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THE MATSUMOTO TRIPLING FOR COMPACT SIMPLY CONNECTED 4-MANIFOLDS

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1. Intoduction. Let W be an oriented simply connected 4-manifold and let x_1, x_2, x_3 be three elements of $H_2(W; \boldsymbol{Z})$ with mutual intersection numbers $x_i \cdot x_j = 0$ $(i \neq j)$. In [3], by analysing Whitney's tricks for intersections of immersed 2-spheres representing x_1, x_2, x_3 in W, Y. Matsumoto introduced a number $\langle x_1, x_2, x_3 \rangle$ as an element of $\boldsymbol{Z}$ modulo an ideal $I = I\{x_1, x_2, x_3\} = \{x_1 \cdot u_1 + x_2 \cdot u_2 + x_3 \cdot u_3 \mid u_1, u_2, u_3 \in H_2(W; \boldsymbol{Z})\}$. The tripling $\langle,,\rangle$ will be referred to as the Matsumoto tripling and the ideal I will be called the intersection ideal of $\{x_1, x_2, x_3\}$.

It has been shown that x_1, x_2, x_3 can be realized by mutually disjoint immersed 2-spheres if and only if $\langle x_1, x_2, x_3 \rangle = 0$, (for the "only if" part see [3] and for the "if" part see [7]). If W is closed, $\langle x_1, x_2, x_3 \rangle$ always vanishes because of the Poincaré duality.

Suppose that the boundary $M = \partial W$ of W is non-empty. For an integer d, a homology class $x \in H_2(W; \boldsymbol{Z})$ has a mod d *boundary reduction* $y \in H_2(M; \boldsymbol{Z}_d)$, if $i_* y = x \bmod d$ for the inclusion map i: $M \to W$.

Our aim in this paper is to prove the following;

THEOREM. *Let (W, M) be a compact oriented simply connected 4-manifold with non-empty boundary $\partial W = M$. Suppose that we are given three elements $x_1, x_2, x_3 \in H_2(W; \boldsymbol{Z})$ with mutual intersection numbers zero and with the intersection ideal $I = (d)$, $d \in \boldsymbol{Z}$. Then each element x_i has a unique* mod d *boundary reduction y_i, $i = 1, 2, 3$ and the following equality holds;*

$$\langle x_1, x_2, x_3 \rangle = -(y_1{}^* \cup y_2{}^* \cup y_3^*) \cap [M] \qquad \text{in } \boldsymbol{Z}_d ,$$

where $y_i^* \in H^1(M; \boldsymbol{Z}_d)$ is the Poincaré dual of y_i in M.

Thus the Matsumoto tripling $\langle,,\rangle$ is completely determined by the multiple cup product of the mod d boundary reductions in the boundary.

An implication of Theorem is

COROLLARY 1 (Invariance of Matsumoto tripling). *Let (W, M) (W', M') and $x_1, x_2, x_3 \in H_2(W; \boldsymbol{Z})$ be 4-manifolds with boundary and* *homology classes as in Theorem. If* $f\colon (W, M)\to(W', M')$ *is a map such that the restriction* $g = f|_M\colon M\to M'$ *is of degree one, i.e.,* $g_*[M]=[M']$, *then we have* $I\{x_1, x_2, x_3\}\supset I\{f_*x_1, f_*x_2, f_*x_3\}$ *and*

$$\langle f_*x_1, f_*x_2, f_*x_3\rangle' \equiv \langle x_1, x_2, x_3\rangle \qquad in\ \boldsymbol{Z}/I\{x_1, x_2, x_3\}\ ,$$

where $\langle f_*x_1, f_*x_2, f_*x_3\rangle'$ *is the reduction of* $\langle f_*x_1, f_*x_2, f_*x_3\rangle$ *in* $\boldsymbol{Z}/I\{x_1, x_2, x_3\}$.

The proof of Theorem will be divided into two cases; $I = (0)$ (§ 2) and $I \neq (0)$ (§ 3). In § 4, we shall give some applications of Theorem as well as the proof of Corollary 1.

