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EXPONENTIAL ASYMPTOTIC STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE RETARDATIONS

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1. Introduction. The development of the theory of functional differential equations with infinite retardations depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space has required a separate development of the theory. In this paper, we consider an abstract phase space B and assume that it satisfies suitable hypotheses. The beginning of such a theory appeared in [3] for the first time. Though there were several confusion and omissions, the ideas have been accepted and several papers have appeared, [6], [7], [8], [9], [11]. The purpose of the present paper is to study the Liapunov theory on the space B, especially exponential asymptotic stability, and we apply our results to show the existence of almost periodic solutions in almost periodic systems.

In Section 2, we introduce the phase space B which satisfies certain general hypotheses, and the fundamental theorems on solutions will be discussed. There are important phase spaces which satisfy all the hypotheses in this paper. One of such phase spaces is the Banach space C_γ, $\gamma > 0$, of continuous functions φ having the limit $\lim_{s\to-\infty} e^{\gamma s}\varphi(s)$ with norm $|\varphi|_C = \sup_{s\leqq 0} e^{\gamma s}|\varphi(s)|$. Another phase space is the following. For $1 \leqq p < \infty$ and $r \geqq 0$, let H be the Banach space of functions φ mapping $(-\infty, 0]$ into R^n which are measurable on $(-\infty, -r]$ and are continuous on $[-r, 0]$ with norm

$$|\varphi|_H = \left[\sup_{-r\leqq\theta\leqq 0}|\varphi(\theta)|^p + \int_{-\infty}^0 |\varphi(\theta)|^p g(\theta)d\theta\right]^{1/p},$$

where $g(\theta)$ is integrable, positive and nondecreasing on $(-\infty, 0]$, and $g(u+v) \leqq g(u)g(v)$ for $u, v \in (-\infty, 0]$. These phase spaces have been considered in [1], [6], [13].

Section 3 is devoted to the construction of a Liapunov functional under the assumption that the zero solution of a system is exponentially asymptotically stable. Such a Liapunov functional plays an important role in discussing perturbed systems (cf. [11]). For linear homogeneous systems, we show that uniform asymptotic stability is equivalent to exponential asymptotic stability. As one of the applications, we obtain a result on stability for a small perturbed system when the unperturbed linear system is uniformly asymptotically stable.

In Section 4, we discuss the existence of an almost periodic solution of an almost periodic system under the assumption that the system has a bounded solution. To do this, we assume the existence of a Liapunov functional which satisfies some conditions. In particular, if a linear homogeneous almost periodic system is uniformly asymptotically stable, we can show the existence of an almost periodic solution for the non-homogeneous almost periodic system by using the Liapunov functional constructed in Section 3. For the case where retardations are finite, see [4], [14].

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2. The phase space. Let R^n be a real n-vector space, and let $R = R^1$. We denote by B a real linear vector space of functions mapping $(-\infty, 0]$ into R^n with semi-norm $|\cdot|$. No confusion should arise if we use the same symbol $|\cdot|$ to denote the norm in R^n. For φ and ψ in B, $\varphi = \psi$ means that $\varphi(\theta) = \psi(\theta)$ for all θ in $(-\infty, 0]$. Then the quotient space $B^* = B/|\cdot|$ is a normed linear space with the norm naturally induced by the semi-norm. The topology of B is defined by the semi-norm, that is, the family $\{U(\varphi, \varepsilon);\ \varphi \in B, \varepsilon > 0\}$ is the open base, where $U(\varphi, \varepsilon) = \{\psi \in B;\ |\psi - \varphi| < \varepsilon\}$. B with this topology is a pseudo-metric space.

For any φ in B and any β in $[0, \infty)$, let φ^β be the restriction of φ to the interval $(-\infty, -\beta]$. This is a function mapping $(-\infty, -\beta]$ into R^n. Denote the space of such functions by B^β and define a semi-norm $|\cdot|_\beta$ in B^β by

$$|\eta|_\beta = \inf\{|\psi|;\ \psi \in B,\ \psi^\beta = \eta\}\ , \qquad \eta \in B^\beta\ .$$

If we let $|\varphi|_\beta = |\varphi^\beta|_\beta$ for $\varphi \in B$, then $|\cdot|_\beta$ is also a semi-norm in B.

If x is an R^n-valued function defined on $(-\infty, \sigma)$, we define the function x_t for each t in $(-\infty, \sigma)$ by the relation $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leqq 0$.

Let Ω be an open set in $R \times B$ and $f\colon \Omega \to R^n$ be a given continuous function. A functional differential equation on Ω is the relation

$$\dot{x}(t) = f(t, x_t)\ , \tag{2.1}$$

where the symbol "$\cdot$" stands for the right hand derivative. For a $(\sigma,$

φ) in Ω, an R^n-valued function x defined on $(-\infty, \sigma + A)$ with $0 < A \leqq \infty$ is said to be a solution of (2.1) through (σ, φ) if $x_\sigma = \varphi$ and if x is continuously differentiable and satisfies (2.1) for $t \in [\sigma, \sigma + A)$. Then we denote $x = x(\sigma, \varphi)$.

We assume that the space B satisfies the following hypotheses:

(B_1) For any φ in B and an A with $0 < A \leqq \infty$, let x be an R^n-valued function defined on $(-\infty, A)$ such that $x_0 = \varphi$ and x is continuous on $[0, A)$. Then x_t is in B for all t in $[0, A)$ and x_t is continuous in t.

(B_2) There is a continuous function $K(\beta) > 0$ such that

$$|\varphi| \leqq K(\beta) \sup_{-\beta \leqq \theta \leqq 0} |\varphi(\theta)| + |\varphi|_\beta$$

for any φ in B and any β in $[0, \infty)$.

Then the hypotheses (B_1) and (B_2) guarantee the existence of a solution of (2.1) through (σ, φ) in Ω. This was proved by Kaminogo [9].

For any β in $[0, \infty)$ and any φ in B, the function $\varphi(\beta + \theta)$, $\theta \in (-\infty, -\beta]$, belongs to B^β by the hypothesis (B_1). Then we can define a linear operator $\tau^\beta\colon B \to B^\beta$ by $[\tau^\beta \varphi](\theta) = \varphi(\beta + \theta)$. We assume that the space B satisfies the additional hypotheses:

(B_3) There is a continuous function $M(\beta) > 0$ such that

$$|\tau^\beta \varphi|_\beta \leqq M(\beta) |\varphi|$$

for any φ in B and any β in $[0, \infty)$.

(B_4) There is a positive number K_1 such that

$$|\varphi(0)| \leqq K_1 |\varphi|$$

for any φ in B.

We state some fundamental properties of solutions of equation (2.1) on Ω.

THEOREM 2.1. *Under the hypotheses* (B_1) *through* (B_4), *we assume that*

$$|f(t, \varphi) - f(t, \psi)| \leqq n(t) |\varphi - \psi| \quad \text{on } \Omega \tag{2.2}$$

for some continuous function $n(t)$. *Then there exists a continuous function* $N(t, s)$ *for which we have*

$$|x_t(\sigma, \varphi) - x_t(\sigma, \psi)| \leqq N(t, \sigma) |\varphi - \psi| \quad \text{for} \quad t \geqq \sigma \,. \tag{2.3}$$

In particular, the solution is unique for any initial value under the condition (2.2). *Moreover, if* $n(t)$ *is a constant, then* $N(t, s)$ *can be* *chosen to depend only on* $t-s$.

