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REMARK ON A CHARACTERIZATION OF BMO-MARTINGALES

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1. Introduction and results. Let (Ω, F, P) be a complete probability space with an increasing family $(F_t)_{t\geqq 0}$ of sub-σ-fields of F which satisfies the habitual conditions. Let M be a local martingale with $M_0 = 0$ and denote by M^c the continuous part of M. Let $\langle M^c\rangle$ be the continuous increasing process such that $(M^c)^2 - \langle M^c\rangle$ is a continuous local martingale and put $\varDelta M_\cdot = M_\cdot - M_{\cdot -}$ and $[M]_\cdot = \langle M^c\rangle_\cdot + \sum_{0\leqq s\leqq \cdot}(\varDelta M_s)^2$. As is well-known, the process

$$(1)\qquad Z_\cdot^{(\lambda)} = \exp\left(\lambda M_\cdot - \frac{\lambda^2}{2}\langle M^c\rangle_\cdot\right)\prod_{0\leqq s\leqq \cdot}(1 + \lambda\varDelta M_s)e^{-\lambda\varDelta M_s}\ ,$$

where λ is real, is a local martingale. If $1 + \lambda\varDelta M_\cdot > 0$, then $Z^{(\lambda)}$ is a strictly positive supermartingale and the limit $Z_\infty^{(\lambda)} = \lim_{t\to\infty} Z_t^{(\lambda)}$ exists almost surely (cf. [6]).

The following theorem was proved by C. Doléans-Dade and P. A. Meyer [2] and by N. Kazamaki [4]. We write simply Z instead of $Z^{(1)}$.

THEOREM 1. *Suppose that Z has the following three properties*:

(i) *Z satisfies the condition (S), that is, there exists a positive constant ε such that*

$$(2)\qquad \varepsilon < Z_\cdot/Z_{\cdot -} < 1/\varepsilon\ ,$$

(ii) $Z_\infty > 0$ *a.s.*

and

(iii) *Z satisfies the condition (A_∞), that is, there exist positive constants a and K such that*

$$(3)\qquad E[(Z_T/Z_\infty)^a|F_T] \leqq K \quad a.s.$$

for any stopping time T.

Then $1 + \varDelta M_\cdot > \varepsilon$ and M is a BMO-*martingale, that is,* $||M||_{\mathrm{BMO}} = (\sup_t ||E[[M]_\infty - [M]_{t-}|F_t]||_\infty)^{1/2} < \infty$.

The converse of Theorem 1 was proved in the above literature [2] and [4] in the case the BMO-norm or the jumps of M is sufficiently small. In this note, our object is to show that the converse is true even if the "sufficient smallness" is removed.

THEOREM 2. *If M is a* BMO-*martingale and there exists a positive constant ε such that $1 + \Delta M. > \varepsilon$, then Z satisfies the conditions* (i), (ii) *and* (iii) *in Theorem* 1. *Furthermore,*

(iv) *Z is a uniformly integrable martingale.*

Theorem 1, Theorem 2 and Gehring's lemma in [1] imply the following corollary.

COROLLARY. *Z satisfies the conditions (S) and (A_∞) if and only if M is a* BMO-*martingale with $1 + \Delta M. > \varepsilon$ for some $\varepsilon > 0$; in this case, Z is a L^p-bounded martingale for some $p > 1$.*

This corollary was proved in [3] in the case M is continuous. We remark also that the continuity condition of local martingales treated in [5] would be supressed. We shall return to this point elsewhere.

2. Proof of Theorem 2. Properties (i) and (ii) are easily checked by using the facts $1 + \Delta M. > \varepsilon$ and $||M||_{\mathrm{BMO}} < \infty$. We may assume without loss of generality that ε is sufficiently small. By an elementary calculation we have the inequality

$$\exp(x - x^2/2\varepsilon^2) \leqq 1 + x \leqq e^x$$

for $1/(1-\varepsilon) > a > 0$ and $-1 + \varepsilon \leqq x \leqq (1-\varepsilon)/2a$, from which we obtain easily

$$(1 + x)^{-a} \leqq (1 - 2ax)^{1/2} \exp(ax^2/2\varepsilon^2) \tag{4}$$

and

$$1 + ax \leqq (1 + x)^a \exp(ax^2/2\varepsilon^2) . \tag{5}$$

Now we shall show the property (iii). We can choose $a > 0$ such that $k_a = (4a^2 + a)/\varepsilon^2 < 1/||M||^2_{\mathrm{BMO}}$. Then we get for any stopping time T, by applying (4) and Schwarz's inequality,

