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ANALYTIC AUTOMORPHISMS AND ALGEBRAIC AUTOMORPHISMS OF C^2

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Introduction. All analytic automorphisms of the Gaussian plane C are given by linear polynomials. On the other hand, the two-dimensional complex Euclidean space C^2 has plentiful analytic automorphisms. An analytic automorphism given by two polynomials is called algebraic and an analytic automorphism given by two entire functions at least one of which is a transcendental entire function is called transcendental. The purpose of this paper is to present some conditions under which an analytic automorphism of C^2 is algebraic.

Recently, Nishino and Suzuki have given an interesting result on the cluster set at $z=0$ of an analytic mapping φ of the punctured disc $\Gamma': 0<|z|<1$ into a two-dimensional complex analytic manifold. By means of their result, we can show the following: an analytic automorphism of C^2 is algebraic if it maps an algebraic curve in C^2, which is not of exceptional type, onto an algebraic curve, and above-mentioned algebraic curves of exceptional type can be determined in the concrete.

This paper consists of four sections. In §1, we show that an analytic automorphism of C^2 is algebraic if it transforms a polynomial of general type into a polynomial. The polynomials of exceptional type and the transcendental automorphisms which transform a polynomial into a polynomial are determined in the concrete.

Consider an algebraic curve S in C^2 and the two-dimensional complex projective space P^2 containing C^2 as its affine part. In §2, we prove the fact that an analytic automorphism which transforms S into an algebraic curve is algebraic if the closure of S in P^2 intersects the line at infinity of P^2 at more than two distinct points. The above result is contained in the principal theorem which is proved in §4. However, we can prove this result without using the theory of cluster sets, and this together with the result in §1 was the starting point of this study.

By a method similar to that in the proof of the principal theorem in §4, we can give a condition under which the complement of an algebraic curve in P^2 allows transcendental automorphisms. This will be published elsewhere.

1. Automorphisms of C^2 and polynomials. An analytic automorphism T of the space C^2 with coordinates x, y is given by two entire functions $f(x, y)$ and $g(x, y)$ of x, y in the form $(x', y') = (f(x, y), g(x, y))$. We say that T is algebraic if both f and g are polynomials of x, y and that T is transcendental if g or f is a transcendental entire function. If T is algebraic, then T transforms an arbitrary polynomial $P(x, y)$ into a polynomial $T^*(P)(x, y) = P(f(x, y), g(x, y))$ and conversely. If the genus of the normalization $\hat{S}$ of an irreducible algebraic curve S in C^2 is p and if $\hat{S}$ has n boundary points, then S is called an algebraic curve of (topological) type (p, n). Consider the level curve S_c: $P = c$ for an arbitrary complex value c, where $P = P(x, y)$ is a polynomial of x, y. An irreducible component of S_c is called a prime surface of P. Types of prime surfaces of P are constant with the exception of a finite number of prime surfaces. If the type of general prime surfaces of P is (p, n), then we call the polynomial P a polynomial of type (p, n) and if, furthermore, $p \geqq 1$ or $n \geqq 3$, then we call P a polynomial of general type. The purpose of this section is to prove the fact that an automorphism of C^2 which transforms "one" polynomial of general type into a polynomial is algebraic and that we can determine the transcendental automorphisms which transform a polynomial into a polynomial.

A polynomial $P(x, y)$ for which the curves S_c: $P(x, y) = c$ are irreducible non-singular and of order one (i.e., $P - c$ has a zero of order one on S_c) with the exception of a finite number of constants c is called primitive. Owing to Stein [7], we know that, for an arbitrary polynomial $P(x, y)$, there corresponds a polynomial $P_0(x, y)$ such that every polynomial $Q(x, y)$ with the jacobian $J(P, Q) \equiv 0$ is decomposed in the form $Q(x, y) = \varphi(P_0(x, y))$, where $\varphi(z)$ is a polynomial of one complex variable z. This polynomial P_0 is primitive and is uniquely determined up to the multiplication and the addition of constants.

Suppose that a polynomial $P(x, y)$ of general type is transformed into a polynomial $Q = T^*(P)$ by an analytic automorphism T of C^2. Then we can assume without loss of generality that both P and Q are primitive polynomials of general type.

Now we consider an affine algebraic variety M_0: $z - Q(x, y) = 0$ in the space $C \times C^2$ of the direct product of the z-plane C and the xy-space C^2. The canonical projection $\tilde{\omega}_0$ of $C \times C^2$ onto the xy-space C^2 defines a birational biregular isomorphism between M_0 and C^2. For each value z', $S^0_{z'}$ denotes the section of M_0 cut out by the complex line $z = z'$. The section $S^0_{z'}$ corresponds biholomorphically to the curve $S_{z'}$: $Q = z'$.

Now we consider a two-dimensional analytic space V and a Riemann surface R. If there exists a proper analytic mapping π of V onto R and if a triple $\mathscr{F} = (V, \pi, R)$ satisfies the following two conditions, then $\mathscr{F}$ is called an analytic family of compact Riemann surfaces of genus p over R.