PROOF. From the hypothesis (B_4), the condition (2.2) and the equation (2.1), it follows easily that

$$|x(\sigma, \varphi)(t) - x(\sigma, \psi)(t)| \leqq K_1 |\varphi - \psi| + \int_\sigma^t n(s)|x_s(\sigma, \varphi) - x_s(\sigma, \psi)|\, ds$$

for $t \geqq \sigma$. Therefore, using the hypotheses (B_2) and (B_3), for $t \geqq \sigma$ we have

$$|x_t(\sigma, \varphi) - x_t(\sigma, \psi)| \leqq K(t-\sigma) \sup_{\sigma \leqq \theta \leqq t} |x(\sigma, \varphi)(\theta) - x(\sigma, \psi)(\theta)| + |\tau^{t-\sigma}(\varphi - \psi)|_{t-\sigma}$$

$$\leqq \{K_1 K(t-\sigma) + M(t-\sigma)\}|\varphi - \psi| + K(t-\sigma)\int_\sigma^t n(s)|x_s(\sigma, \varphi) - x_s(\sigma, \psi)|\, ds$$

$$\leqq \sup_{0 \leqq \theta \leqq t-\sigma} \{K_1 K(\theta) + M(\theta)\}|\varphi - \psi| + \{\sup_{0 \leqq \theta \leqq t-\sigma} K(\theta)\} \int_\sigma^t n(s)|x_s(\sigma, \varphi) - x_s(\sigma, \psi)| ds\,.$$

Set $\sup\{K_1 K(\theta) + M(\theta); 0 \leqq \theta \leqq t-\sigma\} = H(t-\sigma)$ and $\sup\{K(\theta); 0 \leqq \theta \leqq t-\sigma\} = J(t-\sigma)$. $H(t-\sigma)$ and $J(t-\sigma)$ are monotone increasing in t. Applying Gronwall's inequality we have (2.3) with

$$N(t, s) = H(t-s)\exp\left\{J(t-s)\int_s^t n(u)du\right\}\,,$$

which is continuous in (t, s). If $n(t)$ is a constant, clearly $N(t, s)$ depends only on $t-s$. This completes the proof.

THEOREM 2.2. *Under the hypotheses* (B_1) *and* (B_2), *let* $\Omega = R \times U_H$, $0 < H \leqq \infty$, *with* $U_H = \{\varphi \in B; |\varphi| < H\}$, *and let* f *be completely continuous. If the solution* $x = x(\sigma, \varphi)$ *of* (2.1) *satisfies* $|x_t| \leqq h(t) < H$ *as long as* x *exists for a continuous function* $h(t)$ *defined on* $[\sigma, \infty)$, *then* x *exists for all* $t \geqq \sigma$ *and satisfies* $|x_t| \leqq h(t)$ *for all* $t \geqq \sigma$.

PROOF. Suppose that x is noncontinuable at $t = \delta$, $\sigma < \delta < \infty$. Let $S = \{(t, x_t); \sigma \leqq t < \delta\}$ and let $r = \sup\{h(t); \sigma \leqq t \leqq \delta\}$. Then $S \subset [\sigma, \delta] \times \mathrm{Cl}(U_r)$ where $\mathrm{Cl}(U_r)$ is the closure of U_r. Since f is completely continuous, f is bounded on S. Hence $x(t)$ satisfies a Lipschitz condition on $[\sigma, \delta)$, and thus $x(\delta - 0)$ exists. Define $x(\delta)$ by $x(\delta) = x(\delta - 0)$. Then $(\delta, x_\delta) \in R \times \mathrm{Cl}(U_r) \subset \Omega$ by the hypothesis (B_1). Since there exists a solution of (2.1) through (δ, x_δ), x can be extended beyond δ. This is a contradiction.

THEOREM 2.3. *Under the hypotheses* (B_1) *through* (B_4), *if the solution* $x(\sigma, \varphi)$ *of* (2.1) *exists on* $[\sigma, \sigma + A]$, $A > 0$, *and it is unique, then for any* $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$|x_t(s, \psi) - x_t(\sigma, \varphi)| < \varepsilon \quad \textit{for all} \quad t \in [\max\{s, \sigma\}, \sigma + A]$$

if $(s, \psi) \in \Omega$, $|s - \sigma| < \delta(\varepsilon)$ *and* $|\varphi - \psi| < \delta(\varepsilon)$.

This theorem was proved by Hino [7] under stronger conditions, and was modified by Hale and Kato [6]. The phase space considered in [6] is slightly different from ours, but our proof is the same as that of Theorem 2.5 in [6].

3. Exponential asymptotic stability. For H with $0 < H \leqq \infty$ and $I = [0, \infty)$, let $\Omega = I \times U_H$, and let $f: \Omega \to R^n$ be a continuous function with $f(t, 0) \equiv 0$. Consider the system

$$\dot{x}(t) = f(t, x_t) \,. \tag{3.1}$$

DEFINITION 3.1. The zero solution of (3.1) is said to be uniformly stable if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_t(t_0, \varphi)| < \varepsilon$ for any $t_0 \geqq 0$ and for all $t \geqq t_0$ if $|\varphi| < \delta$. The zero solution of (3.1) is said to be uniformly asymptotically stable if it is uniformly stable and if there exist a γ with $0 < \gamma \leqq H$ and a function $T(\eta)$ of $\eta > 0$ such that $|\varphi| < \gamma$ implies $|x_t(t_0, \varphi)| < \eta$ for any $t_0 \geqq 0$ and for all $t \geqq t_0 + T(\eta)$. The zero solution of (3.1) is said to be exponentially asymptotically stable if there are constants $c > 0$, γ with $0 < \gamma \leqq H$ and a nondecreasing function $L(\alpha)$ defined for $\alpha \in [0, \gamma)$ such that $L(\alpha) \to 0$ as $\alpha \to 0$ and

$$|x_t(t_0, \varphi)| \leqq L(|\varphi|)e^{-c(t-t_0)} \tag{3.2}$$

for any $t_0 \geqq 0$ and for all $t \geqq t_0$ if $|\varphi| < \gamma$. Moreover, if $H = \gamma = \infty$, we say that the zero solution of (3.1) is exponentially asymptotically stable in the large.

THEOREM 3.1. *Suppose that the hypotheses* (B_1) *through* (B_4) *are satisfied and assume that for any* $\alpha \in [0, H)$, *there is a function* $n(t, \alpha)$ *which is continuous in* t *such that*

$$|f(t, \varphi) - f(t, \psi)| \leqq n(t, \alpha)|\varphi - \psi| \quad on \quad I \times U_\alpha \,. \tag{3.3}$$

If the zero solution of (3.1) *is exponentially asymptotically stable, that is,* (3.2) *holds and if* $L(\alpha) = O(\alpha)$ *as* $\alpha \to 0$, *then there exists a continuous real-valued functional* $V(t, \varphi)$ *defined on* $I \times U_\gamma$ *which satisfies the following conditions*:

(i) $|\varphi| \leqq V(t, \varphi) \leqq L(|\varphi|)$.

(ii) $|V(t, \varphi^1) - V(t, \varphi^2)| \leqq N(t, \alpha)|\varphi^1 - \varphi^2|$ *for* $\varphi^i \in U_\alpha$, $i = 1, 2$, *where* $N(t, \alpha)$ *is a continuous function of* t.

(iii) $V'_{(3.1)}(t, \varphi) \leqq -qcV(t, \varphi)$, *where* q *is any given constant with* $0 < q < 1$. *Here* $V'_{(3.1)}(t, \varphi)$ *is defined by*

$$V'_{(3.1)}(t, \varphi) = \limsup_{\delta \to 0+} \{V(t+\delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi)\}/\delta ,$$

where $x(t, \varphi)$ *is the solution of* (3.1) *through* $(t, \varphi) \in I \times U_\gamma$.

