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EXISTENCE OF OPTIMAL MARTINGALES

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1. Introduction. Let M be a continuous BMO-martingale with $M_0 = 0$, and $Z(M)$ be the solution of the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_s dM_s . \tag{1}$$

Then, as is proved in [4], $Z(M)$ is an L^p-bounded martingale for some $p > 1$. Now, let us fix a continuous BMO-martingale X. We call $J(M) = E[X_\infty Z_\infty(M)]$ the cost of X associated with M. In this paper we are concerned with the following problem: if $\underline{\underline{S}}$ is a subclass of BMO, then does there exist an element M^o of $\underline{\underline{S}}$ such that $J(M^o) \leqq J(M)$ for all $M \in \underline{\underline{S}}$? In Section 4 we shall determine a class $\underline{\underline{S}}$ for which there exists the optimal martingale M^o achieving the minimum cost, and in the last section we shall give a negative example for this existence problem. We also give an example of the existence of the optimal martingale in dynamical systems subject to random perturbations.

The results are based on two steps. The first is the theory of H^p and BMO-martingales developed by Getoor and Sharpe [2], and Kazamaki and Sekiguchi [4]. The second is the stochastic control theory given by Duncan and Varaiya [1].

The reader is assumed to be familiar with the martingale theory as set forth in Meyer [6].

2. Preliminaries. Let (Ω, F, P) be a complete probability space with a non-decreasing right continuous family $(F_t)_{t \geqq 0}$ of sub σ-fields of F such that F_0 contains all null sets and $F = \bigvee_{t \geqq 0} F_t$. Let L_c be the class of all continuous local martingales X over (F_t) with $X_0 = 0$. If $X \in L_c$, then there exists a unique continuous increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle \in L_c$. If $X, Y \in L_c$, then $\langle X, Y \rangle$ is defined by

$$\langle X, Y \rangle = (\langle X + Y \rangle - \langle X - Y \rangle)/4 .$$

It is well-known that $XY - \langle X, Y \rangle$ belongs to L_c. Let H^p be the Banach space of all $X \in L_c$ such that

$$\|X\|_{H^p} = \|X^*\|_{L^p} < \infty , \qquad p \geqq 1 ,$$

where $X^* = \sup_t |X_t|$. Let BMO be the Banach space of all $X \in L_c$ such that

$$||X||_{\mathrm{BMO}} = \sup_t ||E[\langle X\rangle_\infty - \langle X\rangle_t \,|\, F_t]^{1/2}||_{L^\infty} < \infty\ .$$

As is well-known, the solution $Z(M)$ of (1) is given by the formula

$$Z_t = Z_t(M) = \exp(M_t - \langle M\rangle_t/2)\ .$$

Throughout, we assume that every M belongs to BMO.

DEFINITION 1. For any fixed constants $C \geqq 1$, $r > 1$, let $R^r(C)$ be the class of all $M \in \mathrm{BMO}$ such that

$$\sup_t ||E[(Z_\infty(M)/Z_t(M))^r \,|F_t]\,||_{L^\infty} \leqq C\ . \tag{2}$$

DEFINITION 2. Let $X \in \mathrm{BMO}$ be fixed. Then, the cost $J(M)$ of X associated with $M \in R^r(C)$ is defined by

$$J(M) = E[\langle X, Z(M) - 1\rangle_\infty] = E[X_\infty Z_\infty(M)]\ .$$

Since $Z(M) - 1 \in H^r$ for $M \in R^r(C)$, it is well-defined by Fefferman's inequality [2]. $M^o \in R^r(C)$ is called an optimal martingale if $J(M^o) \leqq J(M)$ for all $M \in R^r(C)$.

3. Transformation of BMO-martingales. We recall in this section the recent results of Kazamaki and Sekiguchi [4]. For any constant $K > 0$, let $B(K)$ denote the class of all $M \in \mathrm{BMO}$ such that $||M||_{\mathrm{BMO}} \leqq K$. For $1 < p < \infty$ and the solution $Z = Z(M)$ of (1), let $A_p(Z)$ denote the constant defined by

$$A_p(Z) = \sup_t ||E[(Z_t/Z_\infty)^{1/(p-1)} \,|\, F_t]\,||_{L^\infty}\ .$$

LEMMA 1.