(1) For an arbitrary point q on R, the fibre $S_q = \pi^{-1}(q)$ is a compact connected one-dimensional analytic surface in V.

(2) For an arbitrary point q on R with the exception of the points in a discrete set on R, the fibre S_q is irreducible, non-singular and of order one and S_q is a Riemann surface of the same genus p. (We call a fibre satisfying this property a regular fibre and a fibre not having this property is called a critical fibre.)

We shall construct an analytic family of compact Riemann surfaces by the compactification of the algebraic variety M_0. Let $\boldsymbol{P}^2$ be the two-dimensional projective space containing $\boldsymbol{C}^2$ as its affine part and let $(x_0 : x_1 : x_2)$ be the homogeneous coordinate of the point with the inhomogeneous coordinate (x, y), i.e., $x = x_1/x_0$, $y = x_2/x_0$. Let $(z_0 : z_1)$ be the homogeneous coordinate of the point with the inhomogeneous coordinate z on the one-dimensional projective space $\boldsymbol{P} = \boldsymbol{C} \cup \{\infty\}$, i.e., $z = z_1/z_0$. Let us denote by m the degree of the given polynomial Q. Consider the projective algebraic variety M defined by the equation $x_0^m\{z_0 Q(x_1/x_0, x_2/x_0) - z_1\} = 0$ in the space $\boldsymbol{P} \times \boldsymbol{P}^2$. The canonical projection $\tilde{\omega}$ of $\boldsymbol{P} \times \boldsymbol{P}^2$ onto $\boldsymbol{P}^2$ defines a birational isomorphism between M and $\boldsymbol{P}^2$. For the canonical projection π of $\boldsymbol{P} \times \boldsymbol{P}^2$ onto $\boldsymbol{P}$ and for each point q on $\boldsymbol{P}$, $S_q = \pi^{-1}(q)$ is a compact, connected, one-dimensional algebraic curve and, if $q \neq \infty$, then $S_q \cap M_0 = S^0_{z'}$, where z' denotes the inhomogeneous coordinate of q. Noting that P is primitive and, if necessary, by applying a suitable birational transformation to M, we see that the triple $\mathscr{F} = (M, \pi, \boldsymbol{P})$ is an analytic family of compact Riemann surfaces over $\boldsymbol{P}$.

Assume that the automorphism T is given by two entire functions $f(x, y)$ and $g(x, y)$. If the holomorphic function $\tilde{\omega}_0^*(f) = f \circ \tilde{\omega}_0$ on M_0 is analytically continued to a rational function on M, then $f(x, y)$ is a polynomial of x, y. If the closure $\bar{s}_c$ in M of s_c: $\tilde{\omega}_0^*(f) = c$ is an algebraic curve on M for at least two complex values c, then a theorem of Thullen [9] implies that $\tilde{\omega}_0^*(f)$ can be analytically continued to a rational function on M (and, therefore, f is a polynomial).

Here we state a theorem due to Nishino on the analytic continuation of an analytic section of a fibre space.

We consider a two-dimensional analytic space V and an analytic projection π of V onto a Riemann surface R. Suppose that a triple $\mathscr{F} = (V, \pi, R)$ is an analytic family of compact Riemann surfaces over R. Let D be a subdomain of R. A finitely multivalent analytic mapping η of D into V satisfying $\pi \circ \eta = \text{id.}$ is called a finitely multivalent analytic section of $\mathscr{F}$ over D.

THEOREM N. (Nishino [3]). *Consider an analytic family* $\mathscr{F} = (V, \pi, \Gamma)$ *of compact Riemann surfaces of genus* p *over the disc* $\Gamma: |z| < r$ *which has no critical fibre except at* $z = 0$. *An unramified finitely multivalent analytic section* η *of* $\mathscr{F}$ *over the punctured disc* $\Gamma': 0 < |z| < r$ *is analytically continued to a finitely multivalent analytic section over* Γ *if* $\mathscr{F}$ *and* η *satisfy one of the following three conditions*:

(1) $p \geqq 2$,

(2) $p = 1$ *and there exists a finitely multivalent analytic section* ξ *over* Γ, *which is unramified on* Γ' *and satisfies* $\xi(z') \neq \eta(z')$ *for all branches of* η *and* ξ *at every* z' *in* Γ',

(3) $p = 0$ *and there exists a finitely multivalent analytic section* ξ *over* Γ, *which is unramified on* Γ' *and has branches* ξ_j, $j = 1, 2, \cdots, \nu(\geqq 3)$ *over* Γ', *satisfying* $\xi_j(z') \neq \xi_k(z')$ $(j \neq k)$ *and* $\eta(z') \neq \xi_j(z')$ *for all branches of* η *at every* z' *in* Γ'.