PROOF. For q with $0 < q < 1$, define $V(t, \varphi)$ by

$$V(t, \varphi) = \sup_{\tau \geqq 0} |x_{t+\tau}(t, \varphi)| e^{qc\tau} .$$

Obviously, we have (i) by (3.2). For any $\alpha \in [0, \gamma)$, we can choose a $T(\alpha) > 0$ so that $L(\beta)e^{-(1-q)c\tau} \leqq \beta$ if $\beta \leqq \alpha$ and $\tau \geqq T(\alpha)$ since $L(\alpha) = O(\alpha)$ as $\alpha \to 0$. Then for $\varphi \in U_\alpha$ and $\tau \geqq T(\alpha)$, we have

$$|x_{t+\tau}(t, \varphi)| e^{qc\tau} \leqq L(|\varphi|)e^{-(1-q)c\tau} \leqq |\varphi|$$

by (3.2). Hence the condition (i) implies that

$$V(t, \varphi) = \sup_{\tau \geqq 0} |x_{t+\tau}(t, \varphi)| e^{qc\tau} = \sup_{0 \leqq \tau \leqq T(\alpha)} |x_{t+\tau}(t, \varphi)| e^{qc\tau}$$

for $\varphi \in U_\alpha$. For $\varphi^i \in U_\alpha$, $i = 1, 2$, we have $|x_{t+\tau}(t, \varphi^i)| < L(\alpha)$ by (3.2), and thus in the same way as in the proof of Theorem 2.1 we have

$$|x_{t+\tau}(t, \varphi^1) - x_{t+\tau}(t, \varphi^2)| \leqq (K_1K^* + M^*)|\varphi^1 - \varphi^2| \exp\left\{K^* \int_0^\tau n(t+s, L(\alpha))ds\right\}$$

for $\tau \in [0, T(\alpha)]$, where $K^* = \sup\{K(\tau); \tau \in [0, T(\alpha)]\}$ and $M^* = \sup\{M(\tau); \tau \in [0, T(\alpha)]\}$. For $\varphi^i \in U_\alpha$, $i = 1, 2$, it follows that

$$\begin{aligned} |V(t, \varphi^1) - V(t, \varphi^2)| &\leqq \left| \sup_{0 \leqq \tau \leqq T(\alpha)} |x_{t+\tau}(t, \varphi^1)| e^{qc\tau} - \sup_{0 \leqq \tau \leqq T(\alpha)} |x_{t+\tau}(t, \varphi^2)| e^{qc\tau} \right| \\ &\leqq \sup_{0 \leqq \tau \leqq T(\alpha)} |x_{t+\tau}(t, \varphi^1) - x_{t+\tau}(t, \varphi^2)| e^{qc\tau} . \end{aligned}$$

Therefore we have (ii) with

$$N(t, \alpha) = e^{qcT(\alpha)}(K_1K^* + M^*) \exp\left\{K^* \int_0^{T(\alpha)} n(t+s, L(\alpha))ds\right\} .$$

Clearly, if $n(t, \alpha)$ is independent of t, so is $N(t, \alpha)$.

Since $x_{t+\delta+\tau}(t+\delta, x_{t+\delta}(t, \varphi)) = x_{t+\delta+\tau}(t, \varphi)$ for $\delta > 0$, we have

$$\begin{aligned} V(t+\delta, x_{t+\delta}(t, \varphi)) &= \sup_{\tau \geqq 0} |x_{t+\delta+\tau}(t+\delta, x_{t+\delta}(t, \varphi))| e^{qc\tau} = \sup_{\tau \geqq 0} |x_{t+\delta+\tau}(t, \varphi)| e^{qc\tau} \\ &= \sup_{\tau \geqq \delta} |x_{t+\delta}(t, \varphi)| e^{qc(\tau-\delta)} \leqq V(t, \varphi)e^{-qc\delta} . \end{aligned}$$

Therefore we have

$$\{V(t+\delta, x_{t+\delta}(t, \varphi)) - V(t, \varphi)\}/\delta \leqq V(t, \varphi)(e^{-qc\delta} - 1)/\delta ,$$

which implies (iii).

Now we show that $V(t, \varphi)$ is continuous. First, the continuity from the left hand side will be proved. Let $(t, \varphi) \in I \times U_\gamma$ be fixed. For a

$(t-\eta, \psi)\in I\times U_\gamma$ with $\eta>0$, we observe that

$$\begin{aligned}|V(t,\varphi)-V(t-\eta,\psi)| &\leqq |V(t-\eta,\varphi)-V(t-\eta,\psi)|\\ &\quad+|V(t,\varphi)-V(t,x_t(t-\eta,\varphi))|\\ &\quad+|V(t,x_t(t-\eta,\varphi))-V(t-\eta,\varphi)|\,.\end{aligned}$$

Since $x_t(t-\eta,\varphi)$ is continuous in η by Theorem 2.3, the condition (ii) implies that the first two terms tend to zero as $\eta\to 0$ and $|\varphi-\psi|\to 0$. Then it is sufficient to see that

$$|V(t,x_t(t-\eta,\varphi))-V(t-\eta,\varphi)|\to 0 \quad\text{as}\quad \eta\to 0\,. \tag{3.4}$$

Since $x_{t+\tau}(t,x_t(t-\eta,\varphi))=x_{t+\tau}(t-\eta,\varphi)$, we have

$$\begin{aligned}&|V(t,x_t(t-\eta,\varphi))-V(t-\eta,\varphi)|\\ &\quad=|\sup_{\tau\geqq 0}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}-\sup_{\tau\geqq 0}|x_{t-\eta+\tau}(t-\eta,\varphi)|e^{qc\tau}|\\ &\quad=\sup_{\tau\geqq-\eta}|x_{t+\tau}(t-\eta,\varphi)|e^{qc(\tau+\eta)}-\sup_{\tau\geqq 0}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}\,.\end{aligned}$$

When

$$\sup_{\tau\geqq-\eta}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}=\sup_{\tau\geqq 0}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}$$

for an η, we have

$$|V(t,x_t(t-\eta,\varphi))-V(t-\eta,\varphi)|\leqq(e^{qc\eta}-1)\sup_{\tau\geqq 0}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}\,. \tag{3.5}$$

On the other hand when

$$\sup_{\tau\geqq-\eta}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}=\sup_{-\eta\leqq\tau\leqq 0}|x_{t+\tau}(t-\eta,\varphi)|e^{qc\tau}$$

for an η, we have

$$\begin{aligned}&|V(t,x_t(t-\eta,\varphi))-V(t-\eta,\varphi)|\\ &\qquad\leqq|x_{t-\eta'}(t-\eta,\varphi)|e^{qc(\eta-\eta')}-|x_t(t-\eta,\varphi)|\end{aligned} \tag{3.6}$$

for some $\eta'\in[0,\eta]$. Hence if we could show that the right hand side of (3.6) tends to zero as $\eta\to 0$, (3.4) would follow from (3.5) and (3.6). Since $|x_t(t-\eta,\varphi)|\to|\varphi|$ as $\eta\to 0$ by Theorem 2.3, we should show that $x_{t-\eta'}(t-\eta,\varphi)\to\varphi$ as $\eta\to 0$. Define $\tilde\varphi$ by

$$\tilde\varphi(s)=\begin{cases}\varphi(s)\,, & s\leqq 0\\ \varphi(0)\,, & s\geqq 0\,.\end{cases}$$

Then $\tilde\varphi_s$ belongs to B for all $s\geqq 0$ by the hypothesis (B_1). Let a be a fixed positive constant and let $r=\sup\{n(s,L(|\varphi|));\ s\in[t-a,t]\}$. If η is sufficiently small and $\theta\in[t-\eta,t-\eta']$, then we have

$$\left|\int_{t-\eta}^{0} f(s, x_s(t-\eta, \varphi))ds\right| \leqq r\int_{t-\eta}^{0} |x_s(t-\eta, \varphi)|\,ds \leqq rL(|\varphi|)(\eta-\eta')$$

by (3.2) and (3.3). Therefore by the hypothesis (B_2),

$$\begin{aligned}|x_{t-\eta'}(t-\eta, \varphi)-\varphi| &\leqq |x_{t-\eta'}(t-\eta, \varphi)-\tilde{\varphi}_{\eta-\eta'}|+|\tilde{\varphi}_{\eta-\eta'}-\varphi| \\ &\leqq K(\eta-\eta')\sup_{t-\eta\leqq\theta\leqq t-\eta'}\left|\int_{t-\eta}^{\theta} f(s, x_s(t-\eta, \varphi))ds\right|+|\tilde{\varphi}_{\eta-\eta'}-\varphi| \\ &\leqq K(\eta-\eta')rL(|\varphi|)(\eta-\eta')+|\tilde{\varphi}_{\eta-\eta'}-\varphi|\,,\end{aligned}$$

which implies $|x_{t-\eta'}(t-\eta, \varphi)-\varphi| \to 0$ as $\eta \to 0$ since $\eta-\eta' \to 0$ and $|\tilde{\varphi}_{\eta-\eta'}-\varphi| \to 0$ as $\eta \to 0$ by the hypothesis (B_1).