$$\begin{aligned}
&E[(Z_T/Z_\infty)^a \,|\, F_T] \\
&\quad = E[\exp\{-a(M_\infty - M_T) + (a/2)(\langle M^c\rangle_\infty - \langle M^c\rangle_T)\} \\
&\qquad \times \prod_{t>T} (1 + \Delta M_t)^{-a} \exp(a\Delta M_t) \,|\, F_T] \\
&\quad \leqq E[\exp\{-a(M_\infty - M_T) + (a/2)(\langle M^c\rangle_\infty - \langle M^c\rangle_T)\} \\
&\qquad \times \prod_{t>T} (1 - 2a\Delta M_t)^{1/2} \exp\{a\Delta M_t + a(\Delta M_t)^2/2\varepsilon^2\} \,|\, F_T] \\
&\quad = E[[\exp\{-a(M_\infty - M_T) - a^2(\langle M^c\rangle_\infty - \langle M^c\rangle_T)\} \\
&\qquad \times \prod_{t>T} (1 - 2a\Delta M_t)^{1/2} \exp(a\Delta M_t)]
\end{aligned}$$

$$\times \exp\{(1/2)(2a^2+a)(\langle M^c\rangle_\infty-\langle M^c\rangle_T)+(a/2\varepsilon^2)\sum_{t>T}(\varDelta M_t)^2\}\,|\,F_T]$$
$$\leqq E[Z_\infty^{(-2a)}/Z_T^{(-2a)}|F_T]^{1/2}E[\exp\{k_a([M]_\infty-[M]_{T-})\}|F_T]^{1/2}\,.$$

The first factor of the last expression is smaller than 1, while, from an inequality of John-Nirenberg's type (see Lemma 2 in [4]), the second factor is dominated by $1/(1-k_a\,||M||_{\mathrm{BMO}}^2)^{1/2}$. Therefore we obtain the property (iii).

Finally we shall show the property (iv). By choosing $a>0$ small enough, we can assume that $Z^{(a)}$ is a uniformly integrable martingale (see [2], p. 386) and $K_a=(4a^2+a)/(1-a)\varepsilon^2\leqq 1/||M||_{\mathrm{BMO}}^2$. Then for any stopping time T,

$$\begin{aligned}
1 &= E[Z_\infty^{(a)}/Z_T^{(a)}\,|\,F_T]\\
&= E[\exp\{a(M_\infty-M_T)-(a^2/2)(\langle M^c\rangle_\infty-\langle M^c\rangle_T)\}\\
&\quad\times\prod_{t>T}(1+a\varDelta M_t)\exp(-a\varDelta M_t)\,|\,F_T]\\
&\leqq E[\exp\{a(M_\infty-M_T)-(a^2/2)(\langle M^c\rangle_\infty-\langle M^c\rangle_T)\}\\
&\quad\times\prod_{t>T}(1+\varDelta M_t)^a\exp(-a\varDelta M_t+a(\varDelta M_t)^2/2\varepsilon^2)\,|\,F_T]\\
&= E[[\exp\{a(M_\infty-M_T)-(a/2)(\langle M^c\rangle_\infty-\langle M^c\rangle_T)\}\\
&\quad\times\prod_{t>T}(1+\varDelta M_t)^a\exp(-a\varDelta M_t)]\\
&\quad\times\exp\{(1/2)(a-a^2)(\langle M^c\rangle_\infty-\langle M^c\rangle_T)+(a/2\varepsilon^2)\sum_{t>T}(\varDelta M_t)^2\}\,|\,F_T]\,,
\end{aligned}$$

where we have made use of (5). Applying Hölder's inequality with exponents $1/a$ and $1/(1-a)$ to the last expression we can obtain:

$$\begin{aligned}
1 &\leqq E[Z_\infty/Z_T\,|\,F_T]E[\exp\{K_a([M]_\infty-[M]_{T-})\}\,|\,F_T]^{(1-a)/a}\\
&\leqq E[Z_\infty/Z_T\,|\,F_T]\{1/(1-K_a\,||M||_{\mathrm{BMO}}^2)\}^{(1-a)/a}\,.
\end{aligned}$$

Therefore

$$Z_T\leqq E[Z_\infty\,|\,F_T]\{1/(1-K_a\,||M||_{\mathrm{BMO}}^2)\}^{(1-a)/a}\,,$$

from which it is seen that Z is uniformly integrable. Thus the proof is completed.

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