Let us continue our discussion. For a fixed complex value c, we consider the surface $s_c: f = c$ which is the inverse image of the analytic plane $x' = c$ by the mapping T and consider the coordinate τ $(|\tau| < +\infty)$ on the analytic plane $x' = c$. We denote by $\Phi(\tau) = (x(\tau), y(\tau))$ an analytic imbedding of the τ-plane into $\boldsymbol{C}^2$ associated with the inverse image s_c. The restriction $P(c, y')$ of $P(x', y')$ to the analytic plane $x' = c$ gives a polynomial of τ and the equation $P(c, y') - z = 0$ for every complex value z has exactly $\tilde{m}$ roots by counting multiplicities, where $\tilde{m}$ denotes the degree of the polynomial $P(c, y')$ with respect to y'. We consider the Riemann surface R_c of the inverse function of the holomorphic function $T^*(P)(\Phi(\tau)) = Q(x(\tau), y(\tau))$ of τ. Then R_c is a ramified $\tilde{m}$-sheeted covering of the z-plane with no relative boundary and with a finite number of ramification points. This means that $s_c^*: \tilde{\omega}_0^*(f) = c$ defines an $\tilde{m}$-valent analytic section η_c of the analytic family $(M - \pi^{-1}(\infty), \pi, \boldsymbol{C})$ over the z-plane $\boldsymbol{C}$ and the Riemann surface defined by η_c has no relative boundary and has only a finite number of ramification points. An algebraic curve Σ, which is the closure of $M - (\pi^{-1}(\infty) \cup M_0)$, defines a finitely multivalent analytic section ξ over the z-sphere $\boldsymbol{P}$ with no relative boundary and with a finite number of ramification points. Hence there exists a sufficiently large positive number ρ such that the analytic family

$\mathscr{F} = (\pi^{-1}(\Gamma), \pi, \Gamma)$ of compact Riemann surfaces over the disc $\Gamma: |z_0/z_1| < \rho^{-1}$ has no critical fibre except at $z_0 = 0$ and multivalent analytic sections ξ and η_c have no ramification point on the punctured disc $\Gamma': 0 < |z_0/z_1| < \rho^{-1}$. Moreover, one of the three conditions (1), (2) and (3) in Theorem N holds, as the polynomial Q is a polynomial of general type. Hence, by Theorem N, the closure $\bar{s}_c^*$ of s_c^* in M is an algebraic curve on M. Since c is an arbitrary complex value except for a finite number of values, f is a polynomial. By the same reasoning, we see that g is also a polynomial. Thus the following theorem has been proved.

THEOREM 1. *An analytic automorphism of C^2 is algebraic if it transforms a polynomial of general type into a polynomial.*

Owing to Wakabayashi, we know that a primitive polynomial of type (0, 1) is transformed into the monomial x by an algebraic automorphism of C^2 and owing to Saitô [6], we also know that a primitive polynomial of type (0, 2) is transformed into the monomial $x^m y^n$ or into the polynomial $x^m(x^l y + P_{l-1}(x))^n$ by an algebraic automorphism of C^2, where l, m and n are positive integers and $P_{l-1}(x)$ is a polynomial of degree at most $l-1$ satisfying $P_{l-1}(0) \neq 0$ (see [6], [8]).

Among the automorphisms of C^2, the one which keeps the monomial x invariant is of the following form:

$$\text{(I)} \qquad x' = x\,, \qquad y' = (\exp(\varphi(x)))\cdot y + \psi(x)\,,$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary entire functions of x.

Let m, n, m' and n' be positive integers. If there exists an analytic automorphism which transforms the monomial $x^{m'}y^{n'}$ into the monomial $x^m y^n$, then $m = m'$ and $n = n'$, or, $m = n'$ and $n = m'$. Any analytic automorphism of C^2 which keeps the monomial $x^m y^n$ invariant is given as follows: If $m \neq n$, then

$$\text{(II)} \qquad x' = (\exp(n\varphi(x^m y^n)))\cdot x\,, \qquad y' = C(\exp(-m\varphi(x^m y^n)))\cdot y\,,$$

where $\varphi(z)$ is an arbitrary entire function of z and C is a constant with $C^n = 1$. If $m = n$, then an automorphism of C^2

$$(\text{II}_0) \qquad x' = y\,, \qquad y' = x$$

keeps the monomial $x^m y^m$ invariant and all the automorphisms of C^2 which keep $x^m y^m$ invariant are generated by the automorphisms of the types (II) and (II_0).

For positive integers l, m, n, l', m', n' and a polynomial $P_{l-1}(x)$ of x of degree at most $l-1$ with $P_{l-1}(0) \neq 0$ and for a polynomial $P_{l'-1}(x)$ of x of degree at most $l'-1$ with $P_{l'-1}(0) \neq 0$, if there exists an analytic automorphism of C^2 which transforms $x^{m'}(x^{l'}y + P_{l'-1}(x))^{n'}$ into $x^m(x^ly + P_{l-1}(x))^n$, then $l = l'$, $m = m'$, $n = n'$ and such an analytic automorphism of C^2 is given by

$$\text{(III)}\qquad \begin{aligned} x' &= (\exp(n\varphi(*)))\cdot x\,, \\ y' &= \{C(\exp(-m\varphi(*)))(x^ly + P_{l-1}(x)) \\ &\quad - P_{l-1}((\exp(n\varphi(*)))\cdot x)\}/((\exp(n\varphi(*)))\cdot x)^l\,, \end{aligned}$$

where $* = x^m(x^ly + P_{l-1}(x))^n$, $\varphi(z)$ is an entire function of z and C is a constant with $C^n = 1$. Here C and φ must satisfy a condition under which the mappings of C^2 given by (III) is biholomorphic, for example, $C = 1$ and $\varphi(z) = z$.

