B(P)$. By Lemma 5, $\hat{X} = X - \langle X, M\rangle$ is a $\hat{P}$-local martingale. Therefore, $X' - \hat{X} = \langle X', M'\rangle + \langle X, M\rangle$ is a continuous $\hat{P}$-local martingale with finite variation on each finite interval. This implies that $X' = \hat{X}$. Thus the theorem is established.

W is a $\hat{P}$-locally square integrable martingale if and only if there is a non-decreasing sequence (T_n) of stopping times with $\lim T_n = \infty$ such that $Z_{T_n}^{-1}$ is P-integrable for each n. When M is continuous, $\hat{M}$ is a continuous local martingale under $d\hat{P}$ and so $W_t = \exp(-\hat{M}_t - \langle\hat{M}\rangle_t/2)$. They are clearly $\hat{P}$-locally square integrable. Then, in the same way as in the proof of Theorem 3, we can show the following.

COROLLARY. *Assume that M is continuous. If Z and W satisfy (A_∞) and $(\hat{A}_\infty)$ respectively, then $B(\hat{P})$ is isomorphic to $B(P)$.*

5. Remarks on the (A_p) condition.

In this section, assuming the sample continuity of the local martingale M, we shall consider the problem: when can one assert that W satisfies $(\hat{A}_\infty)$? By Lemma 3 we know that, if $\hat{M} \in B(\hat{P})$, then W satisfies it.

THEOREM 4. *If M is a continuous local martingale and the inequality*

$$E[\exp\{(\varepsilon + 1/2)(M_\infty - M_t)\}|F_t] \leqq C_\varepsilon , \quad t \geqq 0 \tag{11}$$

holds for some $\varepsilon > 0$, *then the process* W *satisfies* $(\hat{A}_\infty)$.

PROOF. As is remarked in Section 2, to prove the theorem, it suffices to show that Z satisfies the reverse Hölder's inequality $E[Z_\infty^{1+\delta}|F_t] \leqq C_\varepsilon Z_t^{1+\delta}$, $(t \geqq 0)$ where $\delta = 4\varepsilon^2/(1+4\varepsilon)$. Now let us set $p = 1 + 4\varepsilon$. Then the exponent conjugate to p is $q = 1 + 1/4\varepsilon$. Applying Hölder's inequality we have

$$\begin{aligned} E[(Z_\infty/Z_t)^{1+\delta}|F_t] &= E[\exp\{\sqrt{(1+\delta)/p}(M_\infty - M_t) \\ &\qquad - (1+\delta)(\langle M\rangle_\infty - \langle M\rangle_t)/2 \\ &\qquad + (1+\delta - \sqrt{(1+\delta)/p})(M_\infty - M_t)\}|F_t] \\ &\leqq E[\exp\{\sqrt{(1+\delta)p}(M_\infty - M_t) - (1+\delta)p(\langle M\rangle_\infty - \langle M\rangle_t)/2|F_t]^{1/p} \\ &\qquad \times E[\exp\{(1+\delta - \sqrt{(1+\delta)/p})q(M_\infty - M_t)\}|F_t]^{1/q}\,. \end{aligned}$$

By the supermartingale inequality the first term on the right hand side is smaller than 1, and, as $(1+\delta-\sqrt{(1+\delta)/p})q = \varepsilon + 1/2$, from (11) it follows that the second term is bounded. Thus the proof is complete.

For example, if $\|M\|_{B(P)} < \sqrt{2}$, then W satisfies $(\hat{A}_\infty)$. To see this, let ε, δ be two numbers such that $0 < \varepsilon < 1/\sqrt{2(2-\delta)} - 1/2$ and $0 < \delta < 2 - \|M\|_{B(P)}^2$. Then by Lemma 2 we have $E[\exp\{2(\varepsilon+1/2)^2(\langle M\rangle_\infty - \langle M\rangle_t)\}|F_t] \leqq \{1 - \|M\|_{B(P)}^2(2-\delta)\}^{-1}$. On the other hand, by the supermartingale inequality

$$E[\exp\{2(\varepsilon+1/2)(M_\infty - M_t) - 2(\varepsilon+1/2)^2(\langle M\rangle_\infty - \langle M\rangle_t)\}|F_t] \leqq 1\,.$$

Thus we get (11) by using Schwarz's inequality.

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