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BESOV SPACES AND SOBOLEV SPACES ON A NILPOTENT LIE GROUP

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Introduction. In this paper we shall study the theory of Besov spaces (or Lipschitz spaces) and Sobolev spaces on a nilpotent Lie group. To admit a wide variety of applications to more problems we consider the class of "stratified groups" as a class of nilpotent Lie groups. On such Lie groups there is a natural notion of homogeneity which enables one to duplicate many of the standard constructions of Euclidean spaces. But we can not yet duplicate most of results in the theory of Fourier transforms and distributions. Hence fractional integral operators play a fundamental role in our paper. These operators have been extensively studied by G. B. Folland [5], A. Yoshikawa [24] and H. Komatsu [10], [11], [12], [13],]14], [15] in a general setting. By employing the Bessel potential as one of these fractional integral operators we develop the theory of Besov spaces and Sobolev spaces on a stratified group. Our paper is heavily influenced by Flett's paper [4].

The plan of our paper is as follows: In Section 1 we present notations used in later sections and recall the necessary background material concerning homogeneous structures on nilpotent Lie groups. In Section 2, we consider the diffusion semigroup generated by the sub-Laplacian $\mathfrak{J}$ on a stratified group, and we use it to define the Bessel potentials given as fractional powers of the operator $(1+\mathfrak{J})$. Further, we discuss properties of the semigroup, its kernel function and the Bessel potentials. In Section 3 we define an analogue of the classical Besov space in terms of the Bessel potentials and extend several basic theorems from the Euclidean case to our case. Further we investigate several equivalent spaces to this Besov space. In Section 4 we shall see that this Besov space coincides with that defined by use of the Poisson semigroup for positive fractional powers. In Section 5 we define an analogue of the classical Sobolev space in terms of the Bessel potentials. We see that this space has an alternative representation in terms of "the Riesz potentials" and we use it to prove the inclusion theorem in this section and the interpolation theorem in the next section. Several basic theorems for the interpolation space of Besov spaces and Sobolev spaces are discussed in Section 6. In Section 7 we shall give several results concerning the duals of Besov spaces and Sobolev spaces. T. M. Flett [4] has given a long series of lemmas to prove results concerning the duals of certain Besov spaces in n-dimensional Euclidean spaces. But some of these lemmas are not applicable to our case. Hence we shall prove them through a series of lemmas slightly different.

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1. Preliminaries. In this section we will present notations and terminology used in later sections.

If $\mathfrak{G}$ is a nilpotent Lie algebra, a stratification of $\mathfrak{G}$ is a decomposition of $\mathfrak{G}$ as a vector space sum, $\mathfrak{G} = V_1 \oplus \cdots \oplus V_m$ such that $[V_1, V_j] = V_{j+1}$ for $1 \leqq j < m$ and $[V_1, V_m] = \{0\}$. If $\mathfrak{G}$ is stratified, it admits a family of dilations, given by

$$\gamma_r(X_1 + X_2 + \cdots + X_m) = rX_1 + r^2X_2 + \cdots + r^mX_m \quad (X_j \in V_j) .$$

Let G be the corresponding simply connected Lie group. Since $\mathfrak{G}$ is nilpotent, the dilations γ_r lift via the exponential map exp to give a one-parameter group of automorphisms of G, say γ_r. We sometimes denote $\gamma_r x$ simply by rx. Let $||\cdot||$ denote a Euclidean norm on $\mathfrak{G}$ with respect to which the V_j's are mutually orthogonal. We define a homogeneous norm on the corresponding group G by

$$\left| \exp\left(\sum_{j=1}^{m} X_j\right)\right| = \left(\sum_{j=1}^{m} ||X_j||^{2m!/j}\right)^{1/2m!} \quad (X_j \in V_j) .$$

A stratified group means a simply connected nilpotent Lie group G together with a stratification $\mathfrak{G} = \bigoplus_1^m V_j$ of its Lie algebra and the dilations and the homogeneous norm defined above. We fix once and for all a (bi-invariant) Haar measure dx on G which is the lift of Lebesgue measure on $\mathfrak{G}$ via exp. We shall denote the identity element of G by e. C_b denotes the set of all bounded continuous real valued functions on G. The set of all f's in C_b which vanish at infinity is denoted by C_0. The set of all f's in C_b whose support is compact is denoted by C_c. C^∞ denotes the space of real valued indefinitely differentiable functions in G. The set of functions in C^∞ of compact support is denoted by C_c^∞. L^p $(1 \leqq p \leqq \infty)$ will denote the standard L^p-space with respect to the Haar measure dx, with the L^p-norm $||\cdot||_p$. We denote by χ_E a characteristic function of a measurable set E of G. We identify the Lie algebra $\mathfrak{G}$ with the left-invariant vector fields on G. A measurable function f on G will be called homogeneous of degree λ ($\lambda \in \boldsymbol{C}$) if $f\circ\gamma_r = r^\lambda f$ for all $r > 0$. A differential operator D will be called homogeneous of degree λ if $D(f\circ\gamma_r) = r^\lambda(Df)\circ\gamma_r$ for all $f \in C^\infty$, $r > 0$. In particular, $X \in \mathfrak{G}$ is homogeneous of degree j if and only if $X \in V_j$. We choose once and for all a basis $X_1, \cdots, X_n$ for V_1 and set $\mathfrak{J} = -\sum_1^n X_j^2$. $\mathfrak{J}$ is a left-invariant second-order differential operator which is homogeneous of degree 2. If $\alpha = (\alpha_1, \cdots, \alpha_l)$ is an l-tuple of nonnegative integers $\alpha_i \geqq 0$, we then put $|\alpha| = \alpha_1 + \cdots + \alpha_l$ and define D^α to be $X_{i_1}^{\alpha_1} \cdots X_{i_l}^{\alpha_l}$ (where $X_1, \cdots, X_n$ is the basis for V_1), which is a homogeneous differential operator of degree $|\alpha|$.

We recall properties of the homogeneous norm on G (see G. B. Folland [5], A. W. Knapp - E. M. Stein [9] and A. Korányi - S. Vági [16]).

(a) The homogeneous norm $|\cdot|$ is a continuous function from G to $[0, \infty)$ which is of class C^∞ away from e and homogeneous of degree 1.

(b) $|x| = 0$ if and only if $x = e$.

(c) $|x| = |x^{-1}|$ for all $x \in G$.

(d) $\{x \in G\colon |x| \leqq 1\}$ is compact.

(e) There is a constant $C > 0$ such that $|xy| \leqq C(|x| + |y|)$ for all $x, y \in G$.

(f) There exist $C_1, C_2 > 0$ such that $C_1\|X\| \leqq |x| \leqq C_2\|X\|^{1/m}$ whenever $|x| \leqq 1$, where $x = \exp X$.

(g) $d(\gamma_r x) = r^\rho dx$ for each $r > 0$ where ρ is the homogeneous dimension defined by $\sum_1^m j(\dim V_j)$.

(h) If $\alpha \in \boldsymbol{C}$ and $0 < a < b < \infty$, then there exists a constant C such that

$$\int_{a\leqq|x|\leqq b} |x|^{\alpha-\rho}dx = \begin{cases} C\alpha^{-1}(b^\alpha - a^\alpha) & \text{if } \alpha \neq 0 \\ C\log(b/a) & \text{if } \alpha = 0 . \end{cases}$$

We use C to denote a positive constant different in each occasion. It will depend on the parameter appearing in each problem. The same notations C are not necessarily the same on any two occurrences.

2. The Gauss-Weierstrass integral and the Bessel potential. Henceforth we assume that G is a stratified group of homogeneous dimension $\rho > 2$.

In this section we give some properties of the Gauss-Weierstrass integral and the Bessel potential associated with the heat diffusion semigroup on G. We can construct a semigroup $\{H_t\}_{t\geqq 0}$ of linear operators on $L^1 + L^\infty$ with the infinitesimal generator $-\mathfrak{J}$ according to a theorem of G. A. Hunt [7]. These properties are summarized in the following theorem.

THEOREM 1 (G. B. Folland [5], G. A. Hunt [7]).

(i) *$H_t f(x) = h_t * f(x) = \int_G h_t(y^{-1}x) f(y) dy$, $t > 0$, where*

(a) *$h_t(x) = h(x, t)$ is of class C^∞ on $G \times (0, \infty)$,*

(b) *$\int_G h_t(x) dx = 1$ for all $t > 0$,*

(c) *$h(x, t) \geqq 0$ for all $x \in G$ and all $t > 0$,*

(d) *$h_t(x) = h_t(x^{-1})$ for all $x \in G$ and all $t > 0$,*

(e) *$\lim_{t \to 0} \int_U h_t(y) dy = 1$ for any neighborhood U of e,*

(f) *$h_t * h_s = h_{t+s}$ for all $t, s > 0$,*

(g) *$h(rx, r^2 t) = r^{-\rho} h(x, t)$ for all $x \in G$ and all $r, t > 0$.*

(ii) *$\|H_t\|_p \leqq 1$ $(1 \leqq p \leqq \infty)$ and if $1 < p < \infty$, $\{H_t\}$ can be extended to a holomorphic contraction semigroup*

$$\{H_z : |\arg z| < (\pi/2)(1 - |1 - (2/p)|)\} \text{ on } L^p .$$

(iii) *H_t is self-adjoint, and $f \geqq 0$ implies $H_t f \geqq 0$. Moreover, $H_t 1 = 1$.*

(iv) *If $f \in L^p$, $1 \leqq p \leqq \infty$, then $H_t f$ is of class C^∞ on $G \times (0, \infty)$ and $(\partial/\partial t)(H_t f)(x) + \mathfrak{J} H_t f(x) = 0$.*

(v) *Extend $h(x, t)$ to $G \times \boldsymbol{R}$ by setting $h(x, t) = 0$ for $t \leqq 0$. Then h is of class C^∞ on $(G \times \boldsymbol{R}) - \{(e, 0)\}$. In particular, for each $x \neq e$, $h(x, t)$ vanishes rapidly as t decreases to zero.*

(vi) *Let $(-\mathfrak{J}_p)$ be the infinitesimal generator of $\{H_t\}$ on L^p; then*

(a) *$\mathfrak{J}_p$ is a closed operator on L^p whose domain is dense for $p < \infty$, and whose range is dense for $1 < p < \infty$.*

(b) *$\mathfrak{J}_p f = \mathfrak{J} f$ for all $f \in C^\infty \cap L^p$, $1 \leqq p \leqq \infty$. Also, if $p < \infty$, $\mathfrak{J}_p$ is the smallest closed extension of the restriction $\mathfrak{J}|_{C_c^\infty}$ on L^p.*

In later sections we shall require the convergence theorem and the representation theorem associated with the semigroup on G.

THEOREM 2 (K. Saka [19]). *Let f be a measurable function on G. Then $\lim_{t \to 0} H_t f = f$ holds in the following senses:*

(i) *in the L^p-norm if $f \in L^p$, $1 \leqq p < \infty$,*

(ii) *in the weak star topology of L^∞ if $f \in L^\infty$,*

(iii) *uniformly on each compact subset of G if $f \in C_b$,*

(iv) *uniformly if f is uniformly continuous on G, and so in particular, if $f \in C_0$,*

(v) *almost everywhere if $f \in L^p$, $1 < p \leqq \infty$.*

THEOREM 3 (K. Saka [19]). *Suppose that $u(x, t)$ is of class C^∞ on $G \times (0, \infty)$ such that $\sup_{t>0} \|u(\cdot, t)\|_p < \infty$ $(1 \leqq p \leqq \infty)$.*

(i) *If* $1 < p \leqq \infty$, *then it satisfies the heat equation* $(\partial u/\partial t) + \mathfrak{J}u = 0$ *on* $G \times (0, \infty)$ *if and only if it is of the form* $u(x, t) = H_t f(x)$ *where* $f \in L^p$.