The above facts imply the following theorem as a corollary to Theorem 1.

THEOREM 2. *For a transcendental automorphism T of C^2 which transforms a polynomial into a polynomial, there exists a transcendental automorphism $\tilde{T}$ of C^2 of one of the types* (I), (II) *and* (III) *and there exist two algebraic automorphisms T_1 and T_2 of C^2 with the property $T = T_1 \circ \tilde{T} \circ T_2$.*

REMARK. Let $P(x, y)$ be a polynomial of general type. Assume that an analytic automorphism T transforms two curves $P = 0$ and $P = 1$ into algebraic curves. A theorem of Thullen implies that the automorphism T transforms the polynomial P into a polynomial. By Theorem 1, T is algebraic. So there arises the question: Is an analytic automorphism of C^2, which transforms the algebraic curve $P = 0$ into an algebraic curve, algebraic? We will answer this question in §4.

2. Automorphisms of C^2 and algebraic curves in C^2 (Examples). If an algebraic automorphism transforms a polynomial $P(x, y)$ into a polynomial $Q(x, y)$, then the curve $P = c$ for a constant c is transformed into a curve $Q = c$ by this automorphism. Therefore, the examples of transcendental automorphisms (I), (II) and (III) in §1 give us also examples of transcendental automorphisms which transform an algebraic curve into an algebraic curve.

If an algebraic curve is "general", then an analytic automorphism, which transforms this algebraic curve into an algebraic curve, is algebraic. This is the principal theorem of this paper. (See Theorem 3 in §4.) As was seen in the preceeding section, a sum of prime surfaces of a polynomial of type (0, 1) or (0, 2) is "special". Another example of a "special" algebraic curve is as follows: Let $\varphi(x)$ and $\psi(x)$ be two arbitrary polynomials of x. A polynomial $\varphi(x) + \psi(x)y$ is a polynomial of type $(0, n)$, where n is a positive integer. The curve S_0 defined by the equation $\varphi(x) + \psi(x)y = 0$ is invariant under the automorphism

$$\text{(IV)} \qquad x' = x\,, \qquad y' = (\exp(\psi(x))) \cdot y + \{\varphi(x) \cdot ((\exp(\psi(x))) - 1)/\psi(x)\}$$

of C^2. Therefore, the image curve of S_0 by an algebraic automorphism of C^2 is such a "special" curve. In §4 we show that any special curve in this sense belongs to one of the three examples given above. In this section we prove a proposition weaker than the principal theorem.

We consider an algebraic curve S in C^2 and the two-dimensional projective space P^2 containing C^2 as its affine part and we suppose that the closure of S in P^2 intersects the complex line at infinity at more than two distinct points. We assert that an analytic automorphism of C^2 is algebraic if it transforms S into an algebraic curve in C^2. To prove this fact, we make use of the theory of covering surfaces due to Ahlfors.

Suppose that the algebraic curve S is transformed into an algebraic curve S' in C^2 by a transcendental automorphism T: $(x', y') = (f(x, y), g(x, y))$ of C^2. By the assumption we may suppose that f is a transcendental entire function. Because of a theorem of Thullen, for any complex value c with the exception of at most only one complex value, the curve σ_c: $f = c$ is not an algebraic curve. Henceforth, we suppose that σ_c is not algebraic. The inverse image of the analytic plane $x' = c$ by the automorphism T is σ_c which gives an analytic imbedding $\varphi: C \to C^2$ of the w-plane C into C^2.

In the homogeneous coordinate $(x_0 : x_1 : x_2)$ of P^2, the line L_∞ at infinity is defined by $x_0 = 0$ and the inhomogeneous coordinate (x, y) of C^2 is given by $x = x_1/x_0$, $y = x_2/x_0$. We consider the analytic projection $\pi: (x, y) \to (0 : x : y)$ of $C^2 - (0, 0)$ onto L_∞. Then $\pi \circ \varphi: C \to L_\infty$ defines a transcendental meromorphic function. We denote the common points of the closures of S and L_∞ by $p_1, p_2, \cdots, p_m$. By virtue of the hypothesis, we see $m \geqq 3$. We may suppose $p_i \neq (0 : 0 : 1)$, $i = 1, 2, \cdots, m$, so we consider the inhomogeneous coordinate $z = x_2/x_1$ of the points p_i to be z_i for $i = 1, 2, \cdots, m$. There exists a positive number λ_1 such that m discs Γ_i: $|z - z_i| \leqq \lambda_1$ do not intersect each other. Consider the Riemann surface R of the inverse function of $\pi \circ \varphi$ and let R_0 be the subregion of R lying over $D = L_\infty - \bigcup_{i=1}^m \Gamma_i$. Then R_0 is conformally equivalent to a subregion of the w-plane bounded by real analytic curves. For an arbitrary positive number r, we denote by $R(r)$ the subregion of R which conformally corresponds to the subdomain $|w| < r$ of the w-plane. The relative boundary of $R_0(r) = R_0 \cap R(r)$ over D is contained in $\partial R(r)$. Denote by $A(r)$ the mean number of sheets of $R_0(r)$ over D and denote by $L(r)$ the length of the relative boundary C_r of $R_0(r)$ over D. By the inequality of Schwarz, we have