2. The proof of Theorem; part 1. Since the homomorphism $j_*\colon H_2(W; \boldsymbol{Z})\to H_2(W, M; \boldsymbol{Z})$ induced by the inclusion map $j\colon W\to(W, M)$ is represented by the intersection matrix for $H_2(W)$, it follows that a homology class $x \in H_2(W; \boldsymbol{Z})$ has a mod d boundary reduction if and only if the ideal (d) contains the intersection ideal $I\{x\} = \{x\cdot u \mid u \in H_2(W; \boldsymbol{Z})\}$. Hence for an integer d, each $x_i \in H^2(W; \boldsymbol{Z})$, $i = 1, 2, 3$, has a mod d boundary reduction y_i if and only if (d) contains the intersection ideal $I = I\{x_1, x_2, x_3\}$. Since W is simply connected, we have a short exact sequence:

$$H_3(W, M; \boldsymbol{Z}_d) = 0 \longrightarrow H_2(M; \boldsymbol{Z}_d) \xrightarrow{i_*} H_2(W; \boldsymbol{Z}_d) \xrightarrow{j_*} H_2(W, M; \boldsymbol{Z}_d)\ .$$

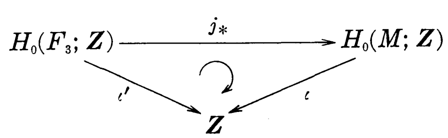
Thus the reduction y_i is unique. In particular, if $I = (0)$, i.e., $j_*x = 0$, then each x_i has a unique integral reduction y_i.

In this section, we shall prove Theorem in this special case $I = (0)$.

Represent y_1, y_2, y_3 by smoothly embedded oriented surfaces $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. We may assume that those surfaces are in general position. Let $j\colon \tilde{F}_3\to M$ be the inclusion map. Then

$$\begin{aligned}(y_1^*\cup y_2^*\cup y_3^*)[M] &= \iota((y_1^*\cup y_2^*\cup y_3^*)\cap[M]) = \iota'((j^*y_1^*\cup j^*y_2^*)\cap[\tilde{F}_3])\\ &= [\gamma_1]\cdot[\gamma_2]\end{aligned}$$

where ι, ι' are augmentations, $\gamma_i = \tilde{F}_i\cap\tilde{F}_3$, $i = 1, 2$, $[\gamma_i]\in H_1(\tilde{F}_3; \boldsymbol{Z})$ is the homology class represented by γ_i (see the diagram below).



Let

$$\tilde{C}_1 = \{\tilde{\alpha}_1', \tilde{\beta}_1'; \cdots ; \tilde{\alpha}_r', \tilde{\beta}_r'\}$$
$$\tilde{C}_2 = \{\tilde{\alpha}_1'', \tilde{\beta}_1''; \cdots ; \tilde{\alpha}_s'', \tilde{\beta}_s''\}$$

$$\widetilde{C}_3 = \{\widetilde{\alpha}_1, \widetilde{\beta}_1; \cdots ; \widetilde{\alpha}_t, \widetilde{\beta}_t\}$$

be the canonical systems of simple closed curves of $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}_3$, respectively, where r (resp. s, t) is the genus of $\widetilde{F}_1$ (resp. $\widetilde{F}_2$, $\widetilde{F}_3$). We can choose systems so that two curves belonging to distinct systems $\widetilde{C}_i$, $\widetilde{C}_j$ have no point in common. Now,

$$[\gamma_1]\cdot[\gamma_2] = \sum_{k=1}^{t}(a_k^1\cdot b_k^2 - a_k^2\cdot b_k^1)$$

where

$$[\gamma_i] = \sum_{k=1}^{t}(a_k^i[\widetilde{\alpha}_k] + b_k^i[\widetilde{\beta}_k]), \quad a_k^i,\ b_k^i \in \boldsymbol{Z}\ .$$

Let $M\times[0, 1]\subset W$ be a collar neighborhood of $\partial W = M = M\times\{0\}$ in W. The classes i_*y_i can be represented by the surface $F_i = \widetilde{F}_i\times(4-i)/4\subset \mathrm{int}\, W$. Translating the systems $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$ along the collar, we obtain the canonical systems of simple closed curves of F_1, F_2, F_3:

$$\begin{aligned} C_1 &= \{\alpha'_1, \beta'_1; \cdots ; \alpha'_r, \beta'_r\} \\ C_2 &= \{\alpha''_1, \beta''_1; \cdots ; \alpha''_s, \beta''_s\} \\ C_3 &= \{\alpha_1, \beta_1; \cdots ; \alpha_t, \beta_t\}\ . \end{aligned}$$

As $W - M\times[0, 1)$ is simply connected, the curves $\widetilde{\alpha}'_i\times 1$, $\widetilde{\alpha}''_j\times 1$, $\widetilde{\alpha}_k\times 1$ in $M\times\{1\}$ bound immersed disks d'_i, d''_j, d_k in $W - M\times[0, 1)$. We set

$$\begin{aligned} D'_i &= \widetilde{\alpha}'_i\times[3/4, 1]\cup d'_i \\ D''_j &= \widetilde{\alpha}''_j\times[2/4, 1]\cup d''_j \\ D_k &= \widetilde{\alpha}_k\times[1/4, 1]\cup d_k\ . \end{aligned}$$