(ii) *If* $p = 1$, *then it satisfies the heat equation and* $\|u(\cdot, t) - u(\cdot, t')\|_1 \to 0$ *as* $t, t' \to 0$, *if and only if it is of the form* $u(x, t) = H_t f(x)$ *where* $f \in L^1$.

Moreover, these representations are unique and $\|f\|_p = \sup_{t>0}\|u(\cdot, t)\|_p$.

THEOREM 4. *Let* $h_t(x) = h(x, t)$ *be the kernel function of the semigroup* $\{H_t\}$. *Then,*

(i) *for all* $t > 0$,

$$|h(x, t)| \leqq \begin{cases} C|x|^{-\rho} & \text{if } |x|^2 \geqq t \\ Ct^{-\rho/2} & \text{if } |x|^2 \leqq t \,. \end{cases}$$

Also, if $\alpha = (\alpha_1, \cdots, \alpha_l)$ *is an* l*-tuple of nonnegative integers* $\alpha_i \geqq 0$ *and* k *is a nonnegative integer, then*

$$\left|\frac{\partial^k}{\partial t^k} D^\alpha h(x, t)\right| \leqq \begin{cases} C|x|^{-(\rho+|\alpha|+2k)} & \text{if } |x|^2 \geqq t \\ Ct^{-(\rho+|\alpha|+2k)/2} & \text{if } |x|^2 \leqq t \,. \end{cases}$$

Further, for all $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha h_t\right\|_p \leqq C\, t^{-(|\alpha|+2k+\rho(1-1/p))/2} \text{ where } 1 \leqq p \leqq \infty \,.$$

(ii) *Let* $1 \leqq p \leqq \infty$, *and put* $u(x, t) = H_t f(x)$, $f \in L^p$. *Let* α, k *be as in* (i). *Then for all* $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, t)\right\|_p \leqq Ct^{-(|\alpha|+2k)/2}\|f\|_p \,.$$

Also, if $1 \leqq p < r \leqq \infty$ *and* $\delta = \rho(1/p - 1/r)$, *then for all* $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, t)\right\|_r \leqq Ct^{-(|\alpha|+2k+\delta)/2}\|f\|_p \,.$$

(iii) *For each* $t > 0$ *the functions* $x \mapsto u(x, t)$ *and* $x \mapsto (\partial^k/\partial t^k)u(x, t)$ *are uniformly continuous on* G. *Moreover, the functions* $t \mapsto \|u(\cdot, t)\|_p$ *and* $t \mapsto \|(\partial^k/\partial t^k)u(\cdot, t)\|_p$ *are decreasing on* $(0, \infty)$ *if* $1 \leqq p \leqq \infty$ *and are continuous on* $(0, \infty)$ *if* $1 \leqq p < \infty$.

PROOF. If $t \leqq |x|^2$, then we have

$$\begin{aligned} |h(x, t)| &= \left|h\left(|x| \cdot \frac{x}{|x|}, \; |x|^2 \cdot \frac{t}{|x|^2}\right)\right| = |x|^{-\rho} \left|h\left(\frac{x}{|x|}, \; \frac{t}{|x|^2}\right)\right| \\ &\leqq |x|^{-\rho} \sup\{|h(y, t_0)| : |y| = 1, 0 < t_0 \leqq 1\} \end{aligned}$$

$$\leqq C|x|^{-\rho} \quad \text{by Theorem 1 (v)} .$$

If $|x|^2 \leqq t$, then we have

$$\begin{aligned}|h(x, t)| &= |h(t^{1/2}\cdot t^{-1/2}x, t)| \\ &= t^{-\rho/2}|h(t^{-1/2}x, 1)| \leqq t^{-\rho/2} \sup_{|y|\leqq 1} |h(y, 1)| \leqq C t^{-\rho/2}\end{aligned}$$

by Theorem 1 (v) which proves the first part of (i).

Since

$$\frac{\partial^k}{\partial t^k} D^\alpha h(ry, r^2 t) = r^{-(\rho+|\alpha|+2k)} \frac{\partial^k}{\partial t^k} D^\alpha h(y, t) ,$$

the second part of (i) follows in the same way.

To prove the last part of (i) we use the property (h) of the homogeneous norm. That is, for each $t > 0$,

$$\begin{aligned}\int_G \left|\frac{\partial^k}{\partial t^k} D^\alpha h_t(y)\right|^p dy &= \int_{|y|^2\geqq t} \left|\frac{\partial^k}{\partial t^k} D^\alpha h_t(y)\right|^p dy \\ &+ \int_{|y|^2<t} \left|\frac{\partial^k}{\partial t^k} D^\alpha h_t(y)\right|^p dy \leqq C\int_{|y|\geqq t^{1/2}} |y|^{-p(\rho+|\alpha|+2k)} dy \\ &+ C\int_{|y|<t^{1/2}} t^{-p(\rho+|\alpha|+2k)/2} dy = Ct^{-p(|\alpha|/2+k)-\rho(p-1)/2} .\end{aligned}$$

This completes the proof of (i).

To prove (ii), we note that

$$\frac{\partial^k}{\partial t^k} D^\alpha u(x, t) = \left(\frac{\partial^k}{\partial t^k} D^\alpha h_t\right) * f(x) .$$

By Young's inequality and (i),

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, t)\right\|_p \leqq \left\|\frac{\partial^k}{\partial t^k} D^\alpha h_t\right\|_1 ||f||_p \leqq Ct^{-(|\alpha|/2+k)} ||f||_p$$

and, if $1/r = 1/p + 1/q - 1 \geqq 0$,

$$\begin{aligned}\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, t)\right\|_r &\leqq \left\|\frac{\partial^k}{\partial t^k} D^\alpha h_t\right\|_q ||f||_p \leqq Ct^{-(|\alpha|/2+k+\rho(1-1/q)/2)}||f||_p \\ &= Ct^{-(|\alpha|+2k+\delta)/2}||f||_p .\end{aligned}$$

To prove (iii) we see that

$$||u(\cdot, s + t)||_p \leqq ||h_t||_1 ||u(\cdot, s)||_p = ||u(\cdot, s)||_p \qquad (1 \leqq p \leqq \infty)$$

and so, $||u(\cdot, t)||_p$ is decreasing on $(0, \infty)$. Since $(\partial^k/\partial t^k)u(x, s + t) = h_t*(\partial^k/\partial t^k)u(x, s)$, $||(\partial^k/\partial t^k)u(\cdot, t)||_p$ is also decreasing on $(0, \infty)$. Further, Theorem 2 implies that both of $||u(\cdot, t)||_p$ and $||(\partial^k/\partial t^k)u(\cdot, t)||_p$ are continuous on $(0, \infty)$ if $1 \leqq p < \infty$.

Since $h_t \in L^{p'}(1/p + 1/p' = 1)$ for each $t > 0$ and $u(x, t) = h_t * f(x)$, $u(x, t)$ is uniformly continuous on G, and so is also $(\partial^k/\partial t^k)u(x, t)$. This completes the proof of the theorem.

THEOREM 5. *Let* $1 \leqq p \leqq \infty$, *and let* u *be a temperature on* $G \times (0, \infty)$, *that is, a solution of the heat equation* $\mathfrak{J}u(x, t) + (\partial u/\partial t)(x, t) = 0$ *on* $G \times (0, \infty)$, *such that the function* $t \mapsto \|u(\cdot, t)\|_p$ *is locally integrable on* $(0, \infty)$. *Then for each* $s > 0$ *and all* $(x, t) \in G \times (0, \infty)$,

$$u(x, s + t) = h_t * u(x, s) = H_t u(x, s) .$$

To prove this theorem we need two lemmas:

LEMMA 1. *Suppose that* $u(x, t)$ *is a continuous function on* $G \times [0, \infty)$ *with* $u(x, 0) = 0$ *such that* $u(x, t)$ *is a temperature on* $G \times (0, \infty)$ *and the function* $t \mapsto \|u(\cdot, t)\|_p$ *is locally integrable on* $[0, \infty)$. *Then* $u(x, t)$ *is identically zero on* $G \times [0, \infty)$.

PROOF. Fix an arbitrary element $(\bar{x}, \bar{t}) \in G \times (0, \infty)$ and let $B(r) = \{y \in G: |y| \leqq r\}$ $(r > 0)$. We choose a function $\varphi(x) \in C^\infty$ having the following properties:

(a) $\varphi(x) = \begin{cases} 1 & \text{if } x \in \bar{x}B(r) \\ 0 & \text{if } x \notin \bar{x}B(r') \end{cases} \quad (r < r')$

(b) $0 \leqq \varphi(x) \leqq 1$

(c) $\sum_{j=1}^n |X_j\varphi(x)| + \sum_{j=1}^n |X_j^2\varphi(x)| \leqq C \quad (x \in G)$,

where C is a constant independent of x.

Since $\varphi(x)u(x, \bar{t}) \in C_b$, we obtain by Theorem 2 that

$$(1) \qquad \begin{aligned} u(\bar{x}, \bar{t}) = \varphi(\bar{x})u(\bar{x}, \bar{t}) &= \lim_{t \to 0} H_t(\varphi(y)u(y, \bar{t}))(\bar{x}) \\ &= \lim_{t \to 0} \int_G h(y^{-1}\bar{x}, t)\varphi(y)u(y, \bar{t})dy \\ &= \lim_{t \uparrow \bar{t}} \int_G h(y^{-1}\bar{x}, \bar{t} - t)\varphi(y)u(y, t)dy . \end{aligned}$$

Put $v(y, t) = h(y^{-1}\bar{x}, \bar{t} - t)\varphi(y)$. Since $u(x, t)$ is a temperature on $G \times (0, \infty)$,

$$u \sum X_j^2 v + u\frac{\partial}{\partial t}v = u \sum X_j^2 v + u\frac{\partial}{\partial t}v - \left(\sum X_j^2 u - \frac{\partial}{\partial t}u\right)v \quad \text{for } 0 < t < \bar{t} .$$

Hence, using (1) and that $u(x, 0) = 0$,

$$(2) \qquad \int_0^{\bar{t}} \int_G \left(u \sum X_j^2 v + u\frac{\partial}{\partial t}v\right)dydt$$

$$
\begin{aligned}
&= \int_0^{\bar t}\int_G \left\{ (u \sum X_j^2 v - \sum X_j^2 u\cdot v) + \frac{\partial}{\partial t}(uv)\right\} dydt \\
&= \int_0^{\bar t}\int_G \frac{\partial}{\partial t}(uv)dydt = \int_G\int_0^{\bar t} \frac{\partial}{\partial t}(uv)dtdy \\
&= \int_G (\lim_{t\uparrow \bar t} u(y, t)v(y, t) - u(y, 0)v(y, 0))dy \\
&= \lim_{t\uparrow \bar t}\int_G u(y, t)\varphi(y)h(y^{-1}\bar x, \bar t - t)dy = u(\bar x, \bar t)\,.
\end{aligned}
$$

If $0 < t < \bar t$,

$$
\begin{aligned}
\sum X_j^2 v + \frac{\partial}{\partial t} v &= \sum X_j^2\varphi(y)\cdot h(y^{-1}\bar x, \bar t - t) \\
&\quad + \sum X_j^2 h(\bar x^{-1}y, \bar t - t)\cdot \varphi(y) \\
&\quad + 2\sum X_j\varphi(y)\cdot X_j h(\bar x^{-1}h, \bar t - t) \\
&\quad - \varphi(y)\frac{\partial}{\partial t} h(y^{-1}\bar x, \bar t - t) \\
&= \sum X_j^2\varphi(y)\cdot h(y^{-1}\bar x, \bar t - t) \\
&\quad + 2\sum X_j\varphi(y)\cdot X_j h(\bar x^{-1}y, \bar t - t)\,.
\end{aligned}
$$

Therefore, using Theorem 4 (i),

$$
\begin{aligned}
\left|\sum X_j^2 v + \frac{\partial}{\partial t} v\right| &\leqq |\sum X_j^2\varphi|\,|h(y^{-1}\bar x, \bar t - t)| \\
&\quad + 2\,|\sum X_j\varphi|\,|\sum X_j h(\bar x^{-1}y, \bar t - t)| \\
&\leqq C(|y^{-1}\bar x|^{-\rho} + |\bar x^{-1}y|^{-(\rho+1)})
\end{aligned}
\tag{3}
$$

if $y \notin \bar x B(r)$ for a large r. By the definition of φ,

$$
X_j^2 v + \frac{\partial}{\partial t} v = 0 \tag{4}
$$

if $y \in \bar x B(r)$.