$$(L(r))^2 = \left(\int_{C_r} \frac{|(\pi\circ\varphi)'(w)|}{1+|\pi\circ\varphi(w)|^2}|dw|\right)^2$$
$$\leqq 2\pi r\int_{C_r} \frac{|(\pi\circ\varphi)'(w)|^2}{(1+|\pi\circ\varphi(w)|^2)^2}|dw| = 2\pi r\frac{dA(r)}{dr}\ .$$

Hence

$$\log\frac{r}{r_0} \leqq 2\pi\int_{r_0}^{r}\frac{dA(r)}{(L(r))^2} \quad \text{and} \quad \int^{\infty}\frac{dA(r)}{(L(r))^2} \to +\infty \quad (r\to+\infty)\ .$$

Therefore it is not possible that, for a positive constant K, $A(r) \leqq KL(r)$ for all r. Hence there exists a sequence $\{r_k\}_{k=1}^{\infty}$ tending monotonously to $+\infty$ such that $L_k/A_k \to 0$ $(k\to+\infty)$, where $L_k = L(r_k)$ and $A_k = A(r_k)$. The theory of covering surfaces due to Ahlfors shows that $\rho_k \geqq (m-2)A_k - hL_k$, where h is a positive constant independent of k and ρ_k is the Euler characteristic of $R_0(r_k)$.

Now we state a lemma.

LEMMA 1. *Let γ be the bicylinder $|x|\leqq 1$, $|y|\leqq 1$ in C^2. Assume that two analytic surfaces σ_1, σ_2 of dimension one defined in a neighborhood of γ satisfy the following conditions*:

$$\sigma_1\cap\{|x|<1,\ |y|=1\}\neq\varnothing\ ,\qquad \sigma_1\cap\{|x|=1\}=\varnothing\ ,$$
$$\sigma_2\cap\{|x|=1,\ |y|<1\}\neq\varnothing\ ,\qquad \sigma_2\cap\{|y|=1\}=\varnothing\ .$$

Then σ_1 and σ_2 have at least one common point in γ.

The proof of this lemma follows directly from the argument principle and may be omitted.

Now there exists a positive number λ_2 such that S does not intersect the set $\{(x, y)\in C^2\colon |(y/x)-z_i| = \lambda_1,\ |1/x|\leqq\lambda_2\}$. Since the analytic plane $x' = c$ intersects the algebraic curve S' at a finite number of points, σ_c intersects S at a finite number of points. By Lemma 1, we see that $R(r)$ has only a finite number of islands (Inseln: subregions of $(\pi\circ\varphi)^{-1}(\Gamma_i)$ with no relative boundary over the disc Γ_i) over the discs Γ_i. Hence the Euler characteristics ρ_k of $R_0(r_k)$ are bounded.

Since $\pi\circ\varphi$ is a transcendental meromorphic function, we see $A_k\to+\infty$ $(k\to+\infty)$. Therefore $0 = \lim\rho_k/A_k \geqq m-2$, which contradicts the condition $m\geqq 3$. Thus we have the following.

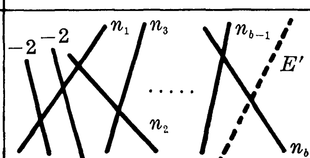
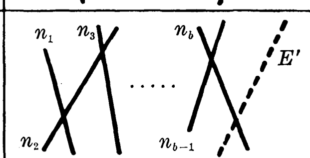
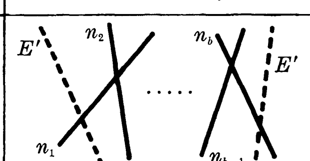
PROPOSITION. *Let S be an algebraic curve in C^2 whose closure in P^2 containing C^2 as its affine part intersects the complex line at infinity* *at more than two distinct points. Then an analytic automorphism of* C^2 *is algebraic, if it transforms* S *into an algebraic curve in* C^2.

REMARK. Wakabayashi [10] proved a result stronger in some sence than this proposition which is also a corollary to Theorem 3 in §4.