By spinning D'_i (D''_j, D_k) around α'_i (α''_j, α_k), if necessary, we may assume that the normal bundle $\nu(\alpha'_i\hookrightarrow F_1)$ extends to a sub-bundle of $\nu(D'_i\to W)$, etc. Using these disks D'_i, D''_j, D_k, perform surgery on F_1, F_2, F_3 in $\mathrm{int}\, W$, and we obtain immersed 2-spheres S_1, S_2, S_3 representing x_1, x_2, x_3. The construction is as follows: using the sub-bundle of $\nu(D'_i\to W)$ (resp. $\nu(D''_j\to W)$, $\nu(D_k\to W)$) mentioned above, we obtain an immersion f'_i (resp. f''_j, f_k): $D^2\times[-1, 1]\to W$, such that $f'_i(D^2\times\{0\}) = D'_i$ (resp. $f''_j(D^2\times\{0\}) = D''_j$, $f_k(D^2\times\{0\}) = D_k$) and $N'_i = f'_i(\partial D^2\times[-1, 1])$ (resp. $N''_j = f''_j(\partial D^2\times[-1, 1])$, $N_k = f_k(\partial D^2\times[-1, 1])$) is a small tubular neighborhood of α'_i (resp. α''_j, α_k) in F_1 (resp. F_2, F_3). Then

$$\begin{aligned} S_1 &= \Big(F_1 - \bigcup_{i=1}^{r} N'_i\Big)\cup\bigcup_{i=1}^{r} f'_i(D^2\times\{\pm 1\}) \\ S_2 &= \Big(F_2 - \bigcup_{j=1}^{s} N''_j\Big)\cup\bigcup_{j=1}^{s} f''_j(D^2\times\{\pm 1\}) \\ S_3 &= \Big(F_3 - \bigcup_{k=1}^{t} N_k\Big)\cup\bigcup_{k=1}^{t} f_k(D^2\times\{\pm 1\})\ . \end{aligned}$$

Now we shall construct Whitney disks $\Delta^{(i,j)}$'s and compute $\langle x_1, x_2, x_3\rangle$.

(1) Whitney disks of type 1. Corresponding to an intersection point of α_k and the double curve γ_1 (resp. γ_2), there occur two intersection points, p and q, of S_3 and S_1 (resp. S_2) with opposite signs. We draw two arcs $\gamma_1^{(3,1)}$, $\gamma_3^{(3,1)}$ (resp. $\gamma_2^{(2,3)}$, $\gamma_3^{(2,3)}$) connecting p and q on S_1 (resp. S_2), S_3. Let the arc $\gamma_1^{(3,1)}$ (resp. $\gamma_2^{(2,3)}$) lie in $f_k(D^2\times[-1,1])$, and if, D_k has not been spun around α_k, let the arc $\gamma_3^{(3,1)}$ (resp. $\gamma_3^{(2,3)}$) go straight down to reach the height of 1/4 and run parallel with β_k on $F_3-\operatorname{int}N_k$ to the other component of ∂N_k and go straight up to the end point.

Now, if D_k has not been spun, the Whitney disk is;

$$\Delta^{(3,1)}=(\gamma_1^{(3,1)}\times[3/4,1])\cup((\gamma_3^{(3,1)}\cap F_3)\times[1/4,1])\cup(\text{an immersed 2-disk in } W-M\times[0,1))$$

$$\Delta^{(2,3)}=(\gamma_2^{(2,3)}\times[2/4,1])\cup((\gamma_3^{(?,3)}\cap F_3)\times[1/4,1])\cup(\text{an immersed 2-disk in } W-M\times[0,1))\ .$$

If D_k has been spun around α_k, change this disk by homotopy to obtain the desired Whitney disk, keeping the part of level higher than 3/8 unchanged. Similarly there exists a Whitney disk $\Delta^{(1,2)}$ corresponding to an intersection point of $\tilde{\alpha}_j''$ and $\tilde{\gamma}$, where $\tilde{\gamma}=\tilde{F}_1\cap\tilde{F}_2$ is the double curve on F_2. We shall call these disks Whitney disks of type 1.