Next we observe that

$$\begin{aligned}|V(t, \varphi)-V(t+\eta, \psi)| &\leqq |V(t+\eta, \varphi)-V(t+\eta, \psi)| \\ &+|V(t+\eta, x_{t+\eta}(t, \varphi))-V(t+\eta, \varphi)|+|V(t, \varphi)-V(t+\eta, x_{t+\eta}(t, \varphi))|\end{aligned}$$

for $\eta > 0$. Then the continuity from the right hand side follows from the condition (ii), the continuity of $x_{t+\eta}(t, \varphi)$ in η and the argument similar to that we had before. This completes the proof.

We now consider the linear system

$$\dot{x}(t) = A(t, x_t)\,, \tag{3.7}$$

where $A(t, \varphi): I \times B \to R^n$ is continuous and linear in φ.

LEMMA 3.1. *There exists a continuous function $n(t)$ such that*

$$|A(t, \varphi)| \leqq n(t)|\varphi| \qquad on \quad I \times B\,.$$

PROOF. Let J be a compact interval on I. We show that $|A(t, \varphi)| \leqq C|\varphi|$ on $J \times B$ for some constant $C > 0$. Otherwise there would exist a sequence $\{(t_k, \varphi^k)\}$ such that $t_k \in J$, $|\varphi^k| = 1$ and $|A(t_k, \varphi^k)| \geqq k$. Set $\psi^k = k^{-1/2}\varphi^k$. Since $|\psi^k| = k^{-1/2} \to 0$ as $k \to \infty$, the set $S = \{\psi^k; k = 1, 2, \cdots\} \cup \{0\}$ is compact. Therefore $A(t, \varphi)$ should be bounded on $J \times S$, but

$$|A(t_k, \psi^k)| = k^{-1/2}|A(t_k, \varphi^k)| \geqq k^{1/2}\,,$$

which implies $|A(t_k, \psi^k)| \to \infty$ as $k \to \infty$, a contradiction. Then it is not difficult to construct a desirable $n(t)$.

THEOREM 3.2. *Suppose that the hypotheses* (B_1) *through* (B_4) *are satisfied. If the zero solution of* (3.7) *is uniformly asymptotically stable, then it is exponentially asymptotically stable in the large. In this case $L(\alpha) = O(\alpha)$ as $\alpha \to 0$, where $L(\alpha)$ is the function in Definition* 3.1.

PROOF. Let $x(t_0, \varphi)$ be a solution of (3.7). By Lemma 3.1 and Theorem 2.1, $x(t_0, \varphi)$ is unique and satisfies

$$|x_t(t_o, \varphi)| \leqq N(t, t_o)\,|\varphi|$$

as long as it exists for a continuous function $N(t, s)$. Thus $x(t_o, \varphi)$ exists for all $t \geqq t_o$ by Theorem 2.2. Therefore we can define a linear operator $T(t, t_o): B \to B$ by

$$T(t, t_o)\varphi = x_t(t_o, \varphi)\,, \qquad \varphi \in B\,,$$

for any $t_o \geqq 0$ and any $t \geqq t_o$.

Since the zero solution of (3.7) is uniformly stable, there exists a $\delta > 0$ such that if $|\varphi| \leqq \delta$, then $|T(t, s)\varphi| < 1$ for $t \geqq s \geqq 0$, and hence

$$\|T(t, s)\| = \sup_{|\varphi|\leqq 1} |T(t, s)\varphi| = \sup_{|\varphi|\leqq \delta} |T(t, s)(\varphi/\delta)| \leqq 1/\delta\,, \tag{3.8}$$

where $\|\cdot\|$ denotes the usual norm of linear operators.

Furthermore, since the zero solution of (3.7) is uniformly asymptotically stable, for any $\eta > 0$ there is a $T(\eta) > 0$ such that $|\varphi| \leqq 1$ implies $|T(t, s)\varphi| < \eta$ for any $s \geqq 0$ and for all $t \geqq s + T(\eta)$. Consequently, $\|T(t, s)\| \leqq \eta$ for all $t \geqq s + T(\eta)$. In particular, we have

$$\|T(t + T(\eta), t)\| \leqq \eta \qquad \text{for any} \quad t \geqq 0\,. \tag{3.9}$$

Choose $\eta < 1$ and let $c = -T(\eta)^{-1}\log\eta$, $M = e^{cT(\eta)}\delta^{-1}$. For any $t_o \geqq 0$ and any $t \geqq t_o$, there is an integer $k \geqq 0$ such that $kT(\eta) \leqq t - t_o < (k+1)T(\eta)$ Then by (3.8) and (3.9), we have

$$\begin{aligned}\|T(t, t_o)\| &\leqq \|T(t, t_o + kT(\eta))\|\,\|T(t_o + kT(\eta), t_o)\| \\ &\leqq \delta^{-1}\|T(t_o + kT(\eta), t_o)\| \leqq \delta^{-1}\eta^k \leqq \delta^{-1}e^{-ckT(\eta)} \leqq Me^{-c(k+1)T(\eta)} \leqq Me^{-c(t-t_o)}\,,\end{aligned}$$

which shows

$$|x_t(t_o, \varphi)| = |T(t, t_o)\varphi| \leqq M|\varphi|\,e^{-c(t-t_o)} \tag{3.10}$$

for any $t_o \geqq 0$ and all $t \geqq t_o$. This completes the proof.

By Theorems 3.1 and 3.2, we have the following.

THEOREM 3.3. *Under the hypotheses* (B_1) *through* (B_4), *if the zero solution of* (3.7) *is uniformly asymptotically stable, then there exists a continuous real-valued functional* $V(t, \varphi)$ *defined on* $I \times B$ *which satisfies the following conditions*:

(i) $|\varphi| \leqq V(t, \varphi) \leqq M|\varphi|$, *where* M *is a constant.*

(ii) $|V(t, \varphi^1) - V(t, \varphi^2)| \leqq M|\varphi^1 - \varphi^2|$.

(iii) $V'_{(3.7)}(t, \varphi) \leqq -cV(t, \varphi)$, *where* $c > 0$ *is a constant.*

PROOF. Since the zero solution of (3.7) is exponentially asymptotically stable by Theorem 3.2, there are constants $c > 0$ and $M > 0$ for which we have (3.10). Define $V(t, \varphi)$ by

$$V(t, \varphi) = \sup_{\tau \geqq 0} |x_{t+\tau}(t, \varphi)| e^{c\tau} .$$

Then we have (i) by (3.10). Since $A(t, \varphi)$ is linear in φ, it follows that

$$|V(t, \varphi^1) - V(t, \varphi^2)| \leqq V(t, \varphi^1 - \varphi^2) \leqq M|\varphi^1 - \varphi^2| .$$

This proves (ii). The remainder of the proof is the same as in the proof of Theorem 3.1.

Now let $V(t, \varphi)$ be a continuous real-valued functional defined on $I \times B$ which is locally Lipschitzian in φ. For an R^n-valued function $z(s)$ such that $z_s \in B$ for $s \geqq t$, we define $V'(t, z_t)$ by

$$V'(t, z_t) = \limsup_{\delta \to 0+} \{V(t + \delta, z_{t+\delta}) - V(t, z_t)\}/\delta .$$

LEMMA 3.2. *Under the hypotheses* (B_1) *and* (B_2), *let* $x(s)$ *and* $y(s)$ *be continuous* R^n*-valued functions of* s *with* $s \geqq t$ *such that* $x_t = y_t = \varphi \in B$. *Then if they have the right hand derivatives* $\dot{x}(t)$ *and* $\dot{y}(t)$ *at* $s = t$, *we have*

$$V'(t, y_t) \leqq V'(t, x_t) + K(0)L|\dot{y}(t) - \dot{x}(t)| ,$$

where L *is a Lipschitz constant of* $V(t, \varphi)$ *in the neighborhood of* (t, φ).