Substituing (3) and (4) to (2) and using Hölder's inequality, we get

$$
\begin{aligned}
|u(\bar x, \bar t)| &\leqq C\int_0^{\bar t}\int_{\bar x B(r')\setminus \bar x B(r)} |u(y, t)|\,|y^{-1}\bar x|^{-\rho}\, dydt \\
&\leqq C\int_0^{\bar t} \|u(\cdot, t)\|_p dt \left(\int_{B(r')\setminus B(r)} |y|^{-\rho p'}dy\right)^{1/p'}
\end{aligned}
$$

if $1/p + 1/p' = 1$.

From the property (h) of the homogeneous norm,

$$\left(\int_{B(r')\setminus B(r)}|y|^{-\rho p'}dy\right)^{1/p'} = \begin{cases} C(r^{-\rho(1-1/p')} - r'^{-\rho(1-1/p')} & \text{if} \quad p' > 1 \\ C(\log(r'/r))^{1/p'} & \text{if} \quad p' = 1 \, . \end{cases}$$

In the case $p' > 1$, this integral converges to zero as $r \to \infty$. In the case $p' = 1$, $u(x, t)$ is bounded on $G \times [0, \bar{t}]$. As in the proof of [K. Saka, 19: Corollary 8], we get $u(x, t) = 0$ on $G \times [0, \bar{t}]$. Hence $u(x, t)$ is identically zero on $G \times [0, \infty)$.

LEMMA 2. *Suppose that a function $f(x)$ is continuous on G and $f \in L^p$, $1 \leqq p \leqq \infty$. Let $u(x, t) = H_t f(x)$. Then $u(x, t)$ converges to $f(x)$ uniformly on each compact subset of G as $t \to 0$.*

PROOF. Obvious from Theorem 1.

PROOF OF THEOREM 5. Let s be a positive number for which $\|u(\cdot, s)\|_p < \infty$. We put $v(x, t) = \int_G h(y^{-1}x, t)u(y, s)dy$, $t > 0$ and $v(x, 0) = u(x, s)$. Since $u(y, s)$ is continuous on G and $u(\cdot, s) \in L^p$, $v(x, t)$ is continuous on $G \times [0, \infty)$ by Lemma 2. Since $\|v(\cdot, t)\|_p \leqq \|u(\cdot, s)\|_p$ for all $t \geqq 0$, the function $t \mapsto \|v(\cdot, t)\|_p$ is locally integrable on $[0, \infty)$. On the other hand, it follows easily that the function $t \mapsto u(x, s + t)$ is continuous on $G \times [0, \infty)$ and the function $t \mapsto \|u(\cdot, s + t)\|_p$ is locally integrable on $[0, \infty)$. By Lemma 1, we obtain

$$v(x, t) = u(x, s + t) \, , \quad \text{that is} \, ,$$

$$u(x, s + t) = \int_G h(y^{-1}x, t)u(y, s)dy \, .$$

The theorem has been proved whenever $\|u(\cdot, s)\|_p < \infty$, i.e., for almost all $s > 0$, and hence for all $s > 0$.

By this theorem, Theorem 4 (ii) (iii) implies the following corollary:

COROLLARY. *Let $1 \leqq p \leqq \infty$ and let u be a temperature on $G \times (0, \infty)$ such that the function $t \mapsto \|u(\cdot, t)\|_p$ is locally integrable on $(0, \infty)$. Then*

(i) *for all $t, s > 0$,*

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, s + t)\right\|_p \leqq C t^{-(|\alpha|+2k)/2} \|u(\cdot, s)\|_p \, .$$

Also, if $1 \leqq p < r \leqq \infty$ and $\delta = \rho(1/p - 1/r)$ then

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot, s + t)\right\|_r \leqq C t^{-(|\alpha|+2k+\delta)/2} \|u(\cdot, s)\|_p \, .$$

(ii) *For each $t > 0$, the functions $x \mapsto u(x, t)$ and $x \mapsto (\partial^k/\partial t^k)u(x, t)$*

are uniformly continuous on G. Moreover, the functions $t \mapsto \|u(\cdot, t)\|_p$ and $t \mapsto \|(\partial^k/\partial t^k)u(\cdot, t)\|_p$ are decreasing on $(0, \infty)$ if $1 \leqq p \leqq \infty$, and are continuous on $(0, \infty)$ if $1 \leqq p < \infty$.

We now define a Bessel potential for a class of certain temperatures.

DEFINITION 1. Let $\mathfrak{T}$ denote the linear space of temperatures u on $G \times (0, \infty)$ with the property that if k is a nonnegative integer, $\alpha = (\alpha_1, \cdots, \alpha_l)$ an l-tuple of nonnegative integers and $c > 0$, then there exists $C > 0$ such that

$$\sup_{x \in G} \left| \frac{\partial^k}{\partial t^k} D^\alpha u(x, t) \right| \leqq C \quad \text{for all } t \geqq c .$$

For any real number α and for any $u \in \mathfrak{T}$ we define $J^\alpha u$ to be the function defined on $G \times (0, \infty)$ as follows:

(a) if $\alpha \geqq 0$,

$$J^{-\alpha} u(x, s) = \frac{1}{\Gamma(k - \alpha/2)} \int_0^\infty t^{k-\alpha/2-1} e^{-t} (1 + \mathfrak{J})^k u(x, s + t) dt$$

where k is a nonnegative integer such that $k > \alpha/2$,

(b) if $\alpha > 0$,

$$J^\alpha u(x, s) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{(\alpha/2)-1} e^{-t} u(x, s + t) dt .$$

Clearly $J^\alpha u(x, s)$ is well-defined for all real numbers α.

PROPOSITION 1. (i) *If $\alpha \geqq 0$, the definition of $J^{-\alpha}$ is independent of the choice of integers $k > \alpha/2$.*

(ii) *If k is a nonnegative integer, then $J^{-2k} = (1 + \mathfrak{J})^k$. In particular, $J^0 =$ the identity map.*

(iii) *For each real number α, J^α is a linear isomorphism of $\mathfrak{T}$ onto itself, with inverse $J^{-\alpha}$ and for all real α, β, $J^\alpha J^\beta = J^{\alpha+\beta}$.*

The proof of this proposition is almost identical to that of Theorem 8 in Flett's paper [4] and we omit it.

PROPOSITION 2. (i) *If $f \in L^p$, $1 \leqq p \leqq \infty$ and $u(x, t) = H_t f(x)$ then $u \in \mathfrak{T}$.*

(ii) *If $u(x, t)$ is a temperature on $G \times (0, \infty)$ and the function $t \mapsto \|u(\cdot, t)\|_p$ $(1 \leqq p \leqq \infty)$ is locally integrable on $(0, \infty)$ then $u \in \mathfrak{T}$.*

PROOF. This follows easily from Theorem 4 (ii) and Corollary (i) of Theorem 5.

PROPOSITION 3. (i) *Let $1 \leqq p \leqq \infty$ and let $f \in L^p$ and $u(x, t) =$*

$H_t f(x)$. *Then for each* $t > 0$

$$\|J^\alpha u(\cdot, t)\|_p \leqq \|f\|_p \qquad \text{if} \quad \alpha > 0\,,$$
$$\|J^{-\alpha} u(\cdot, t)\|_p \leqq C(1 + t^{-\alpha/2})\|f\|_p \qquad \text{if} \quad \alpha \geqq 0\,.$$

(ii) *Let* u *be a temperature on* $G \times (0, \infty)$ *such that* $t \mapsto \|u(\cdot, t)\|_p$ $(1 \leqq p \leqq \infty)$ *is locally integrable on* $(0, \infty)$. *Then for all* $s, t > 0$

$$\|J^\alpha u(\cdot, s + t)\|_p \leqq \|u(\cdot, s)\|_p \qquad \text{if} \quad \alpha > 0\,,$$
$$\|J^{-\alpha} u(\cdot, s + t)\|_p \leqq C(1 + t^{-\alpha/2})\|u(\cdot, s)\|_p \qquad \text{if} \quad \alpha \geqq 0\,.$$

Hence for any real α $\|J^\alpha u(\cdot, t)\|_p$ *is locally integrable and decreasing on* $(0, \infty)$ *if* $1 \leqq p \leqq \infty$ *and continuous on* $(0, \infty)$ *if* $1 \leqq p < \infty$. *Moreover for each real* α

$$J^\alpha u(x, s + t) = H_t J^\alpha u(x, s) = J^\alpha H_t u(x, s)\,.$$

We use Theorem 4 (i) and Theorem 5 to prove this proposition following the arguments of Flett [4: Theorem 10 and its corollary], and the last part of (ii) follows easily from Fubini's theorem and the fact that u is a temperature.

LEMMA 3. *Let* $1 \leqq p \leqq \infty$ *and let* u *be a temperature on* $G \times (0, \infty)$.

(i) *If for* $\beta > 0$ *and* $1 \leqq q < \infty$,

$$\int_0^\infty t^{q\beta-1} e^{-t} \|u(\cdot, t)\|_p^q dt < \infty\,,$$

then for $t > 0$

$$\|u(\cdot, t)\|_p \leqq C(1 + t^{-\beta}) \left\{\int_0^\infty t^{q\beta-1} e^{-t} \|u(\cdot, t)\|_p^q dt\right\}^{1/q}$$

and $\|u(\cdot, t)\|_p = o(t^{-\beta})$ *as* $t \to 0$. *Also, if* $q < r < \infty$ *then*

$$\left\{\int_0^\infty t^{r\beta-1} e^{-t} \|u(\cdot, t)\|_p^r dt\right\}^{1/r} \leqq C \left\{\int_0^\infty t^{q\beta-1} e^{-t} \|u(\cdot, t)\|_p^q dt\right\}^{1/q}.$$

(ii) *Let* $1 \leqq q < \infty$, α *real and* $\alpha < \beta$, *and let* u *be a temperature on* $G \times (0, \infty)$ *such that* $\int_0^\infty t^{q\beta/2-1} e^{-t} \|u(\cdot, t)\|_p^q dt < \infty$ *then* $u \in \mathfrak{T}$ *and*

$$\left\{\int_0^\infty t^{q(\beta-\alpha)/2-1} e^{-t} \|J^\alpha u(\cdot, t)\|_p^q dt\right\}^{1/q} \leqq C \left\{\int_0^\infty t^{q\beta/2-1} e^{-t} \|u(\cdot, t)\|_p^q dt\right\}^{1/q}.$$

(iii) *Let* α *real,* $\alpha < \beta$. *If*

$$\sup_{t>0} \{t^{\beta/2} e^{-t} \|u(\cdot, t)\|_p\} < \infty$$

then $u \in \mathfrak{T}$ *and*

$$\sup_{t>0}\{t^{(\beta-\alpha)/2}e^{-t}\|J^\alpha u(\cdot, t)\|_p\} \leqq C\sup_{t>0}\{t^{\beta/2}e^{-t}\|u(\cdot, t)\|_p\} .$$

Moreover, if in addition $\|u(\cdot, t)\|_p = o(t^{-\beta/2})$ *as* $t\to 0$ *then* $\|J^\alpha u(\cdot, t)\|_p = o(t^{-(\beta-\alpha)/2})$ *as* $t\to 0$.