(a) *If $M \in B(K)$ for a constant $K > 0$, then*

$$A_p(Z) \leqq (1 - K^2/(2(\sqrt{p} - 1)^2))^{-\sqrt{p}/(\sqrt{p}+1)}\ , \tag{3}$$

for p sufficient large.

(b) *If $A_{p-1}(Z) \leqq K'$ for a constant $K' \geqq 1$ and $p > 2$, then*

$$M \in B(K), \quad \textit{where } K = (2(p-2)\log K')^{1/2}\ . \tag{4}$$

The proof is identical with that of Lemma 5 in [4].

LEMMA 2. *For any $K > 0$, there exist constants $C \geqq 1$ and $r > 1$ such that $B(K) \subset R^r(C)$.*

The proof is identical with that of Theorem 1 in [4].

Let $M \in \mathrm{BMO}$ and P_M be the probability measure defined by $dP_M = Z_\infty(M)dP$.

As $Z_\infty(M) > 0$, P_M and P are mutually absolutely continuous. Let E_M denote the expectation with respect to dP_M. Let $\mathrm{BMO}(P_M)$ and $\|\cdot\|_{\mathrm{BMO}(P_M)}$ denote the space BMO and its norm associated with dP_M, respectively. As is stated in [4], if $X \in \mathrm{BMO}$, then $\hat{X} = X - \langle X, M\rangle \in \mathrm{BMO}(P_M)$ and $\langle \hat{X}\rangle = \langle X\rangle$.

LEMMA 3. *Let $C \geqq 1$ and $r > 1$. If $M \in R^r(C)$, then*

$$\|\hat{X}\|_{\mathrm{BMO}(P_M)} \leqq c\,\|X\|_{\mathrm{BMO}}\,, \qquad X \in \mathrm{BMO} \tag{5}$$

where the positive constant c depends only on C and r.

PROOF. Let us assume that $0 < \|X\|_{\mathrm{BMO}} < \infty$ and set $a = 1/(2r'\|X\|^2_{\mathrm{BMO}})$, where $1/r + 1/r' = 1$. As $\|(ar')^{1/2}X\|^2_{\mathrm{BMO}} = 1/2$, Lemma 4 of [4] yields

$$E[\exp(ar'(\langle X\rangle_\infty - \langle X\rangle_t))|F_t] \leqq (1 - \|(ar')^{1/2}X\|^2_{\mathrm{BMO}})^{-1} = 2\,.$$

By using a simple inequality $x \leqq e^{ax}/a$ and Hölder's inequality,

$$\begin{aligned} E_M[\langle \hat{X}\rangle_\infty - \langle \hat{X}\rangle_t | F_t] &= E[(Z_\infty(M)/Z_t(M))(\langle X\rangle_\infty - \langle X\rangle_t)\,|F_t] \\ &\leqq E[(Z_\infty(M)/Z_t(M))\exp(a(\langle X\rangle_\infty - \langle X\rangle_t))|F_t]/a \\ &\leqq E[(Z_\infty(M)/Z_t(M))^r|F_t]^{1/r}E[\exp(ar'(\langle X\rangle_\infty - \langle X\rangle_t))|F_t]^{1/r'}/a \\ &\leqq C^{1/r}2^{1/r'}/a = C^{1/r}2^{1/r'}2r'\,\|X\|^2_{\mathrm{BMO}}. \end{aligned}$$

Thus we obtain (5).

LEMMA 4. *For any constants $C \geqq 1$, $r > 1$, there exists a constant $K > 0$ such that $R^r(C) \subset B(K)$.*

PROOF. Let $M \in R^r(C)$ and define $\tilde{M} = -\hat{M} = -M + \langle M\rangle$. Then we have $\tilde{M} \in \mathrm{BMO}(P_M)$ and $\langle \tilde{M}\rangle = \langle M\rangle$. Under P_M the unique solution $Z' = Z'(\tilde{M})$ of the equation

$$Z'_t = 1 + \int_0^t Z'_s d\tilde{M}_s$$

is given by

$$Z'_t(\tilde{M}) = \exp(\tilde{M}_t - \langle \tilde{M}\rangle_t/2) = \exp(-M_t + \langle M\rangle_t/2) = 1/Z_t(M)\,,$$

so that $Z'_\infty(\tilde{M})dP_M = dP$ and

$$\begin{aligned} E_M[(Z'_t(\tilde{M})/Z'_\infty(\tilde{M}))^{r-1}|F_t] &= E[(Z_\infty(M)/Z_t(M))(Z'_t(\tilde{M})/Z'_\infty(\tilde{M}))^{r-1}|F_t] \\ &= E[(Z_\infty(M)/Z_t(M))^r|F_t] \leqq C\,. \end{aligned}$$