PROOF. By the hypotheses (B_1) and (B_2), we have

$$\begin{aligned} V'(t, y_t) &\leqq \limsup_{\delta \to 0+} \{V(t + \delta, x_{t+\delta}) - V(t, \varphi)\}/\delta \\ &\quad + \limsup_{\delta \to 0+} \{V(t + \delta, y_{t+\delta}) - V(t + \delta, x_{t+\delta})\}/\delta \\ &\leqq V'(t, x_t) + \limsup_{\delta \to 0+} L|y_{t+\delta} - x_{t+\delta}|/\delta \\ &\leqq V'(t, x_t) + \limsup_{\delta \to 0+} \{LK(\delta) \sup_{t \leqq s \leqq t+\delta} |y(s) - x(s)|\}/\delta \\ &\leqq V'(t, x_t) + K(0)L|\dot{y}(t) - \dot{x}(t)| , \end{aligned}$$

which proves the lemma.

Consider the perturbed system of (3.7)

$$\dot{x}(t) = A(t, x_t) + g(t, x_t) , \tag{3.11}$$

where $g(t, \varphi): I \times B \to R^n$ is a continuous function.

THEOREM 3.4. *Under the hypotheses* (B_1) *through* (B_4), *assume that for any* $\varepsilon > 0$, *there exist a* $\delta(\varepsilon) > 0$ *and a continuous function* $b_\varepsilon(t): I \to R$ *such that* $|g(t, \varphi)| \leqq b_\varepsilon(t)|\varphi|$ *for any* $\varphi \in U_{\delta(\varepsilon)}$ *and that*

$$\limsup_{t \to \infty} \int_t^{t+1} b_\varepsilon(s) ds < \varepsilon . \tag{3.12}$$

If the zero solution of (3.7) *is uniformly asymptotically stable, then the zero solution of* (3.11) *is exponentially asymptotically stable.*

PROOF. Since the zero solution of (3.7) is uniformly asymptotically stable, there is a continuous functional $V(t, \varphi)$ defined on $I \times B$ and satisfying the conditions (i), (ii) and (iii) in Theorem 3.3.

Choose an $\varepsilon > 0$ with $c - MK(0)\varepsilon > 0$. Then there exists a $\delta(\varepsilon) > 0$ and a function $b_\varepsilon(t)$ which satisfies the assumptions. We can find a constant $r > 0$ such that

$$\int_{t_0}^{t} b_\varepsilon(s)ds \leqq r + \varepsilon(t - t_0) \tag{3.13}$$

for any $t_0 \geqq 0$ and all $t \geqq t_0$ by (3.12).

Let $\delta_0 = \delta(\varepsilon)/(Me^{K(0)Mr})$. For any φ with $|\varphi| < \delta_0$ and the solution $x = x(t_0, \varphi)$ of (3.11), it follows from Lemma 3.2 and the conditions (i), (ii) and (iii) that

$$\begin{aligned} V'(t, x_t) &\leqq -cV(t, x_t) + K(0)M|g(t, x_t)| \\ &\leqq -cV(t, x_t) + K(0)M\,b_\varepsilon(t)\,|x_t| \\ &\leqq (-c + K(0)M\,b_\varepsilon(t))V(t, x_t) \end{aligned}$$

as long as $|x_t| < \delta(\varepsilon)$. Hence the comparison principle, the condition (i) and (3.13) imply that

$$\begin{aligned} |x_t| \leqq V(t, x_t) &\leqq V(t_0, \varphi)\exp\left\{-c(t - t_0) + K(0)M\int_{t_0}^{t} b_\varepsilon(s)ds\right\} \\ &\leqq Me^{K(0)Mr}[\exp\{-(c - K(0)M\varepsilon)(t - t_0)\}]|\varphi| \end{aligned} \tag{3.14}$$

as long as $|x_t| < \delta(\varepsilon)$. Since $|\varphi| < \delta_0$ and $c - K(0)M\varepsilon > 0$, we have $|x_t| < \delta(\varepsilon)$ by (3.14). Therefore (3.14) holds for any $t_0 \geqq 0$ and for all $t \geqq t_0$ if $|\varphi| < \delta_0$. This completes the proof.

4. Existence of an almost periodic solution. In this section, we make the additional assumptions on the space B.

When a sequence of functions $\{\varphi^k(\theta)\}$, $\varphi^k \in B$, converges to a function $\varphi(\theta)$ uniformly on any compact interval on $(-\infty, 0]$ as $k \to \infty$, we say that the sequence $\{\varphi^k\}$ converges to φ locally uniformly on $(-\infty, 0]$.

The additional hypotheses are the following.

(B_3') $M(\beta) \to 0$ as $\beta \to \infty$, where $M(\beta)$ is the function in the hypothesis (B_3).

(B_5) If a sequence of functions $\{\varphi^k(\theta)\}$, $\varphi^k \in B$, is uniformly bounded and converges to $\varphi(\theta)$ locally uniformly on $(-\infty, 0]$, then φ belongs to B and $|\varphi^k - \varphi| \to 0$ as $k \to \infty$.

Under the hypotheses (B_1) and (B_5), any bounded continuous function mapping $(-\infty, 0]$ into R^n belongs to B. In fact, let φ be a bounded continuous function. Then the functions $\varphi^k(\theta)$, $k = 1, 2, \cdots$, defined by

$$\varphi^k(\theta) = \begin{cases} \varphi(\theta) , & -k \leqq \theta \leqq 0 \\ (\theta + k + 1)\varphi(-k) , & -(k+1) \leqq \theta \leqq -k \\ 0 , & \theta \leqq -(k+1) \end{cases}$$

belong to B by the hypothesis (B_1) and the sequence $\{\varphi^k(\theta)\}$ is uniformly bounded and converges to $\varphi(\theta)$ locally uniformly on $(-\infty, 0]$. Thus $\varphi \in B$ by the hypothesis (B_5).

First of all, we prove the following lemma essentially due to Hale [3].

LEMMA 4.1. *Under the hypotheses* (B_1) *through* (B_3), (B_3') *and* (B_5), *let* $u: R \to R^n$ *be a function which is bounded and uniformly continuous on* I *with* $u_0 \in B$. *Then the set* $S = \mathrm{Cl}\{u_t; t \in I\}$ *is compact in* B.

PROOF. Take any sequence $\{u_{t_k}\}$, $t_k \geqq 0$. Since u_t is continuous in $t \in I$ by the hypothesis (B_1), it is sufficient to consider the case where $t_k \to \infty$ as $k \to \infty$. For any integer $N > 0$, take k so large that $t_k - N \geqq 0$. Then the sequence of functions $\{u(t_k + \theta)\}$ is uniformly bounded and equicontinuous on $[-N, 0]$. Hence we can choose a subsequence which converges uniformly on $[-N, 0]$. Therefore, using the diagonalization procedure, we obtain a subsequence which converges to a function ψ locally uniformly on $(-\infty, 0]$. The limit function ψ is bounded continuous and belongs to B. We may assume that the sequence $\{u(t_k + \theta)\}$ converges to ψ locally uniformly on $(-\infty, 0]$.

Define v and w by

$$v(s) = \begin{cases} u(s) , & s > 0 \\ u(0) , & s \leqq 0 \end{cases}$$

and $w(s) = u(s) - v(s)$ for all $s \in (-\infty, \infty)$. Then v_{t_k} is bounded continuous on $(-\infty, 0]$ and belongs to B. Also, w_{t_k} belongs to B by the hypothesis (B_1). Let $\langle u(0) \rangle$ be a constant function such that $\langle u(0) \rangle(\theta) = u(0)$ for all $\theta \in (-\infty, 0]$. From the hypotheses (B_2) and (B_3) it follows that

$$\begin{aligned} |u_{t_k} - \psi| &\leqq |v_{t_k} + w_{t_k} - \psi| \leqq |v_{t_k} - \psi| + |w_{t_k}| \\ &\leqq |v_{t_k} - \psi| + |\tau^{t_k}(u_0 - \langle u(0) \rangle)|_{t_k} \\ &\leqq |v_{t_k} - \psi| + M(t_k)\,|u_0 - \langle u(0) \rangle| . \end{aligned}$$

Since the sequence $\{v(t_k + \theta)\}$ is uniformly bounded and converges to $\psi(\theta)$ locally uniformly on $(-\infty, 0]$, we have $|v_{t_k} - \psi| \to 0$ as $k \to \infty$ by the hypothesis (B_5). Moreover, $M(t_k) \to 0$ as $k \to \infty$ by the hypothesis (B_3'). Consequently, $|u_{t_k} - \psi| \to 0$ as $k \to \infty$, which proves the lemma.