Flett has proved this lemma in the case of n-dimensional Euclidean space by using the following two lemmas (see [4: Theorem 11, Theorem 12]. Note that β is not necessarily positive). The proof is applicable to our case.

LEMMA 4. *Let* $\delta > 0$ *and let* φ *be a decreasing nonnegative function on* $(0, \infty)$ *such that* $\int_0^\infty t^{\delta-1}e^{-t}\varphi(t)dt < \infty$. *Then*

(i) $\varphi(t) \leqq e\delta(t^{-\delta}+1)\int_0^\infty t^{\delta-1}e^{-t}\varphi(t)dt$ *for all* $t>0$, *and* $\varphi(t) = o(t^{-\delta})$ *as* $t\to 0$.

(ii) *for all* $q \geqq 1$

$$\int_0^\infty t^{\delta-1}e^{-t}\varphi(t)dt \leqq C\int_0^1 t^{\delta-1}e^{-qt}\varphi(t)dt .$$

LEMMA 5 (Hardy's inequality). *If* $1\leqq q<\infty$, $r<1$, *and* h *is measurable and nonnegative on* $(0, \infty)$ *then*

$$\int_0^\infty s^{-r}\left\{\int_s^\infty h(t)dt\right\}^q ds \leqq (q/(1-r))^q \int_0^\infty t^{-r+q}h^q(t)dt .$$

The following lemma is used later.

LEMMA 6 (Flett [3], [4]). (i) *Let* $1<p<r<\infty$, $p\leqq q<\infty$, $\delta = \rho(1/p-1/r)$ *and let* $f\in L^p$ *and* $u(x, t) = H_t f(x)$. *Then*

$$\left\{\int_0^\infty t^{q\delta/2-1}e^{-t}\|u(\cdot, t)\|_r^q dt\right\}^{1/q} \leqq C\|f\|_p .$$

(ii) *Let* $1<p<r<\infty$, $1<q\leqq r$, $\delta=\rho(1/p-1/r)$ *and* $\alpha>0$, $\alpha>\delta$. *Suppose that* $u(x, t)$ *is a temperature on* $G\times(0, \infty)$ *such that*

$$\int_0^\infty t^{q(\alpha-\delta)/2-1}e^{-t}\|u(\cdot, t)\|_p^q dt < \infty .$$

Then $u\in\mathfrak{T}$ *and*

$$\|J^\alpha u(\cdot, s)\|_r \leqq C\left\{\int_0^\infty t^{q(\alpha-\delta)/2-1}e^{-t}\|u(\cdot, t)\|_p^q dt\right\}^{1/q} .$$

Hence there exists a function $f\in L^r$ *such that* $J^\alpha u = H_t f$ *and*

$$\|f\|_r \leqq C\left\{\int_0^\infty t^{q(\alpha-\delta)/2-1}e^{-t}\|u(\cdot, t)\|_p^q dt\right\}^{1/q} .$$

3. Besov spaces. We shall give a definition of the Besov spaces on

G and its alternative representations equivalent to the original definition. Most of these are listed in Flett [4] and Taibleson [22] in the case of n-dimensional Euclidean spaces.

DEFINITION 2. We define the space $\mathfrak{T}\Lambda(\alpha;\ p, q)$, where α is real, $1 \leqq p \leqq \infty$ and $1 \leqq q < \infty$, to be the space of temperatures $u \in \mathfrak{T}$ for which

$$\int_0^\infty (t\| J^{-\alpha-2}u(\cdot, t)\|_p)^q e^{-t} t^{-1} dt < \infty ,$$

equipped with the norm

$$\|u\|_{\alpha;p,q} = \left\{\int_0^\infty (t\,\|J^{-\alpha-2}u(\cdot, t)\|_p)^q e^{-t} t^{-1} dt\right\}^{1/q} .$$

We define also the space $\mathfrak{T}\Lambda(\alpha; p, \infty)$, where α is real and $1 \leqq p \leqq \infty$, to be the space of temperatures $u \in \mathfrak{T}$ for which $\sup_{t>0}\{te^{-t}\|J^{-\alpha-2}u(\cdot, t)\|_p\} < \infty$, equipped with the norm

$$\|u\|_{\alpha;p,\infty} = \sup_{t>0}\{te^{-t}\|J^{-\alpha-2}u(\cdot, t)\|_p\} .$$

We denote by $\mathfrak{T}\lambda(\alpha; p, \infty)$, the subspace of those temperatures $u \in \mathfrak{T}\Lambda(\alpha; p, \infty)$ for which $\|J^{-\alpha-2}u(\cdot, t)\|_p = o(t^{-1})$ as $t \to 0$. It is easily verified that the subspace $\mathfrak{T}\lambda(\alpha; p, \infty)$ is a closed subspace of the space $\mathfrak{T}\Lambda(\alpha; p, \infty)$.

THEOREM 6. *Let α and β be real with $\beta > \alpha$, $1 \leqq p \leqq \infty$. Then*
(i) *if $1 \leqq q < \infty$,*

$$\mathfrak{T}\Lambda(\alpha; p, q) = \left\{u \in \mathfrak{T}\colon \int_0^\infty (t^{(\beta-\alpha)/2}\|J^{-\beta}u(\cdot, t)\|_p)^q e^{-t} t^{-1} dt < \infty\right\} .$$

Moreover, the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\left\{\int_0^\infty (t^{(\beta-\alpha)/2}\|J^{-\beta}u(\cdot, t)\|_p)^q e^{-t} t^{-1} dt\right\}^{1/q} .$$

(ii) $\mathfrak{T}\Lambda(\alpha; p, \infty) = \{u \in \mathfrak{T}\colon \sup_{t>0}\{t^{(\beta-\alpha)/2}e^{-t}\|J^{-\beta}u(\cdot, t)\|_p\} < \infty\}$.
Moreover, the norm $\|\cdot\|_{\alpha;p,\infty}$ is equivalent to

$$\sup_{t>0}\{t^{(\beta-\alpha)/2}e^{-t}\|J^{-\beta}u(\cdot, t)\|_p\}$$

(iii) $\mathfrak{T}\lambda(\alpha; p, \infty) = \{u \in \mathfrak{T}\Lambda(\alpha; p, \infty)\colon \|J^{-\beta}u(\cdot, t)\|_p = o(t^{-(\beta-\alpha)/2})$ *as* $t \to 0\}$.

This theorem follows directly from Lemma 3 (cf. Flett [4: Lemma 12]).

THEOREM 7. *Let α, β be real and let $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$. Then J^β is a linear homeomorphism of $\mathfrak{T}\Lambda(\alpha; p, q)$ onto $\mathfrak{T}\Lambda(\alpha + \beta; p, q)$ and of*

$\mathfrak{T}\lambda(\alpha; p, \infty)$ onto $\mathfrak{T}\lambda(\alpha+\beta; p, \infty)$.

This theorem is immediate from Definition 3 (cf. Flett [4: Theorem 18]).

LEMMA 7. *If $u \in \mathfrak{T}\Lambda(\alpha; p, q)$, where α is real, $1 \leqq p \leqq \infty$ and $1 \leqq q \leqq \infty$, then*

(i) *for each $t > 0$,*

$$\|J^{-\alpha-2}u(\cdot, t)\|_p \leqq C(1+t^{-1})\|u\|_{\alpha;p,q},$$

(ii) *if β is real and $c > 0$, then for all $t \geqq c$,*

$$\|J^{\beta}u(\cdot, t)\|_p \leqq C\|u\|_{\alpha;p,q}.$$

Hence $\|J^{\beta}u(\cdot, t)\|_p$ is locally integrable and decreasing on $(0, \infty)$ if $1 \leqq p \leqq \infty$ and continuous on $(0, \infty)$ if $1 \leqq p < \infty$.

This is verified easily from Proposition 3 (ii) and Lemma 3 (i) (cf. Flett [4: Lemma 10 and Lemma 11]).

THEOREM 8. (i) *Let α be real and $\beta > \alpha$, and let $1 \leqq p \leqq \infty$. Then if $1 \leqq q < \infty$,*

$$\mathfrak{T}\Lambda(\alpha; p, q) = \left\{u \in \mathfrak{T}: \sup_{t \geqq 1/2}\|u(\cdot, t)\|_p + \left\{\int_0^1 (t^{(\beta-\alpha)/2}\|J^{-\beta}u(\cdot, t)\|_p)^q t^{-1}dt\right\}^{1/q} < \infty\right\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\sup_{t \geqq 1/2}\|u(\cdot, t)\|_p + \left\{\int_0^1 (t^{(\beta-\alpha)/2}\|J^{-\beta}(\cdot, t)\|_p)^q t^{-1}dt\right\}^{1/q}.$$

In the case $q = \infty$,

$$\mathfrak{T}\Lambda(\alpha; p, \infty) = \{u \in \mathfrak{T}: \sup_{t \geqq 1/2}\|u(\cdot, t)\|_p + \sup_{0<t\leqq 1}\{t^{(\beta-\alpha)/2}\|J^{-\beta}u(\cdot, t)\|_p\} < \infty\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,\infty}$ is equivalent to

$$\sup_{t \geqq 1/2}\|u(\cdot, t)\|_p + \sup_{0<t\leqq 1}\{t^{(\beta-\alpha)/2}\|J^{-\beta}u(\cdot, t)\|_p\}.$$

(ii) *Let α be real and let k be a nonnegative integer with $k > \alpha/2$ and $1 \leqq p \leqq \infty$. Then if $1 \leqq q < \infty$,*

$$\mathfrak{T}\Lambda(\alpha; p, q) = \left\{u \in \mathfrak{T}: \sup_{t \geqq 1/2}\|u(\cdot, t)\|_p + \left\{\int_0^1 \left(t^{k-\alpha/2}\left\|\frac{\partial^k}{\partial t^k}u(\cdot, t)\right\|_p\right)^q t^{-1}dt\right\}^{1/q} < \infty\right\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\sup_{t\geqq 1/2}||u(\cdot,t)||_p+\left\{\int_0^1\left(t^{k-\alpha/2}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p\right)^q t^{-1}dt\right\}^{1/q}.$$

In the case $q=\infty$,

$$\mathfrak{T}\Lambda(\alpha;p,\infty)=\left\{u\in\mathfrak{T}:\sup_{t\geqq 1/2}||u(\cdot,t)||_p+\sup_{0<t\leqq 1}\left\{t^{k-\alpha/2}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p\right\}<\infty\right\}.$$

Moreover the norm $||\cdot||_{\alpha;p,\infty}$ is equivalent to

$$\sup_{t\geqq 1/2}||u(\cdot,t)||_p+\sup_{0<t\leqq 1}\left\{t^{k-\alpha/2}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p\right\}$$

and

$$\mathfrak{T}\lambda(\alpha;p,\infty)$$
$$=\left\{u\in\mathfrak{T}\Lambda(\alpha;p,\infty):\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p=o(t^{-(k-\alpha/2)})\ \ as\ t\to 0\right\}.$$

Part (i) follows from Proposition 3 (ii), Theorem 6 and Lemma 7 (cf. Flett [4: Lemma 13]). For the proof of part (ii), see Flett [4: Lemma 14].