3. Preparation from the theory of cluster sets.

THEOREM S. ([4]) *Let* V *be a two-dimensional complex manifold and let* E *be a connected curve (i.e., a connected one-dimensional analytic subset) on* V *satisfying the following conditions* (i), (ii) : (i) *any singular point of* E *is an ordinary double point and* (ii) E *has no non-singular rational (compact) component* E_i *with the self-intersection number* $(E_i^2) = -1$ *and having at most two intersection points with the other components of* E. *Suppose that there is a holomorphic mapping* $\varphi\colon \Gamma' \to V - E$ *of the punctured disc* $\Gamma'\colon 0 < |z| < 1$ *into* $V - E$ *such that* $E \supset \mathscr{S}_\varphi = \bigcap_{r>0} \overline{\varphi(\Gamma'_r)}$, *where* $\Gamma'_r = \{0 < |z| < r\}$ *and* $\overline{\varphi(\Gamma'_r)}$ *is the closure of* $\varphi(\Gamma'_r)$ *in* V. *The set* $\mathscr{S}_\varphi$ *is called the cluster set of* φ *in* V *at* $z = 0$. *Assume that* $\mathscr{S}_\varphi$ *is a compact set on* V *containing at least two points. If* $\mathscr{S}_\varphi \neq E$, *then* $\mathscr{S}_\varphi$ *must belong to one of the classes of curves listed in the following table.*

TABLE

<table>
<tr><th>Name of type</th><th>Number of points of $\mathscr{S}_\varphi \cap E'$</th><th colspan="2">Explanation of $\mathscr{S}_\varphi$ and E'</th></tr>
<tr><td>$\gamma'(n_1, n_2, \cdots, n_b)$</td><td>1</td><td>
</td><td>$\max\{n_1 + 1, n_2, \cdots, n_b\} \geqq 0$</td></tr>
<tr><td rowspan="2">$\varepsilon(n_1, n_2, \cdots, n_b)$</td><td>1</td><td>
</td><td rowspan="2">$\max\{n_1, n_2, \cdots, n_b\} \geqq 0$</td></tr>
<tr><td>2</td><td>
</td></tr>
</table>

Here each line represents a component C_i of $\mathscr{S}_\varphi$ and the number n_i attached to each line represents the self-intersection number (C_i^2). Every component of $\mathscr{S}_\varphi$ is non-singular and rational. E' represents the analytic curve consisting of the components of E which do not belong to $\mathscr{S}_\varphi$, that is, $E' = \overline{(E - \mathscr{S}_\varphi)}$.

REMARK. Nishino and Suzuki [4] also classified $\mathscr{S}_\varphi$ in the case $\mathscr{S}_\varphi = E$ and gave a classification table. As we do not use the table in this paper, we do not mention it here.

4. Automorphisms of C^2 and algebraic curves in C^2. Before proving the principal theorem we state the following lemma.

LEMMA 2. *Let M be a two-dimensional projective algebraic complete manifold and let C_0 be an irreducible non-singular rational curve on M with the self-intersection number $(C_0^2) = 0$. Then there exists a holomorphic mapping π of M onto a compact Riemann surface R and the triple $\mathscr{F} = (M, \pi, R)$ is an analytic family of compact Riemann surfaces of genus zero over R which has C_0 as its regular fibre.*

The proof follows by the result of Kodaira and Spencer [1], since $H^0(C_0, \mathscr{O}) = \boldsymbol{C}$, $H^1(C_0, \mathscr{O}) = 0$ and since the normal bundle of C_0 is analytically trivial.

REMARKS. In the same situation as in Lemma 2, if M is rational then $R = \boldsymbol{P}^1$ and a curve C_1 with $C_0 \cap C_1 = \varnothing$ is a sum of irreducible components of fibres of $\mathscr{F}$. A curve C_2 which intersects C_0 transversally at only one point p_0 (i.e., p_0 is an ordinary double point of $C_0 \cup C_2$ and $C_0 \cap C_1 = \{\text{one point}\}$.) intersects each fibre of $\mathscr{F}$ at only one point. The first fact follows, since M is simply connected and the latter two facts are easily obtained from a theorem of Hurwitz.

Suppose that an algebraic curve S in $\boldsymbol{C}^2$ is transformed into an algebraic curve S' in $\boldsymbol{C}^2$ by a transcendental automorphism $T: (x', y') = (f(x, y), g(x, y))$ of $\boldsymbol{C}^2$. We may assume that $f(x, y)$ is a transcendental entire function of x, y. On account of a theorem of Thullen, there exists a constant c for which the curve $s_c: f = c$ is not algebraic. Since s_c is the inverse image of the analytic plane $x' = c$ by the automorphism T, s_c is an analytic imbedding $\psi: \boldsymbol{C} \to \boldsymbol{C}^2$ of the τ-plane $\boldsymbol{C} = \{|\tau| < +\infty\}$ into the xy-space $\boldsymbol{C}^2$. We have the two-dimensional projective space $\boldsymbol{P}^2$ by adding the complex line L_∞ at infinity to $\boldsymbol{C}^2$. The cluster set $\mathscr{S}_\psi = \bigcap_{r<+\infty} \overline{\psi(\{r < |\tau|\})}$ of ψ at $\tau = \infty$ in $\boldsymbol{P}^2$ is L_∞.