Now, in order to discuss almost periodic systems, the space B is assumed to satisfy one more hypothesis that

(B_6) B is separable.

DEFINITION 4.1. Let $f(t, \varphi)$ be a continuous function defined on $R \times D$ into R^n, where D is an open set in B. $f(t, \varphi)$ is said to be almost periodic in t uniformly for $\varphi \in D$, if for any $\varepsilon > 0$ and any compact set S in D, there is a positive number $q(\varepsilon, S)$ such that any interval of length $q(\varepsilon, S)$ contains an h for which

$$|f(t+h, \varphi) - f(t, \varphi)| \leqq \varepsilon \quad \text{for all} \quad t \in R \quad \text{and all} \quad \varphi \in S .$$

Such a number h is called an ε-translation number of $f(t, \varphi)$ and we denote by $E\{\varepsilon, f, S\}$ the set of all ε-translation numbers of f for S.

Let $f(t, \varphi)$ be almost periodic in t uniformly for $\varphi \in D$. Then f is bounded and uniformly continuous on $R \times S$, where S is any compact set in D. Moreover, if the hypothesis (B_6) is satisfied, then the normality theorem holds and the set Λ defined by

$$\Lambda = \Big\{\lambda \in R;\ \lim_{T \to \infty} (1/T) \int_0^T f(t, \varphi) e^{-i\lambda t} dt,\ i = \sqrt{-1}, \text{ is not identically zero for } \varphi \in D\Big\}$$

is countable. The module of f is the set consisting of all real numbers which are finite linear combinations of elements of the set Λ with integer coefficients, and we denote it by $m(f)$. For the details of these facts, we refer the reader to [15]. The space considered in [15] is a separable Banach space, but the completeness of the space is not utilized for the properties of almost periodic functions. See also [10].

Now we show the module containment theorem.

Let $f^k(t, \varphi)$, $k = 1, 2, \cdots$, be almost periodic in t uniformly for $\varphi \in D$. We say that a sequence $\{f^k(t, \varphi)\}$ converges c-uniformly on $R \times D$ when it converges uniformly on $R \times S$ as $k \to \infty$ for any compact set S in D.

THEOREM 4.1. *Under the hypothesis* (B_6), *let* $f(t, \varphi)$ *and* $g(t, \varphi)$ *be almost periodic in* t *uniformly for* $\varphi \in D$. *Then the following five statements are equivalent*:

(i) $m(g) \subset m(f)$.

(ii) *For any* $\varepsilon > 0$ *and any compact set* S *in* D, *there exist a* $\delta = \delta(\varepsilon, S) > 0$ *and a compact set* S^* *in* D *such that* $S \subset S^*$ *and* $E\{\delta, f, S^*\} \subset E\{\varepsilon, g, S\}$.

(iii) $\{g(t + h_k, \varphi)\}$ *converges c-uniformly on* $R \times D$ *for any real* *sequence* $\{h_k\}$ *such that* $\{f(t+h_k, \varphi)\}$ *converges c-uniformly on* $R\times D$.

(iv) $\{g(t+h_k, \varphi)\}$ *converges to* $g(t, \varphi)$ *c-uniformly on* $R\times D$ *for any real sequence* $\{h_k\}$ *such that* $\{f(t+h_k, \varphi)\}$ *converges to* $f(t, \varphi)$ *c-uniformly on* $R\times D$.

(v) *For any real sequence* $\{h_k\}$ *such that* $\{f(t+h_k, \varphi)\}$ *converges to* $f(t, \varphi)$ *c-uniformly on* $R\times D$, *there exists a subsequence* $\{h_{k_j}\}$ *of* $\{h_k\}$ *such that* $\{g(t+h_{k_j}, \varphi)\}$ *converges to* $g(t, \varphi)$ *c-uniformly on* $R\times D$.

PROOF. As in the case where $f(t)$ and $g(t)$ are almost periodic in t, we can prove the following implications: (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Therefore we only show (v) $\Rightarrow$ (ii).

There is a countable set $E=\{\varphi^i; i=1, 2, \cdots\}$ in D such that $\mathrm{Cl}(E)\supset D$ by the hypothesis (B_6). Suppose (ii) were false. Then for some $\varepsilon_0>0$ and some compact set S_0 in D, there would exist sequences $\{\delta_k\}$, $\{t_k\}$, $\{h_k\}$ and $\{\psi^k\}$ such that $\delta_k\to 0$ as $k\to\infty$,

$$|f(t+h_k, \varphi)-f(t, \varphi)|\leqq \delta_k \quad \text{for all } t\in R \text{ and any } \varphi\in S_0\cup\{\varphi^1, \cdots, \varphi^k\}\,, \tag{4.1}$$

and

$$|g(t_k+h_k, \psi^k)-g(t_k, \psi^k)|\geqq \varepsilon_0 \qquad \text{for } \psi^k\in S_0\,. \tag{4.2}$$

Let S be any compact set in D. For an integer $m>0$, let $\{U_1^m, \cdots, U_{j_m}^m\}$ denote a $(1/m)$-net of S. U_n^m contains a $\varphi^{p(m,n)}\in E$ for any $m>0$ and n with $1\leqq n\leqq j_m$ since E is dense in D. The set $X=S\cup\{\varphi^{p(m,n)}; m=1, 2, \cdots, n=1, \cdots, j_m\}$ is compact, because for any $\sigma>0$, the number of $\varphi\in X$ with $d(\varphi, S)\geqq\sigma$ is finite. Since $f(t, \varphi)$ is uniformly continuous on $R\times X$, for any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon, X)>0$ such that

$$|f(t, \varphi)-f(t, \psi)|\leqq \varepsilon/3 \qquad \text{for all } t\in R \tag{4.3}$$

if $\varphi\in X$, $\psi\in X$ and $|\varphi-\psi|<\delta$. Choose an m so large that $1/m<\delta/2$. Then there is an integer $N=N(\varepsilon, S)>0$ such that $\delta_k<\varepsilon/3$ and $\{\varphi^1, \cdots, \varphi^k\}\supset\{\varphi^{p(m,n)}; n=1, \cdots, j_m\}$ if $k\geqq N$. Since $U(\varphi, \delta)$ contains a $\varphi^{p(m,n)}$ for any φ in S, it follows from (4.1) and (4.3) that

$$\begin{aligned}|f(t+h_k, \varphi)-f(t, \varphi)| &\leqq |f(t+h_k, \varphi)-f(t+h_k, \varphi^{p(m,n)})|\\ &\quad+|f(t+h_k, \varphi^{p(m,n)})-f(t, \varphi^{p(m,n)})|\\ &\quad+|f(t, \varphi^{p(m,n)})-f(t, \varphi)|\leqq \varepsilon/3+\delta_k+\varepsilon/3<\varepsilon\end{aligned}$$

for all $\varphi\in S$ and all $t\in R$ if $k\geqq N(\varepsilon, S)$. This shows that $\{f(t+h_k, \varphi)\}$ converges to $f(t, \varphi)$ c-uniformly on $R\times D$. Therefore there is a subsequence $\{h_{k_j}\}$ of $\{h_k\}$ such that $\{g(t+h_{k_j}, \varphi)\}$ converges to $g(t, \varphi)$ c-uniformly on $R\times D$ by our assumption (v), a contradiction to (4.2). Thus (v) implies (ii). This completes the proof.