We orient W as follows:

$$[W]=[M]\times[n]\ ,$$

where n is the outward normal vector and [ ] is the orientation. Let the sign of the intersection point of α_k and γ_1 be $\varepsilon(=\pm1)$, i.e., $[\alpha_k]\times[\gamma_1]=\varepsilon[F_3]$. This is equivalent to saying that $[\gamma_1]=\varepsilon[\beta_k]$ near the intersection point. The curves γ_1, γ_2 are oriented as follows (see Figure 1):

$$[S_1]=[\gamma_1]\times[v]$$
$$[S_2]=[\gamma_2]\times[v]\ ,$$

where v is a normal vector field on F_3 in M such that $[F_3]\times[v]=[M]$. Let p be the intersection point of S_1 and S_3 on the side where $[\beta_k]$ is the inward vector of F_3-N_k, and let q be the other point. Then near p, $[S_3]=[\alpha_k]\times[n]$ and near q, $[S_3]=[\alpha_k]\times(-[n])$. Therefore, at p,

$$\begin{aligned}[S_1]\times[S_3]&=[\gamma_1]\times[v]\times[\alpha_k]\times[n]=\varepsilon[\beta_k]\times[v]\times[\alpha_k]\times[n]\\&=\varepsilon[\alpha_k]\times[\beta_k]\times[v]\times[n]=\varepsilon[F_3]\times[v]\times[n]=\varepsilon[W]\end{aligned}$$

and the sign of p is ε. So the orientation of $\gamma^{(3,1)}$ is chosen in such a way that

$$[\gamma_3^{(3,1)}\cap F_3]=\varepsilon[\beta_k]$$

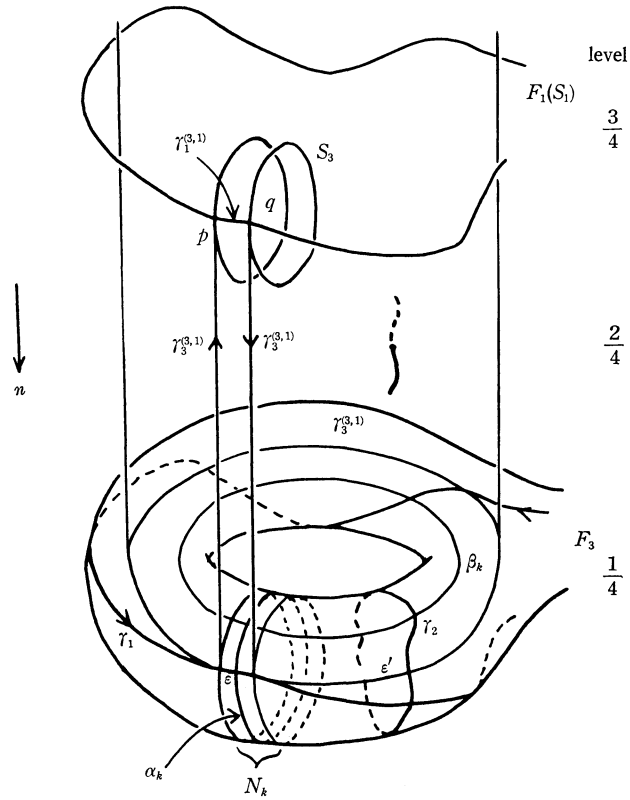


FIGURE 1

and that the orientation of the Whitney disk is

$$[\Delta^{(3,1)}] = [n] \times \varepsilon[\beta_k] \,.$$

If there is an intersection point of γ_2 and $\gamma_3^{(3,1)}$ on F_3, there is a corresponding intersection point of $\Delta^{(3,1)}$ and S_2. Let the orientation of γ_2 be $\varepsilon'[\alpha_k]$ ($\varepsilon' = \pm 1$) near the point. Then the sign of the resulting intersection of $\Delta^{(3,1)}$ and S_2 can be computed as follows:

$$\begin{aligned}[\Delta^{(3,1)}] \times [S_2] &= [n] \times \varepsilon[\beta_k] \times [\gamma_2] \times [v] = [n] \times \varepsilon[\beta_k] \times \varepsilon'[\alpha_k] \times [v] \\ &= \varepsilon\varepsilon'[\alpha_k] \times [\beta_k] \times [v] \times [n] = \varepsilon\varepsilon'[W] \,.\end{aligned}$$

This Whitney disk $\Delta^{(3,1)}$ of type 1 may intersect S_2 outside the collar neighborhood of M, but the intersection occurs in pair and the algebraic sum is zero. Therefore

$$\sum_{\text{type 1}} \Delta^{(3,1)} \cdot S_2 = \sum \varepsilon \cdot \varepsilon' = \sum_{k=1}^{t} a_k^2 \cdot b_k^1 \,.$$