Now consider the closure $\bar{S}$ of S in $\boldsymbol{P}^2$. A singular point of $\bar{S} \cup L_\infty$ is called a non-normal crossing singular point of $\bar{S} \cup L_\infty$, if it is not an ordinary double point of $\bar{S} \cup L_\infty$. After the resolution of non-normal crossing singularities among the singular points of $\bar{S} \cup L_\infty$ on L_∞ by a finite sequence of σ-processes, we obtain a compact rational manifold M satisfying the following three conditions (i), (ii), (iii): (i) There is a birational isomorphism $\sigma: \boldsymbol{P}^2 \to M$ whose restriction to $\boldsymbol{C}^2$ gives a biregular isomorphism between $\boldsymbol{C}^2$ and a quasiprojective manifold $M_0 = \sigma(\boldsymbol{C}^2)$. (ii) Let Θ be the closure of $\sigma(S)$ in M. Any common point of Θ and $\Sigma = M - M_0$ is an ordinary double point of $E = \Theta \cup \Sigma$. (iii) Σ has no irreducible component C with the self-intersection number $(C^2) = -1$ and having at most two intersection points with the other components of E.

The mapping $\varphi = \sigma \circ \psi$ of the τ-plane $\boldsymbol{C}$ into M_0 is holomorphic. Since the analytic complex line $x' = c$ intersects S' at a finite number of points in the $x'y'$-space $\boldsymbol{C}^2$, $\psi(\boldsymbol{C})$ intersects S at a finite number of points and for a sufficiently large positive number r_0, $\psi(\{r_0 < |\tau|\}) \cap S = \varnothing$. Hence the restriction of $\varphi: \boldsymbol{C} \to M$ to $\{r_0 < |\tau|\}$ gives a holomorphic mapping of $\{r_0 < |\tau|\}$ into $M - E$. The cluster set $\mathscr{S}_\varphi = \bigcap_{r<+\infty} \overline{\varphi(\{r < |\tau|\})}$ of φ at $\tau = \infty$ in M consists of several components of Σ. Now consider a neighborhood V of the algebraic curve Σ. Since $\mathscr{S}_\varphi \Subset V$ and $\mathscr{S}_\varphi \neq E$, Theorem S implies that $\mathscr{S}_\varphi$ and $E' = \overline{E - \mathscr{S}_\varphi}$ belong to one of the classes listed in Table in §3.

(1) The case of γ'.

By a compactification of a two-dimensional complex manifold W we mean a two-dimensional compact complex manifold N together with a compact analytic curve C on N such that $N - C$ is holomorphically equivalent to W. A compactification (N, C) of W is called a minimal normal compactification if it satisfies the following two conditions: (i) Any singular point of C is an ordinary double point. (ii) C has no nonsingular rational component C_i with the self-intersection number $(C_i^2) = -1$ having at most two intersection points with the other components of C.

Owing to Ramanujam, we know that the curve C of a minimal normal compactification (N, C) of $\boldsymbol{C}^2$ is a linear tree of rational curves. Since M is a compactification of $\boldsymbol{C}^2$ and $\mathscr{S}_\varphi$ is not exceptional, the graph of $\mathscr{S}_\varphi$ must be linear and it is a contradiction. Therefore, this case does not take place. (See Ramanujam [5], Morrow [2].)

(2) The case of ε.

There exists at least one irreducible component C of $\mathscr{S}_\varphi$ with the self-intersection number $(C^2) \geqq 0$. Assume that $(C^2) = 0$. By Lemma 2, there exists a rational function h with no indetermination point such that C is the curve of poles of h. Since there is no compact curve in $\boldsymbol{C}^2$, C must intersect another component E_1 of Σ. By virtue of Table in §3, we see that C must intersect at most two components of E.

(i) If C intersects only E_1, then $\Theta \cap C = \varnothing$ and the remark to Lemma 2 shows that Θ is a sum of irreducible components of fibres of $\mathscr{F}$, that is, the level curves of the rational function h on M, and that every fibre of $\mathscr{F}$ intersects E_1 at only one point. Since $\sigma|_{C^2}$ is birational and biregular, S consists of several prime surfaces of the polynomial $h \circ \sigma$ of x, y of type $(0, 1)$. Owing to Wakabayashi's result, we know that $h \circ \sigma$ is transformed into the monomial x by an algebraic automorphism of $\boldsymbol{C}^2$. Therefore, S is transformed into a surface $\prod_{i=1}^{k}(x - a_i) = 0$ by an algebraic automorphism of $\boldsymbol{C}^2$ for some constants a_i $(i=1, 2, \cdots, k)$ and a positive integer k.