For $0 < H \leqq \infty$, let $f: R \times U_H \to R^n$ be almost periodic in t uniformly for $\varphi \in U_H$. Assume that

(4.4) $$|f(t, \varphi)| \leqq n(|\varphi|) \quad \text{on} \quad R \times U_H$$

for an increasing function $n(\alpha)$ defined on $[0, H)$. Consider the system

(4.5) $$\dot{x}(t) = f(t, x_t) ,$$

and its associated product system

(4.6) $$\dot{x}(t) = f(t, x_t) , \qquad \dot{y}(t) = f(t, y_t) .$$

THEOREM 4.2. *Under the hypotheses* (B_1) *through* (B_6) *and* (B_3'), *assume that there is a continuous real-valued functional* $V(t, \varphi, \psi)$ *defined on* $I \times U_H \times U_H$ *which satisfies the following conditions*:

(i) $a(|\varphi - \psi|) \leqq V(t, \varphi, \psi) \leqq b(|\varphi - \psi|)$, *where* $a(s)$ *and* $b(s)$ *are continuous increasing functions such that* $a(0) = b(0) = 0$ *and* $a(s) > 0$ *for* $s > 0$.

(ii) *For any* $\alpha \in [0, H)$, *there is a constant* L_α *such that*

$$|V(t, \varphi^1, \psi^1) - V(t, \varphi^2, \psi^2)| \leqq L_\alpha\{|\varphi^1 - \varphi^2| + |\psi^1 - \psi^2|\}$$

on $I \times U_\alpha \times U_\alpha$.

(iii) $V'_{(4.6)}(t, \varphi, \psi) \leqq -cV(t, \varphi, \psi)$, *where* $c > 0$ *is a constant.*

If there is a solution $u = u(t_0, \varphi^0)$, $t_0 \geqq 0$, *of* (4.5) *such that* $|u_t| \leqq r < H$ *for all* $t \geqq t_0$ *and for some constant* r, *then there exists a uniformly asymptotically stable almost periodic solution* $P(t)$ *of* (4.5) *such that* $|p_t| \leqq r$ *for all* $t \in R$ *and* $m(p)$ *is contained in* $m(f)$. *In particular, when* $f(t, \varphi)$ *is periodic in* t *of period* ω, *there exists a periodic solution of* (4.5) *of period* ω *which is uniformly asymptotically stable.*

PROOF. First of all, we show the existence of an asymptotically almost periodic solution of (4.5). We may assume $t_0 = 0$. Define S_0 by $S_0 = \mathrm{Cl}\{u_t; t \in I\}$. For $t \in I$, we have $|\dot{u}(t)| = |f(t, u_t)| \leqq n(|u_t|) \leqq n(r)$ by (4.4), and $|u(t)| \leqq k_1 |u_t| \leqq K_1 r$ by the hypothesis (B_4). Therefore S_0 is compact by Lemma 4.1. Then the condition (ii) implies

(4.7) $$|V(t, \varphi^1, \psi^1) - V(t, \varphi^2, \psi^2)| \leqq L\{|\varphi^1 - \varphi^2| + |\psi^1 - \psi^2|\} \quad \text{on} \quad I \times S_0 \times S_0$$

for a constant L.

Let $\{h_k\}$ be any real sequence such that $h_k \to \infty$ as $k \to \infty$. We may assume that $\{f(t+h_k, \varphi)\}$ converges uniformly on $R \times S_0$ as $k \to \infty$. Then for any $\varepsilon > 0$, choose an integer $N = N(\varepsilon, S_0) > 0$ such that if $m \geqq k \geqq N$, we have

(4.8) $$b(2r)e^{-ch_k} < a(\varepsilon)/2$$

and

(4.9) $|f(t+h_k, \varphi) - f(t+h_m, \varphi)| \leqq a(\varepsilon)c/(2K(0)L)$

for all $t \in R$ and any $\varphi \in S_o$.

Using the condition (iii), (4.7) and (4.9), and applying Lemma 3.2, we have

$$\begin{aligned} V'(t+h_k, u_{t+h_k}, u_{t+h_m}) &\leqq -cV(t+h_k, u_{t+h_k}, u_{t+h_m}) \\ &\quad + K(0)L|f(t+h_m, u_{t+h_m}) - f(t+h_k, u_{t+h_m})| \\ &\leqq -cV(t+h_k, u_{t+h_k}, u_{t+h_m}) + a(\varepsilon)/2 \end{aligned}$$

for all $t \in I$, if $m \geqq k \geqq N$. Then the comparison principle and (4.8) imply that

$$V(t+h_k, u_{t+h_k}, u_{t+h_m}) \leqq e^{-c(t+h_k)}\{V(0, \varphi^o, u_{h_m-h_k}) - a(\varepsilon)/2\} + a(\varepsilon)/2 < a(\varepsilon)$$

for all $t \in I$. Therefore we see that

$$|u_{t+h_m} - u_{t+h_k}| < \varepsilon \qquad \text{for all} \quad t \in I$$

by the condition (i), which then implies that

$$|u(t+h_m) - u(t+h_k)| < K_1\varepsilon \qquad \text{for all} \quad t \in I \quad \text{if} \quad m \geqq k \geqq N,$$

by the hypothesis (B_4). Thus $\{u(t+h_k)\}$ converges uniformly on I as $k \to \infty$, and $u(t)$ is asymptotically almost periodic.

Now we show the existence of an almost periodic solution. Let $\{h_k\}$ be any real sequence such that $h_k \to \infty$ and $\{f(t+h_k, \varphi)\}$ converges to $f(t, \varphi)$ uniformly on $R \times S_o$ as $k \to \infty$. Since $u(t)$ is asymptotically almost periodic, it has the decomposition $u(t) = s(t) + q(t)$, where $s(t)$ is almost periodic in t and $q(t)$ is a function which is continuous on I and tends to zero as $t \to \infty$. Taking a subsequence, if necessary, we can assume that $\{s(t+h_k)\}$ converges to an almost periodic function $p(t)$ uniformly on R. Since $p(t)$ is bounded continuous on R, p_t belongs to B for all $t \in R$.

Define v and w by

$$v(\sigma) = \begin{cases} u(\sigma), & \sigma > 0 \\ \varphi^o(0), & \sigma \leqq 0, \end{cases}$$

and $w(\sigma) = u(\sigma) - v(\sigma)$ for all $\sigma \in (-\infty, \infty)$. For a fixed $t \in R$, let k be so large that $t + h_k \geqq 0$. Then v_{t+h_k} and w_{t+h_k} belong to B for all k by the same argument as in the proof of Lemma 4.1. For a compact interval J on $(-\infty, 0]$ and any $\varepsilon > 0$, there is an integer $N = N(\varepsilon, J) > 0$ such that $J \subset [-(t+h_k), 0]$, $|p(t+\theta) - s(t+h_k+\theta)| < \varepsilon/2$ and $|q(t+h_k+\theta)| < \varepsilon/2$ on J if $k \geqq N$. Thus we have $|p(t+\theta) - v(t+h_k+\theta)| < \varepsilon$ on J if $k \geqq N$, since

$$v(t+h_k+\theta)=s(t+h_k+\theta)+q(t+h_k+\theta)$$

for $\theta\in[-(t+h_k), 0]$. This implies that the sequence $\{v(t+h_k+\theta)\}$ converges to $p(t+\theta)$ locally uniformly on $(-\infty, 0]$. Clearly, the sequence is uniformly bounded on $(-\infty, 0]$. Hence $|v_{t+h_k}-p_t|\to 0$ as $k\to\infty$ by the hypothesis (B_5). Using the hypotheses (B_2) and (B_3), we see that

$$\begin{aligned}|u_{t+h_k}-p_t| &= |v_{t+h_k}+w_{t+h_k}-p_t| \leqq |v_{t+h_k}-p_t|+|w_{t+h_k}| \\ &\leqq |v_{t+h_k}-p_t|+|\tau^{t+h_k}(\varphi^0-\langle\varphi^0(0)\rangle)|_{t+h_k} \\ &\leqq |v_{t+h_k}-p_t|+M(t+h_k)|\varphi^0-\langle\varphi^0(0)\rangle| .\end{aligned}$$

Since $|v_{t+h_k}-p_t|\to 0$ as $k\to\infty$ and $M(t+h_k)\to 0$ as $k\to\infty$ by the hypothesis (B_3'), we have $|u_{t+h_k}-p_t|\to 0$ as $k\to\infty$. Thus $p_t\in S_0$ for any $t\in R$.