DEFINITION 3. Let α be a positive number and $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$. We define the space $\Lambda(\alpha;p,q)$ by

$$\Lambda(\alpha;p,q)=\left\{f\in L^p:\int_0^\infty(t\,||J^{-\alpha-2}H_tf||_p)^q e^{-t}t^{-1}dt<\infty\right\}$$

if $1\leqq q<\infty$, and

$$\Lambda(\alpha;p,\infty)=\{f\in L^p:\sup_{t>0}\{te^{-t}\,||J^{-\alpha-2}H_tf||_p\}<\infty\}$$

equipped with the norms

$$||f||_{\alpha;p,q}=\left\{\int_0^\infty(t\,||J^{-\alpha-2}H_tf\,||_p)^q e^{-t}t^{-1}dt\right\}^{1/q}$$

and

$$||f||_{\alpha;p,\infty}=\sup_{t>0}\{te^{-t}\,||J^{-\alpha-2}H_tf\,||_p\}$$

respectively.

We also define the subspace $\lambda(\alpha;p,\infty)$ of $\Lambda(\alpha;p,\infty)$ by

$$\lambda(\alpha;p,\infty)=\{f\in\Lambda(\alpha;p,\infty):||J^{-\alpha-2}H_tf\,||_p=o(t^{-1})\ \text{ as }\ t\to 0\}.$$

REMARK. The referee showed me that the space $\Lambda(\alpha;\infty,\infty)$ coincides with the Lipschitz space Γ_α defined in Folland [5, p. 193].

THEOREM 9. *Let α be a positive number and let $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$.*

Then the map $f \mapsto H_t f$ is an isometric isomorphism of $\Lambda(\alpha; p, q)$ onto $\mathfrak{T}\Lambda(\alpha; p, q)$.

PROOF. Although this proof is similar to the one of Flett [4: Theorem 19], we exhibit it in order to show a corollary below.

If $f \in \Lambda(\alpha; p, q)$ and $u(x, t) = H_t f(x)$, then by Proposition 2 (i) we have $u \in \mathfrak{T}$ and $\|u\|_{\alpha;p,q} = \|f\|_{\alpha;p,q}$. Conversely, suppose that $u \in \mathfrak{T}\Lambda(\alpha; p, q)$. Put $v = J^{-\alpha-2}u$, then

$$u(x, s) = J^{\alpha+2}v(x, s) = \frac{1}{\Gamma(\alpha/2+1)}\int_0^\infty t^{\alpha/2}e^{-t}v(x, s+t)dt .$$

Hence, using Lemma 7 (i),

$$\begin{aligned} \|u(\cdot, s)\|_p &\leqq C\int_0^\infty t^{\alpha/2}e^{-t}\|v(\cdot, s+t)\|_p dt \\ &\leqq C\int_0^\infty t^{\alpha/2}e^{-t}(1+(s+t)^{-1})dt\cdot\|u\|_{\alpha;p,q} = C\|u\|_{\alpha;p,q} . \end{aligned}$$

From Theorem 3 there exists $f \in L^p$ such that $u(x, t) = H_t f(x)$ and $\|f\|_p \leqq C\|u\|_{\alpha;p,q} = C\|f\|_{\alpha;p,q}$ if $1 < p$. To apply Theorem 3 for $p = 1$ we have to check that $\|u(\cdot, t) - u(\cdot, t')\|_1 \to 0$ as $t, t' \to 0$. For this, see Flett [4].

COROLLARY. *Let α be a positive number and $\beta > \alpha$, and let $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$. If $f \in \Lambda(\alpha; p, q)$ then*

$$\|f\|_p \leqq C\|f\|_{\alpha;p,q} .$$

THEOREM 10. *Let $\alpha > 0$ and let $\beta > \alpha$, $1 \leqq p \leqq \infty$. Then,*
(i) *if $1 \leqq q < \infty$,*

$$\Lambda(\alpha; p, q) = \left\{f \in L^p\colon \int_0^\infty (t^{(\beta-\alpha)/2}\|J^{-\beta}H_t f\|_p)^q e^{-t}t^{-1}dt < \infty\right\} .$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\left\{\int_0^\infty (t^{(\beta-\alpha)/2}\|J^{-\beta}H_t f\|_p)^q e^{-t}t^{-1}dt\right\}^{1/q} .$$

If $q = \infty$,

$$\Lambda(\alpha; p, \infty) = \{f \in L^p\colon \sup_{t>0}\{t^{(\beta-\alpha)/2}e^{-t}\|J^{-\beta}H_t f\|_p\} < \infty\} .$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\sup_{t>0}\{t^{(\beta-\alpha)/2}e^{-t}\|J^{-\beta}H_t f\|_p\} ,$$

and

$$\lambda(\alpha; p, \infty) = \{f \in L^p\colon \|J^{-\beta}H_t f\|_p = o(t^{-(\beta-\alpha)/2}) \text{ as } t \to 0\} .$$

(ii) *Let k be a positive integer with $k > \alpha/2$. If $1 \leqq q < \infty$,*

$$\Lambda(\alpha; p, q) = \left\{ f \in L^p \colon \int_0^\infty \left(t^{k-\alpha/2} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p \right)^q t^{-1} dt < \infty \right\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ is equivalent to

$$\|f\|_p + \left\{ \int_0^\infty \left(t^{k-\alpha/2} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p \right)^q t^{-1} dt \right\}^{1/q}.$$

If $q = \infty$,

$$\Lambda(\alpha; p, \infty) = \left\{ f \in L^p \colon \sup_{t>0} \left\{ t^{k-\alpha/2} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p \right\} < \infty \right\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,\infty}$ is equivalent to

$$\|f\|_p + \sup_{t>0} \left\{ t^{k-\alpha/2} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p \right\},$$

and

$$\lambda(\alpha; p, \infty) = \left\{ f \in L^p \colon \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p = o(t^{-(k-\alpha/2)}) \text{ as } t \to 0 \right\}.$$

PROOF. The part (i) follows from Lemma 3.

To prove the part (ii) let $1 \leqq p < \infty$ and $f \in L^p$. Put $u(x, t) = H_t f(x)$. We suppose that $\int_0^\infty t^{q(k-\alpha/2)-1} \|(\partial^k/\partial t^k) u(\cdot, t)\|_p^q dt < \infty$ where $k > \alpha/2$ is an integer. Then

$$\sup_{t \geqq 1/2} \|u(\cdot, t)\|_p + \left\{ \int_0^1 t^{q(k-\alpha/2)-1} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p^q dt \right\}^{1/q}$$
$$\leqq \|f\|_p + \left\{ \int_0^\infty t^{q(k-\alpha/2)-1} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p^q dt \right\}^{1/q} < \infty.$$

By Theorem 8,

$$\|f\|_{\alpha;p,q} \leqq \|f\|_p + \left\{ \int_0^\infty t^{q(k-\alpha/2)-1} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p^q dt \right\}^{1/q}.$$

On the other hand, we suppose $f \in \Lambda(\alpha; p, q)$ and $u(x, t) = H_t f(x)$. By Corollary (i) of Theorem 5, we get

$$\left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p \leqq C t^{-k} \sup_{t \geqq 1/2} \|u(\cdot, t)\|_p.$$

Hence, using Theorem 8,

(1) $$\left\{ \int_1^\infty t^{q(k-\alpha/2)-1} \left\| \frac{\partial^k}{\partial t^k} u(\cdot, t) \right\|_p^q dt \right\}^{1/q}$$

$$\leqq C\left\{\int_1^\infty t^{q(k-\alpha/2)-1}t^{-kq}dt\right\}^{1/q}\sup_{t\geqq 1/2}\|u(\cdot,t)\|_p$$
$$= C\sup_{t\geqq 1/2}\|u(\cdot,t)\|_p \leqq C\|f\|_{\alpha;p,q}$$

and

$$(2)\qquad \left\{\int_0^1 t^{q(k-\alpha/2)-1}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p^q dt\right\}^{1/q} \leqq C\|f\|_{\alpha;p,q}\,.$$

Combining (1) and (2), we get

$$\left\{\int_0^\infty t^{q(k-\alpha/2)-1}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p^q dt\right\}^{1/q} \leqq C\|f\|_{\alpha;p,q}$$

so that by Corollary of Theorem 9,

$$\|f\|_p + \left\{\int_0^\infty t^{q(k-\alpha/2)-1}\left\|\frac{\partial^k}{\partial t^k}u(\cdot,t)\right\|_p^q dt\right\}^{1/q} \leqq C\|f\|_{\alpha;p,q}\,.$$

This completes the proof for $1\leqq p<\infty$. For $p=\infty$, the proof is similar to the above.

LEMMA 8 (Muramatsu [17]). *Let (M_1,μ_1), (M_2,μ_2) be two σ-finite measure spaces, and let $K(x,y)$ be a $(\mu_1\times\mu_2)$-measurable function such that*

$$\int_{M_1}|K(x,y)|^r d\mu_1(x)\leqq C_1^r \quad \textit{for almost all } y\in M_2\,,$$
$$\int_{M_2}|K(x,y)|^r d\mu_2(y)\leqq C_2^r \quad \textit{for almost all } x\in M_1\,,$$

$(1\leqq r\leqq\infty)$.

Then the integralo perator $T\colon f\mapsto\int_{M_2}K(x,y)f(y)d\mu_2(y)$ is a bounded linear operator from $L^p(M_2,\mu_2)$ into $L^q(M_1,\mu_1)$ with $\|T\|\leqq C_1^{1-r/q}C_2^{r/q}$, where $1/r+1/p-1/q=1$. In particular if

$$|K(x,y)|\leqq C \textit{ for almost all } (x,y)\in M_1\times M_2\,,$$
$$\int_{M_1}|K(x,y)|d\mu_1(x)\leqq C \textit{ for almost all } y\in M_2\,,$$

and

$$\int_{M_2}|K(x,y)|d\mu_2(y)\leqq C \textit{ for almost all } x\in M_1\,,$$

then the integral operator T with the kernel $K(x,y)$ is bounded from $L^p(M_2,\mu_2)$ into $L^q(M_1,\mu_1)$, where $1\leqq p\leqq q\leqq\infty$.