Then by (4), taking $1/(p-2) = r - 1$,

$$\|\tilde{M}\|_{\mathrm{BMO}(P_M)} \leqq (2(\log C)/(r-1))^{1/2} = C'\,. \tag{6}$$

Thus by Lemma 2,

$$\sup_t \| E_M[(Z'_\infty(\tilde{M})/Z'_t(\tilde{M}))^{r''} | F_t] \|_{L^\infty} \leqq C''$$

for some constants $C'' \geqq 1$, $r'' > 1$ which depend only on C'. Furthermore, by Lemma 3 we have

$$(7) \qquad \| Y' - \langle Y', \tilde{M} \rangle \|_{\mathrm{BMO}(P)} \leqq c \| Y' \|_{\mathrm{BMO}(P_M)}, \quad Y' \in \mathrm{BMO}(P_M)$$

where the positive constant c depends only on C'' and r''. Then (7) yields, taking $Y' = \tilde{M}$,

$$\| M \|_{\mathrm{BMO}} \leqq c \| \tilde{M} \|_{\mathrm{BMO}(P_M)} .$$

Therefore, combining this inequality with (6), the lemma is proved.

4. Existence of optimal martingales.

LEMMA 5. *Let $1 < p < \infty$. Then the Banach space H^p is reflexive.*

PROOF. As is well-known, the class $\underline{\underline{H}}^p$ of all L^p-bounded right continuous martingales X with $X_0 = 0$ is a reflexive Banach space with the norm $\|X\|_{\underline{\underline{H}}^p} = \|X^*\|_{L^p}$. Clearly, H^p is a closed subspace of $\underline{\underline{H}}^p$. Then it follows from the theorems of Eberlein-Shmulyan and Mazur [8] that H^p is also reflexive.

From now on, we fix the constants $C \geqq 1$ and $r > 1$.

THEOREM 1. *The set $D^r(C) = \{Z(M) - 1; M \in R^r(C)\}$ is weakly compact in H^r.*

PROOF. To prove the theorem, we show that $D^r(C)$ has the following properties: (a) boundedness, (b) convexity and (c) closedness. The details are as follows.

(a) By Doob's inequality and (2),

$$E[(\sup_t |Z_t(M)|)^r] \leqq (r/(r-1))^r E[Z_\infty(M)^r] \leqq (r/(r-1))^r C .$$

Thus the boundedness of $D^r(C)$ follows.

(b) Let $M^{(i)} \in R^r(C)$, $\lambda_i \geqq 0$, $i = 1, 2$, with $\lambda_1 + \lambda_2 = 1$, and let $Z^{(i)} - 1 = Z(M^{(i)}) - 1 \in D^r(C)$. Define the process $M \in L_c$ by

$$M_t = \sum_{i=1}^{2} \int_0^t \Big(\lambda_i Z_s^{(i)} / \sum_{j=1}^{2} \lambda_j Z_s^{(j)} \Big) dM_s^{(i)} .$$

Then it is easy to see that $(\sum_{i=1}^{2} \lambda_i Z_t^{(i)})$ is a solution of (1), that is,

$$\sum_{i=1}^{2} \lambda_i Z^{(i)} = Z(M) .$$

Also, since

$$E\Big[\int_t^\infty \Big(\lambda_i Z_s^{(i)} \Big/ \sum_{j=1}^{2} \lambda_j Z_s^{(j)} \Big)^2 d\langle M^{(i)} \rangle_s | F_t \Big] \leqq \| M^{(i)} \|_{\mathrm{BMO}}^2, \quad i = 1, 2 ,$$

we have $M \in \text{BMO}$. From Minkowski's inequality it follows that

$$E[(Z_\infty(M))^r | F_t]^{1/r} \leqq \sum_{i=1}^{2} E[(\lambda_i Z_\infty^{(i)})^r | F_t]^{1/r} = \sum_{i=1}^{2} \lambda_i E[(Z_\infty^{(i)})^r | F_t]^{1/r}$$
$$\leqq \sum_{i=1}^{2} \lambda_i (C(Z_t^{(i)})^r)^{1/r} = C^{1/r} Z_t(M) .$$