For any fixed $t\in R$, choose k so large that $t+h_k\geqq 0$. Since $\{f(s+h_k, \varphi)\}$ converges to $f(s, \varphi)$ uniformly on $R\times S_0$ and $f(s, \varphi)$ is uniformly continuous on $R\times S_0$, we see that $\lim_{k\to\infty}f(s+h_k, u_{s+h_k})=f(s, p_s)$ for any $s\in R$. Also, $u(t+h_k)\to p(t)$ as $k\to\infty$ by the hypothesis (B_4). Therefore, by Lebesgue's dominant convergence theorem, we have

$$\begin{aligned}p(t) &= \lim_{k\to\infty} u(t+h_k)=\lim_{k\to\infty}\left\{u(h_k)+\int_{h_k}^{t+h_k} f(s, u_s)ds\right\} \\ &= \lim_{k\to\infty}\left\{u(h_k)+\int_0^t f(s+h_k, u_{s+h_k})ds\right\} \\ &= p(0)+\int_0^t \lim_{k\to\infty} f(s+h_k, u_{s+h_k})ds \\ &= p(0)+\int_0^t f(s, p_s)ds\end{aligned}$$

for any fixed $t\in R$. Thus $p(t)$ is an almost periodic solution of (4.5).

By using the functional $V(t, \varphi, \psi)$, it is easily shown that $p(t)$ is uniformly asymptotically stable and every solution remaining in U_H approaches $p(t)$ as $t\to\infty$, which implies the uniqueness of $p(t)$.

Finally we show that $m(p)\subset m(f)$. Let $\{h_k\}$ be any real sequence for which $\{f(t+h_k, \varphi)\}$ converges to $f(t, \varphi)$ c-uniformly on $R\times U_H$. Since $p(t)$ is almost periodic, there exists a subsequence $\{h_{k_j}\}$ of $\{h_k\}$ for which $\{p(t+h_{k_j})\}$ converges uniformly on R as $j\to\infty$. Let $Q(t)=\lim_{j\to\infty} p(t+h_{k_j})$. Then by the same argument as in the above, $Q(t)$ is an almost periodic solution of (4.5). The uniqueness of $p(t)$ implies $Q(t)=p(t)$, and so we see $m(p)\subset m(f)$ by (v) in Theorem 4.1. This completes the proof.

Now we consider the linear system

$$\dot{x}(t)=A(t, x_t)\,, \tag{4.10}$$

and the nonhomogeneous system

$$x(t) = A(t, x_t) + b(t) , \tag{4.11}$$

where $A(t, \varphi)$: $R \times B \to R^n$ is continuous and linear in φ, and $b(t)$: $R \to R^n$ is continuous and $|b(t)| \leqq r$ for all $t \in R$ and for a constant r. Then we have the following.

THEOREM 4.3. (I) *Suppose that the hypotheses* (B_1) *through* (B_4) *are satisfied. If the zero solution of* (4.10) *is uniformly asymptotically stable on I, that is,* (3.10) *holds for some constants* $M > 0$ *and* $c > 0$, *then all solutions of* (4.11) *are bounded on I. In particular, we have* $|x_t(t_o, \varphi)| \leqq K(0)Mr/c$ *for any* $t_o \geqq 0$ *and for all* $t \geqq t_o$ *if* $|\varphi| \leqq K(0)r/c$, *where* $x(t_o, \varphi)$ *is the solution of* (4.11).

(II) *Suppose that the hypotheses* (B_1) *through* (B_6) *and* (B_3') *are satisfied, and assume that* $A(t, \varphi)$ *and* $b(t)$ *are almost periodic in t uniformly for* $\varphi \in B$. *If the zero solution of* (4.10) *is uniformly asymtotically stable, then there exists an almost periodic solution of* (4.11), *which is uniformly asymptotically stable and bounded by* $K(0)Mr/c$, *and its module is contained in the module of* $A(t, \varphi) + b(t)$. *Particularly, when* $A(t, \varphi)$ *and* $b(t)$ *are periodic in t of period* ω, *there exists a periodic solution of* (4.11) *of period* ω *which is uniformly asymptotically stable.*

PROOF. (I) Since the zero solution of (4.10) is uniformly asymptotically stable, there is a continuous functional $V(t, \varphi)$ satisfying the conditions (i), (ii) and (iii) in Theorem 3.3. Then the condition (iii) and Lemma 3.2 imply that

$$\begin{aligned} V'(t, x_t) &\leqq V'_{(4.10)}(t, y_t) + K(0)M|\dot{x}(t) - \dot{y}(t)| \\ &\leqq -cV(t, y_t) + K(0)M|b(t)| , \end{aligned}$$

where x is a solution of (4.11) through $(t_o, \varphi) \in I \times B$ and y is a solution of (4.10) through (t, x_t). Since $x_t = y_t$, we have

$$V'(t, x_t) \leqq -cV(t, x_t) + K(0)M|b(t)| .$$

Then, using the comparison principle and the condition (i), we have

$$\begin{aligned} |x_t| \leqq V(t, x_t) &\leqq e^{-c(t-t_o)}V(t_o, \varphi) + \int_{t_o}^{t} e^{-c(t-s)}K(0)M|b(s)|\,ds \\ &\leqq Me^{-c(t-t_o)}|\varphi| + (K(0)Mr/c)(1 - e^{-c(t-t_o)}) . \end{aligned}$$

Therefore x is bounded. In particular, $|x_t| \leqq K(0)Mr/c$ for all $t \geqq t_o$ if $|\varphi| \leqq K(0)r/c$.

(II) Since $A(t, \varphi)$ is bounded on $R \times S$ for any compact set S in B, we see that $|A(t,\varphi)| \leqq N|\varphi|$ on $R \times B$ for a constant $N > 0$ by the argument similar to that in Lemma 3.1. Then we have

$$|A(t,\varphi) + b(t)| \leqq N|\varphi| + r \qquad \text{on } R \times B\,,$$

that is, (4.4) holds for $f(t,\varphi) = A(t,\varphi) + b(t)$.

Let $V(t,\varphi)$ be the continuous functional satisfying the conditions (i), (ii) and (iii) in Theorem 3.3. Define $W(t,\varphi,\psi)\colon I \times B \times B \to R$ by

$$W(t,\varphi,\psi) = V(t,\varphi-\psi)\,.$$

Then we have

$$|\varphi-\psi| \leqq W(t,\varphi,\psi) \leqq M|\varphi-\psi|$$

and

$$|W(t,\varphi^1,\psi^1) - W(t,\varphi^2,\psi^2)| \leqq M\{|\varphi^1-\varphi^2| + |\psi^1-\psi^2|\}$$

on $I \times B \times B$. Let x and y be solutions of (4.11) and (4.10), respectively. Then the condition (iii) implies that

$$\begin{aligned} W'_{(4.11)}(t,\varphi,\psi) &= \limsup_{\delta\to 0+} \{V(t+\delta, x_{t+\delta}(t,\varphi) - x_{t+\delta}(t,\psi)) - V(t,\varphi-\psi)\}/\delta \\ &= \limsup_{\delta\to 0+} \{V(t+\delta, y_{t+\delta}(t,\varphi-\psi)) - V(t,\varphi-\psi)\}/\delta \\ &= V'_{(4.10)}(t,\varphi-\psi) \leqq -cV(t,\varphi-\psi) \leqq -cW(t,\varphi,\psi)\,. \end{aligned}$$

Therefore $W(t,\varphi,\psi)$ satisfies the conditions (i), (ii) and (iii) in Theorem 4.2. Moreover, the conclusion in (I) shows the existence of a solution of (4.11) bounded by $K(0)Mr/c$. Thus we can apply Theorem 4.2 to obtain the conclusion in (II).

REMARK. For an almost periodic functional differential equation with infinite delay, Hino [8] also showed the existence of an almost periodic solution under the existence assumption of a bounded solution with some stability property, but the phase space considered in [8] is slightly different from ours. His hypothesis corresponding to (B_2) is stronger than ours, but (B'_3) is stronger than his corresponding hypothesis.

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