THEOREM 11. *If $0<\alpha<2$ and $1\leqq p\leqq\infty$, then for $1\leqq q<\infty$,*

$$\Lambda(\alpha; p, q) = \left\{f \in L^p\colon \int_0^\infty (t^{-\alpha/2}\|H_t f(x) - f(x)\|_p)^q \frac{dt}{t} < \infty\right\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ *is equivalent to*

$$\|f\|_p + \left(\int_0^\infty (t^{-\alpha/2}\|H_t f(x) - f(x)\|_p)^q \frac{dt}{t}\right)^{1/q}.$$

For $q = \infty$,

$$\Lambda(\alpha; p, \infty) = \{f \in L^p\colon \sup_{t>0}(t^{-\alpha/2}\|H_t f(x) - f(x)\|_p) < \infty\}.$$

Moreover the norm $\|\cdot\|_{\alpha;p,\infty}$ *is equivalent to*

$$\|f\|_p + \sup_{t>0}(t^{-\alpha/2}\|H_t f(x) - f(x)\|_p).$$

PROOF (cf. Taibleson [22: Theorem 4]). If $f \in L^p$ and $u(x, t) = H_t f(x)$, then from Theorem 2

$$\|u(x, t) - f(x)\|_p \leqq \int_0^t \left\|\frac{\partial u}{\partial t}(x, s)\right\|_p ds, \quad 1 \leqq p \leqq \infty. \tag{1}$$

Let $1 \leqq p < \infty$ and let f be any element in $\Lambda(\alpha; p, q)$ $(0 < \alpha < 2)$. Then substituting (1) to the below,

$$\begin{aligned}\left(\int_0^\infty (t^{-\alpha/2}\|u(x, t) - f(x)\|_p)^q \frac{dt}{t}\right)^{1/q} &\leqq \left(\int_0^\infty \left(t^{-\alpha/2}\int_0^t \left\|\frac{\partial u}{\partial t}(x, s)\right\|_p ds\right)^q \frac{dt}{t}\right)^{1/q} \\ &= \left\{\int_0^\infty \left(\int_0^t t^{-\alpha/2}s^{\alpha/2}\left(s^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, s)\right\|_p\right)\frac{ds}{s}\right)^q \frac{dt}{t}\right\}^{1/q}.\end{aligned}$$

Taking $K(t, s) = t^{-\alpha/2}s^{\alpha/2}\chi_{(0,t]}$, $d\mu_1 = t^{-1}dt$ and $d\mu_2 = s^{-1}ds$ in Lemma 8 we obtain

$$\begin{aligned}&\left(\int_0^\infty (t^{-\alpha/2}\|u(x, t) - f(x)\|_p)^q \frac{dt}{t}\right)^{1/q} \\ &\qquad \leqq C\left(\int_0^\infty \left(s^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, s)\right\|_p\right)^q \frac{ds}{s}\right)^{1/q} < \infty.\end{aligned}$$

Conversely, suppose that

$$\int_0^\infty (t^{-\alpha/2}\|u(x, t) - f(x)\|_p)^q \frac{dt}{t} < \infty.$$

Since $\|(\partial u/\partial t)(x, t)\|_p \to 0$ as $t \to \infty$ by Theorem 4(ii),

$$\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p = \lim_{N\to\infty}\left\|\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(x, 2^N t)\right\|_p. \tag{2}$$

By Theorem 4 (i), we observe that

$$
\begin{aligned}
&t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(x, 2^N t)\right\|_p \\
&\leqq t^{1-\alpha/2}\sum_{k=1}^{N}\left\|\frac{\partial u}{\partial t}(x, 2^{k-1}t) - \frac{\partial u}{\partial t}(x, 2^k t)\right\|_p \\
&= t^{1-\alpha/2}\sum_{k=1}^{N}\left\|\frac{\partial h}{\partial t}(x, 2^{k-1}t)*(f(x) - u(x, 2^{k-1}t))\right\|_p \\
&\leqq t^{1-\alpha/2}\sum_{k=1}^{N}\left\|\frac{\partial h}{\partial t}(x, 2^{k-1}t)\right\|_1 \|f(x) - u(x, 2^{k-1}t)\|_p \\
&\leqq Ct^{1-\alpha/2}\sum_{k=1}^{N} 2^{1-k}t^{-1}\|f(x) - u(x, 2^{k-1}t)\|_p \\
&= C\sum_{k=1}^{N}(2^{\alpha/2-1})^{k-1}(2^{k-1}t)^{-\alpha/2}\|f(x) - u(x, 2^{k-1}t)\|_p\,.
\end{aligned}
$$

From the fact that

$$
\begin{aligned}
&\left(\int_0^\infty((2^{k-1}t)^{-\alpha/2}\|f(x) - u(x, 2^{k-1}t)\|_p)^q\frac{dt}{t}\right)^{1/q} \\
&\qquad = \left(\int_0^\infty(t^{-\alpha/2}\|f(x) - u(x, t)\|_p)^q\frac{dt}{t}\right)^{1/q}
\end{aligned}
$$

for all $k = 1, 2, \cdots$, it follows that

$$
\begin{aligned}
&\left(\int_0^\infty\left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(x, 2^N t)\right\|_p\right)^q\frac{dt}{t}\right)^{1/q} \\
&\qquad \leqq C\sum_{k=1}^{\infty}(2^{\alpha/2-1})^{k-1}\left(\int_0^\infty\left(t^{-\alpha/2}\|f(x) - u(x, t)\|_p\right)^q\frac{dt}{t}\right)^{1/q} < \infty\,.
\end{aligned}
$$

Hence by Fatou's lemma and (2),

$$
\begin{aligned}
&\left(\int_0^\infty\left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)^q\frac{dt}{t}\right)^{1/q} \\
&\qquad \leqq C\left(\int_0^\infty(t^{-\alpha/2}\|f(x) - u(x, t)\|_p)^q\frac{dt}{t}\right)^{1/q} < \infty
\end{aligned}
$$

so that $f \in \Lambda(\alpha; p, q)$ $(0 < \alpha < 2)$. Moreover the norm $\|f\|_{\alpha;p,q}$ is equivalent to $\|f\|_p + \left(\int_0^\infty(t^{-\alpha/2}\|u(x, t) - f(x)\|_p)^q t^{-1}dt\right)^{1/q}$. The proof for $p = \infty$ is similar to the above.

LEMMA 9 (R. Johnson [8]). *Let $h(t)$ be a nonnegative, decreasing function on $(0, \infty)$, α real, and $0 < p \leqq q < \infty$. Then*

$$
\left\{\int_0^\infty(t^\alpha h(t))^q\frac{dt}{t}\right\}^{1/q} \leqq C\left\{\int_0^\infty(t^\alpha h(t))^p\frac{dt}{t}\right\}^{1/p}\,.
$$

For $q = \infty$,

$$h(t) \leqq Ct^{-\alpha} \left\{ \int_0^\infty (t^\alpha h(t))^p \frac{dt}{t} \right\}^{1/p} .$$

LEMMA 10. *Let* $1 \leqq p \leqq \infty$, *if* $f \in L^p \cap C^\infty$ *such that* $X_j f \in L^p$, $j = 1, \cdots, n$ *then*

$$||f(xy) - f(x)||_p \leqq C|y| \sum_{j=1}^n ||X_j f||_p .$$

PROOF (cf. G. B. Folland [5: Proposition (5, 4)]). Suppose $y = \exp Y$ with $Y \in V_1$, then $f(xy) - f(x) = \int_0^1 Yf(x \exp tY)dt$, so that $||f(xy) - f(x)||_p \leqq ||Yf||_p \leqq |y| \sum ||X_j f||_p$. Next, given any $y \in G$, write $y = \prod_1^N y_i$ with $y_i \in \exp V_1$ and $|y_i| \leqq C|y|$, $i = 1, \cdots, N$. Then

$$\begin{aligned} f(xy) - f(x) = (f(xy_1 \cdots y_N) - f(xy_1 \cdots y_{N-1})) \\ + \cdots + (f(xy_1y_2) - f(xy_1)) + (f(xy_1) - f(x)) \end{aligned}$$

so that

$$||f(xy) - f(x)||_p \leqq C \sum_i^N |y_i| \left(\sum_j^n ||X_j f||_p \right) \leqq C|y| \sum_j ||X_j f||_p .$$

THEOREM 12. *Let* $1 \leqq p \leqq \infty$.

(i) *For* $0 < \alpha < 1$,

$$\Lambda(\alpha; p, q) = \left\{ f \in L^p \colon \int_G (|y|^{-\alpha} ||f(xy) - f(x)||_p)^q |y|^{-\rho} dy < \infty \right\}$$

when $1 \leqq q < \infty$

and

$$\Lambda(\alpha; p, \infty) = \{ f \in L^p \colon \sup_{|y|>0} \{|y|^{-\alpha} ||f(xy) - f(x)||\}_p < \infty \} \text{ when } q = \infty .$$

Moreover the norms $||\cdot||_{\alpha;p,q}$ *and* $||\cdot||_{\alpha;p,\infty}$ *are equivalent to*

$$||f||_p + \left(\int_G (|y|^{-\alpha} ||f(xy) - f(x)||_p)^q |y|^{-\rho} dy \right)^{1/q}$$

and

$$||f||_p + \sup_{|y|>0} \{|y|^{-\alpha} ||f(xy) - f(x)||_p\}$$

respectively, and

$$\lambda(\alpha; p, \infty) = \{ f \in L^p \colon ||f(xy) - f(x)||_p = o(|y|^\alpha) \text{ as } |y| \to 0 \} .$$

(ii) *For* $0 < \alpha < 2$,

$$\Lambda(\alpha; p, q) = \left\{ f \in L^p \colon \int_G (|y|^{-\alpha} ||f(xy) + f(xy^{-1}) - 2f(x)||_p)^q |y|^{-\rho} dy < \infty \right\}$$

when $1 \leqq q < \infty$ *and*

$$\Lambda(\alpha; p, \infty) = \{f \in L^p: \sup_{|y|>0} \{|y|^{-\alpha}\|f(xy) + f(xy^{-1}) - 2f(x)\|_p\} < \infty\}$$

when $q = \infty$.

Moreover the norms $\|\cdot\|_{\alpha;p,q}$ *and* $\|\cdot\|_{\alpha;p,\infty}$ *are equivalent to*

$$\|f\|_p + \left(\int_G (|y|^{-\alpha}\|f(xy) + f(xy^{-1}) - 2f(x)\|_p)^q |y|^{-\rho} dy\right)^{1/q}$$

and

$$\|f\|_p + \sup_{|y|>0} \{|y|^{-\alpha}\|f(xy) + f(xy^{-1}) - 2f(x)\|_p\}$$

respectively, and

$$\lambda(\alpha; p, \infty) = \{f \in L^p: \|f(xy) + f(xy^{-1}) - 2f(x)\|_p = o(|y|^\alpha) \text{ as } |y| \to 0\} .$$

PROOF (cf. E. M. Stein [20: Chapter V §4, §5]). Let $f \in L^p$ and put $u(x, t) = H_t f(x)$. To prove the part (i) we assume $0 < \alpha < 1$. Note that $\int_G (\partial h_t/\partial t)(x) dx = 0$. Thus

$$\frac{\partial u}{\partial t}(x, t) = \int_G \frac{\partial h_t}{\partial t}(y)(f(xy) - f(x)) dy .$$

Hence

$$\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p \leqq \int_G \left|\frac{\partial h_t}{\partial t}(y)\right| \|f(xy) - f(x)\|_p dy . \tag{1}$$

Put $\omega_p(y) = \|f(xy) - f(x)\|_p$ and suppose that $\int_G (|y|^{-\alpha}\|f(xy) - f(x)\|_p)^q |y|^{-\rho} dy < \infty$. We see from Theorem 4 (i) and (1) that

$$\begin{aligned} t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p &\leqq t^{1-\alpha/2}\left(\int_{|y|^2 \geq t} \left|\frac{\partial h_t}{\partial t}(y)\right| \|f(xy) - f(x)\|_p dy \right. \\ &\qquad \left. + \int_{|y|^2 < t} \left|\frac{\partial h_t}{\partial t}(y)\right| \|f(xy) - f(x)\|_p dy\right) \\ &\leqq C t^{1-\alpha/2} \int_{|y|^2 \geq t} |y|^{-(\rho+2)} \omega_p(y) dy \\ &\qquad + C t^{-(\rho+\alpha)/2} \int_{|y|^2 < t} \omega_p(y) dy . \end{aligned}$$