Similarly

$$\sum_{\text{type 1}} \Delta^{(2,3)} \cdot S_1 = -\sum_{k=1}^{t} a_k^1 \cdot b_k^2$$

and

$$\sum_{\text{type 1}} \Delta^{(1,2)} \cdot S_3 = 0 .$$

(2) Whitney disks of type 2. In $W - M \times [0, 1)$, the intersection of S_i and S_j occurs as in Figure 2, corresponding to the intersection of the immersed disks d_i and d_j. We call the Whitney disks obtained from this intersection as the Whitney disks of type 2. Clearly, Whitney disks of type 2 occur in pairs and

$$\sum_{\text{type 2}} \Delta^{(i,j)} \cdot S_k = 0 .$$

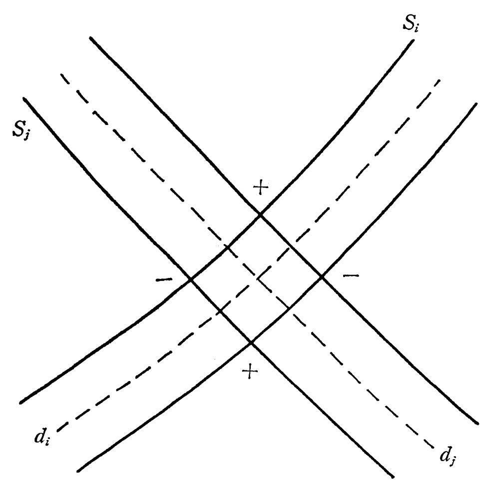


FIGURE 2

Combining (1) and (2), we obtain

$$\langle x_1, x_2, x_3 \rangle = \sum_{k=1}^{t} (a_k^2 \cdot b_k^1 - a_k^1 \cdot b_k^2)$$

by the definition of the Matsumoto tripling and Theorem is proved in case $I = (0)$.

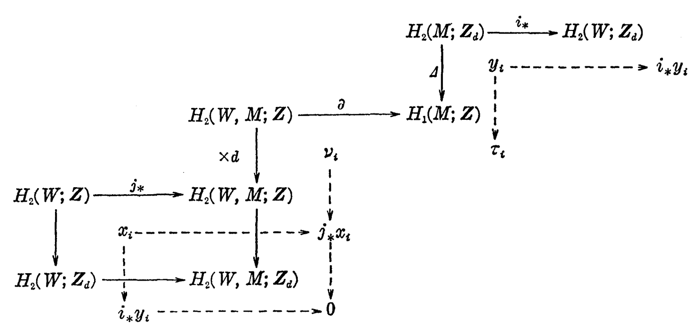
3. The proof of Therem; part 2. Our plan of the proof is entirely the same as in the case $I = (0)$. However, an element of $H_2(M; \mathbf{Z}_d)$ cannot be realized by an embedded surface in general. This is the only point to study.

Now, we shall show how to construct a nice complex representing y_i. The Bockstein homomorphism Δ: $H_2(M; \boldsymbol{Z}_d) \to H_1(M; \boldsymbol{Z})$ maps y_i to τ_i. Assume τ_i is non-trivial. For, otherwise, y_i can be realized by an embedded surface. Let l_i be a closed curve representing τ_i, and N be a tubular neighborhood of l_i. Then $H_1(\partial N; Z) \approx \boldsymbol{Z} \oplus \boldsymbol{Z}$ is generated by a longitude ξ and a meridian η. Since $d\tau_i = 0$, the element $d\cdot\xi + e\cdot\eta$ is null-homologous in $M - \mathring{N}$ for some integer e. Here, this number e is related to the linking number in the sense of Seifert [5] by $V(\tau_i, \tau_i) = e/d \pmod 1$. Then the element $d\cdot\xi + e\cdot\eta$ is represented by a torus link of type (d, e) on ∂N. By the Pontrjagin-Thom construction, L_i bounds a properly embedded punctured surface T_i in $(M - \mathring{N}, \partial N)$ (which is possibly non-connected). Let L_i^* be the fiberwise join of L_i and l_i. Then $T_i \cup L_i^*$ represents an element $y_i' \in H_2(M; \boldsymbol{Z}_d)$ and $\Delta(y_i') = \tau_i$. This implies that $y_i - y_i' = z_i$ can be considered as an element of $H_2(M; \boldsymbol{Z})$. Moreover, the Mayer-Vietoris exact sequence shows that z_i is also an element of $H_2(M - \mathring{N}; \boldsymbol{Z})$. Thus we can represent the homology class $y_i = y_i' + z_i \in H_2(M; \boldsymbol{Z}_d)$ again by the union of a properly embedded surface T_i' with $\partial T_i' = L_i$ and L_i^*.