(ii) Next suppose that C intersects E_1 and E_2 other than C. If E_2 is a component of Σ, then $\Theta \cap C = \varnothing$ and the same reasoning as in (i) shows that S consists of several prime surfaces of the polynomial $h \circ \sigma$ of x, y of type $(0, 2)$. Owing to Saitô's result, S is transformed into a sum of several prime surfaces of a monomial $x^m y^n$ or of a polynomial $x^m(x^l y + P_{l-1}(x))^n$ by an algebraic automorphism of $\boldsymbol{C}^2$, where l, m and n are positive integers and $P_{l-1}(x)$ is a polynomial of x with $P_{l-1}(0) \neq 0$. (See [6], [8].)

If E_2 is a component of Θ, then $\sigma^{-1}(E_2 \cap M_0)$ is a component of S and $h \circ \sigma$ is a polynomial of type $(0, 1)$. Since E_2 intersects C at only one point which is an ordinary double point of E, the remark to Lemma 2 implies that each fibre of $\mathscr{F}$ intersects E_2 at only one point. Therefore $\sigma^{-1}(E_2 \cap M_0)$ is transformed into an algebraic curve $y - R(x) = 0$ by the algebraic automorphism of $\boldsymbol{C}^2$ which transforms the polynomial $h \circ \sigma$ into the monomial x, where $R(x)$ is a rational function of x. If S is irreducible, then S is clearly $\sigma^{-1}(E_2 \cap M_0)$. If S is reducible, then by the same discussion as in (i) we see that S is transformed into an algebraic curve $(y - R(x)) \cdot \prod_{i=1}^{k}(x - a_i) = 0$ by an algebraic automorphism of $\boldsymbol{C}^2$. Thus S is transformed into an algebraic curve $\varphi(x) + \psi(x)y = 0$ by an algebraic automorphism, where φ and ψ are polynomials of x.

If $(C^2) > 0$, after a suitable sequence of σ-processes, the algebraic image C' of C intersects at most two other components of the total image of E and the self-intersection number $(C'^2) = 0$. Hence, we can determine S as in the case $(C^2) = 0$. Thus we have the following.

THEOREM 3. *If an algebraic curve S in $\boldsymbol{C}^2$ is transformed into an algebraic curve by a transcendental automorphism of $\boldsymbol{C}^2$, then an algebraic automorphism of $\boldsymbol{C}^2$ transforms S into an algebraic curve S_0 with one of the following forms*: (i) *An algebraic curve $\varphi(x) + \psi(x)y = 0$, where $\varphi(x)$ and $\psi(x)$ are polynomials of x.* (ii) *A sum of several prime surfaces of a monomial $x^m y^n$, where m and n are positive integers.*

(iii) *A sum of several prime surfaces of a polynomial* $x^m(x^l y + P_{l-1}(x))^n$, *where* l, m *and* n *are positive integers and* $P_{l-1}(x)$ *is a polynomial of degree at most* $l-1$ *of* x *with* $P_{l-1}(0) \neq 0$.

For each curve belonging to one of the classes (i), (ii) and (iii), we have examples of transcendental automorphisms (I), (II), (III) in §1 or (IV) in §2 of C^2 under which the curve is invariant.

As is easily seen, the proposition in §2 is a corollary to this theorem.

REFERENCES

[1] K. KODAIRA AND D. C. SPENCER, A theorem of completeness of characteristic systems of complete continuous systems, Amer. J. Math. 81 (1959), 477-500.

[2] J. A. MORROW, Compactification of C^2, Bull. Amer. Math. Soc. 78 (1972), 813-816. See also Rice Univ. Studies 59 (1973), 97-112.

[3] T. NISHINO, Nouvelles recherches sur les fonctions entières de plusieurs variables complexes (I), J. Math. Kyoto Univ. 8 (1968), 49-100, (II) Fonctions entières qui se réduisent à celles d'une variable, J. Math. Kyoto Univ. 9 (1969), 221-274, (V) Fonctions qui se réduisent aux polynômes, J. Math. Kyoto Univ. 15 (1975), 527-553.

[4] T. NISHINO ET M. SUZUKI, Sur les singularités isolées des applications holomorphes à valeurs dans une surface complexe, to appear in Publ. Res. Inst. Math. Sci.

[5] C. P. RAMANUJAM, A topological characterization of the affine plane as an algebraic variety, Ann. of Math. (2) 94 (1971), 69-88.

[6] H. SAITÔ, Fonctions entières qui se réduisent à certains polynômes (I), Osaka J. Math. 9 (1972), 293-332.

[7] K. STEIN, Analytische Projection komplexer Mannigfaltigkeiten, Colloque Bruxell 1953, 87-107.

[8] M. SUZUKI, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace C^2, J. Math. Soc. Japan 26 (1974), 241-257.

[9] P. THULLEN, Uber die wesentlichen Singlaritäten analytischer Funktionen und Flächen in Raume von n komplexen veränderlichen, Math. Ann. 111 (1934), 137-156.

[10] I. WAKABAYASHI, On the logarithmic Kodaira dimension of the complement of a curve in P^2, Proc. Japan Acad. (A) 54 (1978), 157-162.

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