Hence

$$\begin{aligned} &\left(\int_0^\infty \left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \\ &\leqq C\left(\int_0^\infty \left(t^{1-\alpha/2}\int_{|y|^2 \geq t} |y|^{-(\rho+2)} \omega_p(y) dy\right)^q \frac{dt}{t}\right)^{1/q} \end{aligned}$$

$$+ C\Big(\int_0^\infty\Big(t^{-(\rho+\alpha)/2}\int_{|y|^2<t}\omega_p(y)dy\Big)^q \frac{dt}{t}\Big)^{1/q}$$

$$= C\Big(\int_0^\infty\Big(\int_G t^{1-\alpha/2}\chi_{\{|y|^2\geqq t\}}|y|^{-2+\alpha}(\omega_p(y)|y|^{-\alpha})|y|^{-\rho}dy\Big)^q \frac{dt}{t}\Big)^{1/q}$$

$$+ \Big(\int_0^\infty\Big(\int_G t^{-(\rho+\alpha)/2}\chi_{\{|y|^2<t\}}|y|^{\alpha+\rho}(\omega_p(y)|y|^{-\alpha})|y|^{-\rho}dy\Big)^q \frac{dt}{t}\Big)^{1/q} .$$

By Lemma 8,

$$\Big(\int_0^\infty\Big(t^{1-\alpha/2}\Big\|\frac{\partial u}{\partial t}(x, t)\Big\|_p\Big)^q \frac{dt}{t}\Big)^{1/q} \leqq C\Big(\int_G (|y|^{-\alpha}\omega_p(y))^q|y|^{-\rho}dy\Big)^{1/q} .$$

By Theorem 10 (ii) we obtain

$$\|f\|_{\alpha;p,q} \leqq C\Big(\|f\|_p + \Big(\int_G\Big(|y|^{-\alpha}\|f(xy) - f(x)\|_p\Big)^q |y|^{-\rho}dy\Big)^{1/q}\Big) .$$

(This inequality includes the corresponding inequality to $q = \infty$.) Therefore $f \in \Lambda(\alpha; p, q)$.

Conversely, let $f \in \Lambda(\alpha; p, q)$ and $u(x, t) = H_t f(x)$. By Theorem 2, for each $t > 0$

$$f(xy) - f(x)$$
$$= \lim_{\varepsilon\to 0}\Big\{-\int_\varepsilon^t \frac{\partial u}{\partial t}(xy, s)ds + \int_\varepsilon^t \frac{\partial u}{\partial t}(x, s)ds + (u(xy, t) - u(x, t))\Big\}$$

in the L^p-norm ($1 \leqq p < \infty$) or for almost all $x \in G$ ($p = \infty$). Hence

$$(2) \quad \|f(xy) - f(x)\|_p \leqq 2\int_0^t \Big\|\frac{\partial u}{\partial t}(x, s)\Big\|_p ds + \|u(x, t) - u(xy, t)\|_p , \qquad (1 \leqq p \leqq \infty) .$$

From Theorem 4 (i) and Theorem 5 it follows that

$$(3) \quad \Big\|\frac{\partial}{\partial t}X_j u(x, t)\Big\|_p \leqq \|X_j h_{t/2}\|_1 \Big\|\frac{\partial u}{\partial t}(x, t/2)\Big\|_p$$
$$\leqq Ct^{-1/2}\Big\|\frac{\partial u}{\partial t}(x, t/2)\Big\|_p , \quad j = 1, \cdots, n .$$

Since $\|(\partial/\partial t)X_j u\|_\infty \to 0$ as $t \to \infty$, $j = 1, \cdots, n$ by Theorem 4 (ii), we get

$$X_j u(x, t) = -\int_t^\infty \frac{\partial}{\partial t}X_j u(x, s)ds , \quad j = 1, \cdots, n .$$

Hence (3) gives that

$$\|X_j u(x, t)\|_p \leqq \int_t^\infty \Big\|\frac{\partial}{\partial t}X_j u(x, s)\Big\|_p ds \leqq C\int_t^\infty s^{-1/2}\Big\|\frac{\partial u}{\partial t}(x, s/2)\Big\|_p ds$$

$$= C\int_{2t}^{\infty} s^{-1/2} \left\|\frac{\partial}{\partial t} u(x, s)\right\|_p ds, \quad j = 1, \cdots, n .$$

From Lemma 10,

$$\|u(xy, t) - u(x, t)\|_p \leqq C|y| \sum_j^n \|X_j u\|_p \leqq C|y| \int_{2t}^{\infty} s^{-1/2} \left\|\frac{\partial u}{\partial t}(x, s)\right\|_p ds ,$$

so that (2) implies that

$$(4) \quad \|f(xy) - f(x)\|_p$$
$$\leqq 2\int_0^t \left\|\frac{\partial u}{\partial t}(x, s)\right\|_p ds + C\int_{2t}^{\infty} |y| s^{-1/2} \left\|\frac{\partial u}{\partial t}(x, s)\right\|_p ds .$$

Taking $t = |y|^2$ we get

$$\left(\int_G (|y|^{-\alpha} \|f(xy) - f(x)\|_p)^q |y|^{-\rho} dy\right)^{1/q}$$
$$\leqq C\left(\int_G \left(\int_0^{|y|^2} |y|^{-\alpha} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p dt\right)^q |y|^{-\rho} dy\right)^{1/q}$$
$$+ C\left(\int_G \left(\int_{2|y|^2}^{\infty} |y|^{1-\alpha} t^{-1/2} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p dt\right)^q |y|^{-\rho} dy\right)^{1/q}$$
$$= C\left(\int_G \left(\int_0^{\infty} |y|^{-\alpha} \chi_{\{|y|^2 \geqq t\}} t^{\alpha/2} \left(t^{1-\alpha/2} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right) \frac{dt}{t}\right)^q |y|^{-\rho} dy\right)^{1/q}$$
$$+ C\left(\int_G \left(\int_0^{\infty} |y|^{1-\alpha} \chi_{\{2|y|^2 \leqq t\}} t^{-(1-\alpha)/2} \left(t^{1-\alpha/2} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right) \frac{dt}{t}\right)^q |y|^{-\rho} dy\right)^{1/q} .$$

By Lemma 8 we obtain

$$\left(\int_G (|y|^{-\alpha} \|f(xy) - f(x)\|_p)^q |y|^{-\rho} dy\right)^{1/q}$$
$$\leqq C\left(\int_0^{\infty} \left(t^{1-\alpha/2} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} .$$

(This inequality includes the corresponding inequality to $q = \infty$.) This proves the first parts of (i). To prove the last part of (i) we have to see that $\omega_p(y) = o(|y|^\alpha)$ as $|y| \to 0$ if and only if $\|(\partial u/\partial t)(x, t)\|_p = o(t^{-1+\alpha/2})$ as $t \to 0$. First suppose that $\omega_p(y) = o(|y|^\alpha)$ as $|y| \to 0$, that is, given $\varepsilon > 0$ there is a positive number t_0 such that $\omega_p(y) < \varepsilon |y|^\alpha$ if $|y|^2 < t_0$. (1) gives that

$$t^{1-\alpha/2} \left\|\frac{\partial u}{\partial t}(x, t)\right\|_p \leqq t^{1-\alpha/2} \int_G \left|\frac{\partial h_t}{\partial t}(y)\right| \omega_p(y) dy$$
$$= t^{1-\alpha/2} \int_{|y|^2 < t} \left|\frac{\partial h_t}{\partial t}(y)\right| \omega_p(y) dy$$

$$+ t^{1-\alpha/2}\int_{t\leqq|y|^2<t_0}\left|\frac{\partial h_t}{\partial t}(y)\right|\omega_p(y)dy$$

$$+ t^{1-\alpha/2}\int_{t_0\leqq|y|^2}\left|\frac{\partial h_t}{\partial t}(y)\right|\omega_p(y)dy$$

$$= I_1 + I_2 + I_3 \quad (\text{if} \quad t < t_0) .$$

We will estimate integrals I_1, I_2 and I_3 using Theorem 4 (i):

$$I_1 \leqq C\varepsilon t^{1-\alpha/2}\int_{|y|^2<t} t^{-(\rho+2)/2}|y|^\alpha dy = C\varepsilon ,$$

$$I_2 \leqq C\varepsilon t^{1-\alpha/2}\int_{t\leqq|y|^2} |y|^{-(\rho+2)}|y|^\alpha dy = C\varepsilon ,$$

and

$$I_3 = Ct^{1-\alpha/2}\int_{t_0\leqq|y|^2}|y|^{-(\rho+2)+\alpha}dy = Ct^{1-\alpha/2}t_0^{-1+\alpha/2} \to 0$$

$$(\text{as } t \to 0) .$$

Since ε is arbitrary, $||(\partial u/\partial t)(x, t)||_p = o(t^{-1+\alpha/2})$ as $t\to 0$.

Conversely, suppose that $||(\partial u/\partial t)(x, t)||_p = o(t^{-1+\alpha/2})$ as $t\to 0$, that is, given $\varepsilon > 0$ there is a positive number t_0 such that $t^{1-\alpha/2}||(\partial u/\partial t)(x, t)||_p < \varepsilon$ for all $0 < t < t_0$. By Theorem 4 (i), (4) gives that

$$|y|^{-\alpha}\omega_p(y) \leqq 2|y|^{-\alpha}\int_0^{|y|^2} t^{-1+\alpha/2}\left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)dt$$

$$+ C|y|^{1-\alpha}\int_{2|y|^2}^{t_0} t^{\alpha/2-3/2}\left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)dt$$

$$+ C|y|^{1-\alpha}\int_{t_0}^{\infty} t^{\alpha/2-3/2}\left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)dt$$

$$\leqq \varepsilon + \varepsilon + |y|^{1-\alpha}t_0^{\alpha/2-1/2} \qquad \text{if} \quad 2|y|^2 < t_0 .$$

Hence $\omega_p(y) = o(|y|^\alpha)$ as $|y|\to 0$. This completes the proof of (i).

Next we shall prove part (ii). Let $0 < \alpha < 2$ and let $f \in L^p$, $1 \leqq p \leqq \infty$ and $u(x, t) = H_t f(x)$. Put $\omega_p^{(2)}(y) = ||f(xy) + f(xy^{-1}) - 2f(x)||_p$. We omit the proof of the case $q = \infty$ since the resulting inequality includes the corresponding inequality for $q = \infty$. First, suppose that $\int_G (|y|^{-\alpha}\omega_p^{(2)}(y))^q|y|^{-\rho}dy < \infty$. Note that $\int_G (\partial^2/\partial t^2)h_t(y)dy = 0$. Therefore,

$$\frac{\partial^2}{\partial t^2}u(x, t) = (1/2)\int_G \frac{\partial^2 h_t}{\partial t^2}(y)(f(xy) + f(xy^{-1}) - 2f(x))dy$$

so that

$$\left\|\frac{\partial^2 u}{\partial t^2}(x, t)\right\|_p \leqq (1/2)\int_G\left|\frac{\partial^2 h_t}{\partial t^2}(y)\right|\omega_p^{(2)}(y)dy .$$

From Theorem 4 (i),

$$t^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, t)\right\|_p \leqq Ct^{2-\alpha/2}\int_{|y|^2\geqq t}|y|^{-(\rho+4)}\omega_p^{(2)}(y)dy$$
$$+ Ct^{2-\alpha/2}\int_{|y|^2<t}t^{-(\rho+4)/2}\omega_p^{(2)}(y)dy\ .$$