The next lemma plays a key role in the proof.

LEMMA. *$V(\tau_i, \tau_j) = 0$ for $i \neq j$.*

PROOF. We shall consider the following diagram.



The element $\nu_i \in H_2(W, M; \boldsymbol{Z})$ is mapped to $j_* x_i$ by the homomorphism $\times d$, the multiplication by the number d. Then $\partial(\nu_i) = \tau_i$ by definition. Now, the linking number can be calculated in terms of an intersection number in the 4-manifold W. Namely,

$$V(\tau_i, \tau_j) = x_i \cdot \nu_j / d \pmod 1 .$$

Our assumption $x_i \cdot x_j = 0$ is valid with rational coefficients. Thus $V(\tau_i, \tau_j) = 0$ for $i \neq j$, completing the proof.

REMARK. This lemma states that for some 2-chain C_i such that $\partial C_i = d\tau_i$, the intersection number $C_i \cdot \tau_j = 0 \pmod d$. However, we can choose a nice loop l_j representing τ_j in such a way that $C_i \cdot l_j = 0$. Then for a 2-chain C_j with $\partial C_j = d \cdot l_j$, the intersection number $C_j \cdot l_i = 0$, where l_i is a closed curve defined by C_i.

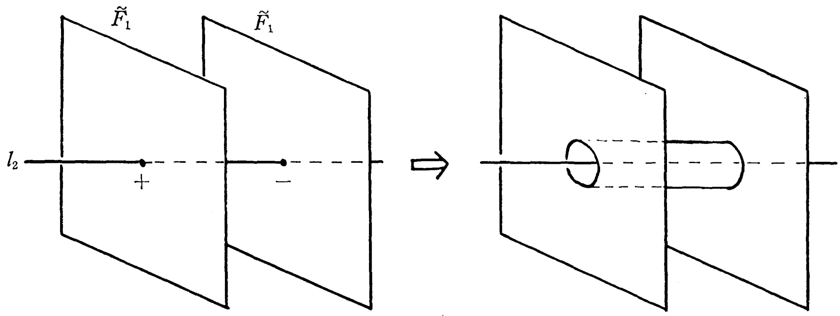


FIGURE 3

PROOF OF THEOREM IN CASE $I \neq (0)$. Take a complex $\tilde{F}_1$ which represents y_1 and contains a loop l_1, representing τ_1, as a singularity. Then there is a loop l_2, representing τ_2, with $\tilde{F}_1 \cdot l_2 = 0$ by the remark above. By performing surgery on $\tilde{F}_1$ as in Figure 3, if necessary, we may assume that $\tilde{F}_1 \cap l_2 = \emptyset$. The complex $\tilde{F}_2$ represents y_2 and contains a loop l_2 as a singularity. For the same reason as above, it may be assumed that $\tilde{F}_2 \cap l_1 = \emptyset$. The loop l_3 and the complex $\tilde{F}_3$ are similarly defined.

Now, we shall count the algebraic sum of the triple points $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$. By the construction of $\tilde{F}_i$, we see that $\tilde{F}_1 \cap \tilde{F}_3$ and $\tilde{F}_2 \cap \tilde{F}_3$ represent cycles on $\tilde{F}_3 - l_3$ in the sense of integral coefficients. Set them as

$$\sum_k (a_1^k[\alpha_k] + b_k^1[\beta_k]) + \sum_j c_j^1[\gamma_j]$$
$$\sum_k (a_k^2[\alpha_k] + b_k^2[\beta_k]) + \sum_j c_j^2[\gamma_j] \,,$$

where $[\alpha_k]$, $[\beta_k[$, $[\gamma_j]$ is the canonical system of generators of $H_1(\tilde{F}_3 - l_3; \boldsymbol{Z})$ as in Figure 4. Then by the definition of the cup product,

$$((y_1^* \cup y_2^* \cup y_3^*) \cap [M]) = \sum (a_k^1 \cdot b_k^2 - a_k^2 \cdot b_k^1) \,.$$