Therefore, $M \in R^r(C)$ and

$$\sum_{i=1}^{2} \lambda_i (Z^{(i)} - 1) = \sum_{i=1}^{2} \lambda_i Z^{(i)} - 1 = Z(M) - 1 \in D^r(C) .$$

(c) Let $\{M^{(n)}\}$ be a sequence from $R^r(C)$ and let $Z^{(n)} - 1 = Z(M^{(n)}) - 1 \in D^r(C)$. Let $Z - 1$ be in H^r such that

$$\lim_{n \to \infty} \| Z^{(n)} - Z \|_{H^r} = 0 ,$$

and

$$\lim_{n \to \infty} \sup_t |Z_t^{(n)} - Z_t| = 0 \qquad \text{a.s. } P .$$

Since by Lemma 4 there exists some constant $K > 0$ such that every $M^{(n)}$ belongs to $B(K)$, we have for all n,

$$E[(M_\infty^{(n)})^2] = E[\langle M^{(n)} \rangle_\infty] \leqq \| M^{(n)} \|_{\text{BMO}}^2 \leqq K^2 .$$

Hence,

$$E[| \lim_{n \to \infty} (M_\infty^{(n)} - \langle M^{(n)} \rangle_\infty / 2) |] \leqq \liminf_{n \to \infty} E[| M_\infty^{(n)} - \langle M^{(n)} \rangle_\infty / 2 |]$$
$$\leqq \liminf_{n \to \infty} \{ E[| M_\infty^{(n)} |] + E[\langle M^{(n)} \rangle_\infty]/2 \}$$
$$\leqq \liminf_{n \to \infty} \{ E[(M_\infty^{(n)})^2]^{1/2} + E[\langle M^{(n)} \rangle_\infty]/2 \}$$
$$\leqq K + K^2/2 .$$

Thus we obtain

$$P(\lim_{n \to \infty} (M_\infty^{(n)} - \langle M^{(n)} \rangle_\infty / 2) = -\infty) = 0 ,$$

so that $Z_\infty = \lim_{n \to \infty} \exp(M_\infty^{(n)} - \langle M^{(n)} \rangle_\infty / 2) > 0$ a.s. P. By Theorem 15 of [5, § VI], $Z_t > 0$ for all t a.s. P. Thus we can define the process $M \in L_c$ by $M_t = \int_0^t Z_s^{-1} \, dZ_s$. Then Z is a solution of (1), i.e., $Z = Z(M)$. By (3), for p sufficient large,

$$E[(Z_t / Z_\infty)^{1/(p-2)} | F_t] \leqq \liminf_{n \to \infty} E[(Z_t^{(n)} / Z_\infty^{(n)})^{1/(p-2)} | F_t]$$
$$\leqq \liminf_{n \to \infty} A_{p-1}(Z^{(n)})$$
$$\leqq (1 - K^2 / (2(\sqrt{p-1} - 1)^2))^{-\sqrt{p-1}/(\sqrt{p-1}+1)} ,$$

which implies $M \in \mathrm{BMO}$ by (4). Furthermore, it is clear that

$$E[(Z_\infty/Z_t)^r | F_t] \leqq \liminf_{n\to\infty} E[(Z_\infty^{(n)}/Z_t^{(n)})^r | F_t] \leqq C .$$

Therefore, $M \in R^r(C)$ and then, $Z - 1 = Z(M) - 1 \in D^r(C)$.

Now we prove that $D^r(C)$ is a weakly compact subset of H^r. Let $\{Y_n\}$ be a sequence from $D^r(C)$. By (a) and Lemma 5, $D^r(C)$ is a bounded subset of the reflexive Banach space H^r $(r > 1)$. Then it follows from Eberlein and Shmulyan's theorem that there exists a subsequence $\{Y_{n_k}\}$ of $\{Y_n\}$ such that $\{Y_{n_k}\}$ converges weakly to an element Y of H^r. On the other hand, by Mazur's theorem, there exists a convex combination $Y^{(m)} = \sum_{k=1}^m \mu_k^{(m)} Y_{n_k}$ $(\mu_k^{(m)} \geqq 0,\ \sum_{k=1}^m \mu_k^{(m)} = 1)$ of Y_{n_k}'s such that $\{Y^{(m)}\}$ converges strongly to Y. Therefore, by (b) and (c), $Y \in D^r(C)$. This completes the proof.