The next step of the proof is to perform surgery on $\tilde{F}_i$ in W so that it is realized by an immersed 2-sphere. Take a properly immersed 2-disk D_i which represents $\nu_i \in H_2(W, M; \boldsymbol{Z})$ and satisfies $\partial D_i = l_i$. Then $F_i = \tilde{F}_i \cup d \cdot D_i$ is regarded as the image from a closed surface. And by

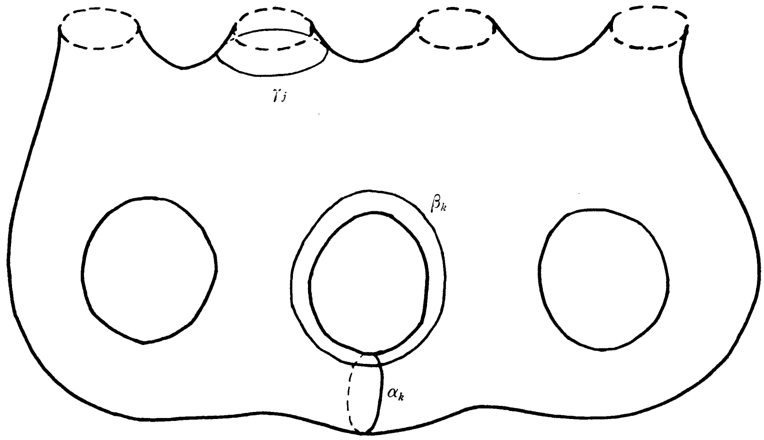


FIGURE 4

assumption, $D_i \cdot D_j = 0$ for $i \neq j$. The rest of the proof is entirely the same as before. Because in the computation of the Matsumoto tripling, the intersection number of S_k and Whitney disks $\Delta^{(i,j)}$'s, which lie in the complement of the collar of M in W, is zero modulo d for the same reason as in the case $d = 0$. This completes the proof of Theorem.

4. **Some applications of Theorem.** First of all we give

PROOF OF COROLLARY 1. Since W and W' are simply connected, $g_*[M] = [M']$ implies that f is of degree one, i.e., $f_*[W] = [W']$. Hence $f_*: H_2(W) \to H_2(W')$ is an epimorphism and the intersection ideal of $\{f_*x_1, f_*x_2, f_*x_3\}$ is contained in $I = I\{x_1, x_2, x_3\}$. Thus we have the reduction $\langle f_*x_1, f_*x_2, f_*x_3\rangle'$ of $\langle f_*x_1, f_*x_2, f_*x_3\rangle$ as an element of $\boldsymbol{Z}/I$. We put $K^i(M; \boldsymbol{Z}_d) = \operatorname{Coker}(g^*: H^i(M'; \boldsymbol{Z}_d) \to H^i(M; \boldsymbol{Z}_d))$ and $K_i(M; \boldsymbol{Z}_d) = \operatorname{Ker}(g_*: H_i(M; \boldsymbol{Z}_d) \to H_i(M'; \boldsymbol{Z}_d))$. Since the map $g: M \to M'$ is of degree one, we have direct sum decompositions $H^i(M; \boldsymbol{Z}_d) = H^i(M'; \boldsymbol{Z}_d) \oplus K^i(M; \boldsymbol{Z}_d)$ and $H_{3-i}(M; \boldsymbol{Z}_d) = H_{3-i}(M'; \boldsymbol{Z}_d) \oplus K_{3-i}(M; \boldsymbol{Z}_d)$, which are compatible with the Poincaré duality isomorphism $\cap [M]$ and orthogonal with respect to the pairing $\langle , \rangle: H^i(M; \boldsymbol{Z}_d) \otimes H^{3-i}(M; \boldsymbol{Z}_d) \to \boldsymbol{Z}_d$ given by $\langle \xi, \eta \rangle = (\xi \cup \eta) \cap [M]$, see [1, pp. 9-12].

For an element $x \in H_2(W; \boldsymbol{Z})$, let $y \in H_2(M; \boldsymbol{Z}_d)$ be a mod d boundary reduction of x and let y^* be the Poincaré dual of y. If y' is a mod d boundary reduction of $x' = f_*x$, then we have $g_*y = y'$, since $i'_*: H_2(M'; \boldsymbol{Z}_d) \to H_2(W'; \boldsymbol{Z}_d)$ is monic. In order to prove Corollary 1, it suffices to show that

$$g^*(y_1'^* \cup y_2'^* \cup y_3'^*) \cap [M] = (y_1^* \cup y_2^* \cup y_3^*) \cap [M] ,$$

where $y_i^* \in H^1(M; \boldsymbol{Z}_d)$ and $y_i'^* \in H^1(M'; \boldsymbol{Z}_d)$ are the Poincaré duals of $y_i \in H_2(M; \boldsymbol{Z}_d)$ and $y_i' \in H_2(M'; \boldsymbol{Z}_d)$ such that $g_* y_i = y_i'$. Since the direct sum decompositions of $H_i(M)$ above are compatible with the Poincaré duality, we may write $g^* y_i'^* = y_i^* + u_i$ for some $u_i \in K^1(M)$. Moreover, they are orthogonal with respect to the pairing $\langle , \rangle$. It follows that

$$\begin{aligned} g^*(y_1'^* \cup y_2'^* \cup y_3'^*) \cap [M] \\ &= ((y_1^* + u_1) \cup (y_2^* + u_2) \cup (y_3^* + u_3)) \cap [M] \\ &= (y_1^* \cup y_2^* \cup y_3^*) \cap [M] . \end{aligned}$$

This completes the proof of Corollary 1.

Kaplan's number and Milnor's higher linking number. Let $L = \bigcup_{i=1}^3 k_i$ be a link in S^3 of three components k_1, k_2, k_3 with the mutual linking numbers zero. Regard S^3 as the boundary of a 4-ball B^4 and attach a 2-handle to S^3 by a 0-framing of each k_i. This gives us a compact simply connected 4-manifold W_L with boundary $M_L = \partial W_L$. Let $x_1, x_2, x_3 \in H_2(W_L; \boldsymbol{Z})$ be the homology classes of W_L corresponding to the cores of the attached 2-handles. Then there exist unique integral boundary reductions $y_i \in H_2(M_L; \boldsymbol{Z})$ of x_i, $i = 1, 2, 3$. Matsumoto [3] observed that $-\langle x_1, x_2, x_3 \rangle$ in W_L coincides with Milnor's higher linking number $\bar{\mu}(L)$ of L, [4]. Meanwhile, Kaplan [2] has defined an invariant $T(L)$ and proved that $T(L) = (y_1^* \cup y_2^* \cup y_3^*) \cap [M_L]$. Thus we have

COROLLARY 2. *Let $L = \bigcup_{i=1}^3 k_i$ be a link as above. Then we have $T(L) = \bar{\mu}(L)$.*

Normal singularity of complex algebraic surfaces. Let V be a complex algebraic surface in an open neighborhood of the origin $\boldsymbol{O}$ in a complex N-space $\boldsymbol{C}^N$. Suppose that V contains $\boldsymbol{O}$ as a normal singularity. For sufficiently small $\varepsilon > 0$, a closed oriented 3-manifold $K_\varepsilon(\boldsymbol{O}) = \{z \in V \mid |z| = \varepsilon\}$ is called a link of $\boldsymbol{O}$ in V. Sullivan [6] has proved that $(y_1^* \cup y_2^* \cup y_3^*) \cap [K_\varepsilon(\boldsymbol{O})] = 0$ for any elements $y_1, y_2, y_3 \in H_2(K_\varepsilon(\boldsymbol{O};) \boldsymbol{Z})$. Hence we have

COROLLARY 3. *Let K be a link of a normal singularity in a complex algebraic surface. Then in a compact simply connected 4-manifold W with $\partial W = K$, three homology classes $x_1, x_2, x_3 \in H_2(W; \boldsymbol{Z})$ can be realized as mutually disjoint immersed 2-spheres in W if $j_* x_i = 0$ for $i = 1, 2, 3$, where j_* is induced by the inclusion map $j: W \to (W, M)$. In particular, if W is the so-called Milnor fiber of an isolated hypersurface singularity in $\boldsymbol{C}^3$ with the monodromy $h_*: H_2(W, \boldsymbol{Z}) \to H_2(W; \boldsymbol{Z})$, then three invariant homology classes $x_1, x_2, x_3 \in H_2(W; \boldsymbol{Z})$ of h_* (i.e., $h_*(x_i) = x_i$) can be realized by mutually disjoint immersed 2-spheres.*

REMARK. We cannot expect the Z_d-analogue of Sullivan's result. In fact, consider the isolated hypersurface singularity; $z_1^d + z_2^d + z_3^d = 0$, $(d \geqq 3)$, whose link $K_\varepsilon(O)$ is an S^1-bundle of degree d over a surface of genus $(d-1)(d-2)/2$. Then there exist three elements $y_1, y_2, y_3 \in H_2(K_\varepsilon(O); Z_d)$ such that $y_1^* \cup y_2^* \cup y_3^* \neq 0$.

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