THEOREM 2. *There exists an optimal martingale M^o in $R^r(C)$.*

PROOF. Let us fix $X \in \mathrm{BMO}$. We first show that the cost $J(M) = E[X_\infty Z_\infty(M)]$ of X associated with $M \in R^r(C)$ is bounded. Let us assume that it is not bounded. Then there exists a sequence $\{M^{(n)}\} \subset R^r(C)$ such that $|J(M^{(n)})| > n$ for each n. By Theorem 1, the sequence $\{Z(M^{(n)}) - 1\} \subset D^r(C)$ contains a subsequence $\{Z(M^{(n_k)}) - 1\}$ which converges weakly to $Z(M) - 1 \in D^r(C)$ for some $M \in R^r(C)$. Hence,

$$\begin{aligned} |J(M)| &= |E[X_\infty Z_\infty(M)]| = \lim_{k\to\infty} |E[X_\infty Z_\infty(M^{(n_k)})]| \\ &= \lim_{k\to\infty} |J(M^{(n_k)})| = \infty , \end{aligned}$$

which is a contradiction.

Next, set $J^o = \inf\{J(M);\ M \in R^r(C)\}$ and let $\{M^{(n)}\} \subset R^r(C)$ be a sequence such that $\lim_{n\to\infty} J(M^{(n)}) = J^o$. By the above argument, taking a subsequence $\{M^{(n_k)}\}$ of $\{M^{(n)}\}$ we can find $M^o \in R^r(C)$ such that $\lim_{k\to\infty} J(M^{(n_k)}) = J(M^o)$. Thus the theorem is proved.

5. Examples. Let (B_t, P) be a Brownian motion with $B_0 = 0$, and let F_t be the σ-field generated by $(B_s, s \leqq t)$. Let G be the class of all predictable processes f with

$$\sup_t \left\| E\left[\int_t^\infty f_s^2 ds \,|\, F_t\right]^{1/2} \right\|_{L^\infty} < \infty$$

and

$$\sup_t \left\| E\left[\left(\exp\left(\int_t^\infty f_s dB_s - \int_t^\infty f_s^2 ds/2\right)\right)^r |F_t\right]\right\|_{L^\infty} \leqq C .$$

By the integral representation theorem of martingales, $R^r(C)$ can be identified with the class of all processes $M=\left(\int_0^t f_s dB_s\right)$ for every $f\in G$. Let $\lambda(t,\cdot)$ be a bounded measurable function on $[0,1]\times R^1$ and let $T=\inf\{t\leqq 1; B_t\in\Gamma\}$ for a Borel set Γ of R^1. Define the cost $J(M)$ associated with $M\in R^r(C)$ by

$$J(M)=E_M\left[\int_0^T \lambda(s,B_s)ds\right].$$

Then, by Theorem 2 there exists an $f\in G$ such that f minimizes $J(M)$.

Finally we give a negative example for the existence problem. Let Φ be the class of all processes $M^{(a)}\in L_c$ defined by $M_t^{(a)}=aB_{t\wedge 1}$ for $a\in R^1$. Then, $\Phi\subset\mathrm{BMO}$ and $\|M^{(a)}\|_{\mathrm{BMO}}=|a|$, because

$$\langle M^{(a)}\rangle_t=a^2(t\wedge 1)$$

and

$$E[\langle M^{(a)}\rangle_\infty-\langle M^{(a)}\rangle_t|F_t]=a^2-a^2(t\wedge 1).$$

Let $Z^{(a)}$ be the unique solution of

$$Z_t^{(a)}=1+\int_0^t Z_s^{(a)}dM_s^{(a)}.$$

Then the cost $J(M^{(a)})=E[\langle M^{(1)},Z^{(a)}-1\rangle_\infty]$ associated with $M^{(a)}\in\Phi$ has no minimum. Indeed, since $Z^{(a)}$ is a martingale, we have

$$J(M^{(a)})=E\left[\int_0^\infty Z_s^{(a)}d\langle M^{(1)},M^{(a)}\rangle_s\right]=E\left[a\int_0^1 Z_s^{(a)}ds\right]=a.$$

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