Hence, using Lemma 8,

$$\left(\int_0^\infty\left(t^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, t)\right\|_p\right)^q\frac{dt}{t}\right)^{1/q}$$
$$\leqq C\left(\int_0^\infty\left(\int_G t^{2-\alpha/2}\chi_{\{|y|^2\geqq t\}}|y|^{-4+\alpha}(\omega_p^{(2)}(y)|y|^{-\alpha})|y|^{-\rho}dy\right)^q\frac{dt}{t}\right)^{1/q}$$
$$+ C\left(\int_0^\infty\left(\int_G t^{-(\rho+\alpha)/2}\chi_{\{|y|^2<t\}}|y|^{\alpha+\rho}(\omega_p^{(2)}(y)|y|^{-\alpha})|y|^{-\rho}dy\right)^q\frac{dt}{t}\right)^{1/q}$$
$$\leqq C\left(\int_G(|y|^{-\alpha}\omega_p^{(2)}(y))^q|y|^{-\rho}dy\right)^{1/q} < \infty\ .$$

Next, we shall prove the converse inequality. Since

$$u(x, \varepsilon) = -\int_\varepsilon^t s\frac{\partial^2 u}{\partial t^2}(x, s)ds + t\frac{\partial u}{\partial t}(x, t) + u(x, t) - \varepsilon\frac{\partial u}{\partial t}(x, \varepsilon)$$

for $0<\varepsilon<t$, we have

$$u(xy, \varepsilon) - 2u(x, \varepsilon) + u(xy^{-1}, \varepsilon)$$
$$= -\int_\varepsilon^t s\frac{\partial^2 u}{\partial t^2}(xy, s)ds + t\frac{\partial u}{\partial t}(xy, t) + u(xy, t) - \varepsilon\frac{\partial u}{\partial t}(xy, \varepsilon)$$
$$-\int_\varepsilon^t s\frac{\partial^2 u}{\partial t^2}(xy^{-1}, s)ds + t\frac{\partial u}{\partial t}(xy^{-1}, t) + u(xy^{-1}, t)$$
$$-\varepsilon\frac{\partial u}{\partial t}(xy^{-1}, \varepsilon) + 2\int_\varepsilon^t s\frac{\partial^2 u}{\partial t^2}(x, s)ds - 2t\frac{\partial u}{\partial t}(x, t)$$
$$-2u(x, t) + 2\varepsilon\frac{\partial u}{\partial t}(x, \varepsilon)\ .$$

Hence,

$$\|u(xy, \varepsilon) - 2u(x, \varepsilon) + u(xy^{-1}, \varepsilon)\|_p \tag{5}$$
$$\leqq 4\int_\varepsilon^t s\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds + 4\varepsilon\left\|\frac{\partial u}{\partial t}(x, \varepsilon)\right\|_p$$
$$+ 2t\left\|\frac{\partial u}{\partial t}(xy, t) - \frac{\partial u}{\partial t}(x, t)\right\|_p$$
$$+ \|u(xy, t) - 2u(x, t) + u(xy^{-1}, t)\|_p\ .$$

Now, let $f \in \Lambda(\alpha; p, q)$. Then by Theorem 10

$$\left\{\int_0^\infty \left(t^{1-\alpha/2}\left\|\frac{\partial u}{\partial t}(x, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} = D < \infty .$$

Applying Lemma 9 with $h(t) = \|(\partial u/\partial t)(x, t)\|_p$, we get

$$\varepsilon \left\|\frac{\partial u}{\partial t}(x, \varepsilon)\right\|_p \leqq C\varepsilon \cdot \varepsilon^{-(1-\alpha/2)} D = C\varepsilon^{\alpha/2} D \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Hence (5) implies that

$$\begin{aligned} (6) \qquad & \|f(xy) - 2f(x) + f(xy^{-1})\|_p \\ & \leqq 4\int_0^t s\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds + 2t\left\|\frac{\partial u}{\partial t}(xy, t) - \frac{\partial u}{\partial t}(x, t)\right\|_p \\ & \quad + \|u(xy, t) - 2u(x, t) + u(xy^{-1}, t)\|_p . \end{aligned}$$

From Theorem 4 (i),

$$\begin{aligned} (7) \qquad \left\|\frac{\partial^2}{\partial t^2} X_j u(x, t)\right\|_p & \leqq \|X_j h_{t/2}\|_1 \left\|\frac{\partial^2}{\partial t^2} u\left(x, \frac{t}{2}\right)\right\|_p \\ & \leqq C t^{-1/2}\left\|\frac{\partial^2 u}{\partial t^2}\left(x, \frac{t}{2}\right)\right\|_p , \quad j = 1, \cdots, n \end{aligned}$$

and so

$$\begin{aligned} (8) \qquad \left\|\frac{\partial}{\partial t} X_j u(x, t)\right\|_p & \leqq \int_t^\infty \left\|\frac{\partial^2}{\partial t^2} X_j u(x, s)\right\|_p ds \\ & \leqq C\int_t^\infty s^{-1/2}\left\|\frac{\partial^2 u}{\partial t^2}\left(x, \frac{s}{2}\right)\right\|_p ds, \quad j = 1, \cdots, n . \end{aligned}$$

Since by Theorem 4 (i), $\|t(\partial^2/\partial t^2)X_j u(x, t)\|_\infty \to 0$ as $t \to 0$, $j = 1, \cdots, n$, we have

$$\begin{aligned} X_j u(x, t) & = \int_t^\infty s\frac{\partial^2}{\partial t^2} X_j u(x, s) ds + t\frac{\partial}{\partial t} X_j u(x, t) \\ & = \int_t^\infty s\frac{\partial^2}{\partial t^2} X_j u(x, s) ds + t\int_t^\infty \frac{\partial^2}{\partial t^2} X_j u(x, s) ds , \end{aligned}$$

$j = 1, \cdots, n$, so that (7) gives that

$$\begin{aligned} (9) \qquad \|X_j u(x, t)\|_p & \leqq \int_t^\infty s\left\|\frac{\partial^2}{\partial t^2} X_j u(x, s)\right\|_p ds + t\int_t^\infty \left\|\frac{\partial^2}{\partial t^2} X_j u(x, s)\right\|_p ds \\ & \leqq C\left(\int_{2t}^\infty s^{1/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds + t\int_{2t}^\infty s^{-1/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\right) , \\ & \qquad j = 1, \cdots, n . \end{aligned}$$

By Lemma 10, (9) gives that

$$
\begin{aligned}
(10)\quad & \|u(xy, t) - 2u(x, t) + u(xy^{-1}, t)\|_p \\
& \leqq \|u(xy, t) - u(x, t)\|_p + \|u(x, t) - u(xy^{-1}, t)\|_p \\
& \leqq C|y| \left(\sum_j^n \|X_j u\|_p\right) \\
& \leqq C|y| \left(\int_{2t}^\infty s^{1/2} \left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds + t\int_{2t}^\infty s^{-1/2} \left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\right).
\end{aligned}
$$

Further, using (8), Lemma 10 gives that

$$
\begin{aligned}
(11)\qquad \left\|\frac{\partial u}{\partial t}(xy, t) - \frac{\partial u}{\partial t}(x, t)\right\|_p & \leqq C|y| \sum_j^n \left\|X_j \frac{\partial u}{\partial t}(x, t)\right\|_p \\
& \leqq C|y| \int_{2t}^\infty s^{-1/2} \left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\,.
\end{aligned}
$$

Combining (10) and (11) with $t = |y|^2$, (6) implies that

$$
\begin{aligned}
& \left(\int_G (|y|^{-\alpha} \|f(xy) - 2f(x) + f(xy^{-1})\|_p)^q |y|^{-\rho} dy\right)^{1/q} \\
& \leqq C\left(\int_G \left(|y|^{-\alpha} \int_0^{|y|^2} s\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\right)^q |y|^{-\rho} dy\right)^{1/q} \\
& \quad + 2C\left(\int_G \left(|y|^{3-\alpha} \int_{2|y|^2}^\infty s^{-1/2} \left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\right)^q |y|^{-\rho} dy\right)^{1/q} \\
& \quad + C\left(\int_G \left(|y|^{1-\alpha} \int_{2|y|^2}^\infty s^{1/2} \left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p ds\right)^q |y|^{-\rho} dy\right)^{1/q} \\
& = C\left(\int_G \left(\int_0^\infty |y|^{-\alpha} s^{\alpha/2} \chi_{\{|y|^2 \geqq s\}} \left(s^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p\right) \frac{ds}{s}\right)^q |y|^{-\rho} dy\right)^{1/q} \\
& \quad + C\left(\int_G \left(\int_0^\infty |y|^{3-\alpha} s^{-(3-\alpha)/2} \chi_{\{2|y|^2 < s\}} \left(s^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p\right) \frac{ds}{s}\right)^q |y|^{-\rho} dy\right)^{1/q} \\
& \quad + C\left(\int_G \left(\int_0^\infty |y|^{1-\alpha} s^{-(1-\alpha)/2} \chi_{\{2|y|^2 < s\}} \left(s^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, s)\right\|_p\right) \frac{ds}{s}\right)^q |y|^{-\rho} dy\right)^{1/q}.
\end{aligned}
$$

By Lemma 8 we obtain the inequality required:

$$
\begin{aligned}
& \left(\int_G (|y|^{-\alpha} \|f(xy) - 2f(x) + f(xy^{-1})\|_p)^q |y|^{-\rho} dy\right)^{1/q} \\
& \qquad \leqq C\left(\int_0^\infty \left(t^{2-\alpha/2}\left\|\frac{\partial^2 u}{\partial t^2}(x, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

The proof of the remaining part is similar to the corresponding proof of (i).

THEOREM 13. *Let $\alpha > 0$ and $\beta > \alpha$, and let $1 < p < \infty$. Then*

$$
\Lambda(\alpha; p, q) = \left\{f \in L^p: \int_0^\infty (t^{(\beta-\alpha)/2} \| I^{-\beta} H_t f\|_p)^q \frac{dt}{t} < \infty\right\} \quad (1 \leqq q < \infty)
$$

and

$$\Lambda(\alpha; p, \infty) = \{f \in L^p: \sup_{t>0} \{t^{(\beta-\alpha)/2}\|I^{-\beta}H_t f\|_p\} < \infty\} .$$

Moreover the norm $\|\cdot\|_{\alpha;p,q}$ *and* $\|\cdot\|_{\alpha;p,\infty}$ *are equivalent to* $\|f\|_p + \left\{\int_0^\infty (t^{(\beta-\alpha)/2}\|I^{-\beta}H_t f\|_p)^q t^{-1}dt\right\}^{1/q}$ *and* $\|f\|_p + \sup_{t>0}\{t^{(\beta-\alpha)/2}\|I^{-\beta}H_t f\|_p\}$ *respectively, where*

$$I^{-\beta}H_t f(x) = \frac{1}{\Gamma(k-\beta/2)}\int_0^\infty s^{k-\beta/2-1}\mathfrak{J}^k H_{t+s}f(x)ds\,, \quad k = [\beta/2]+1\,.$$

PROOF. The space $B^{\alpha/2}_{q,\beta/2}(\mathfrak{J}_p)$, $\beta > \alpha > 0$, defined by Komatsu[11: Definition 5.1] coincides with the space $\left\{f \in L^p: \int_0^\infty (t^{(\beta-\alpha)/2}\|I^{-\beta}H_t f\|_p)^q t^{-1}dt < \infty\right\}$ in the case discussed in our paper. Komatsu [11: Proposition 2.5] has proved that $B^{\alpha/2}_{q,\beta/2}(\mathfrak{J}_p) = B^{\alpha/2}_{q,\beta/2}(1+\mathfrak{J}_p)$. The fact that $-(1+\mathfrak{J})$ generates a semigroup $e^{-t}H_t$ implies that

$$B^{\alpha/2}_{q,\beta/2}(1+\mathfrak{J}_p) = \left\{f \in L^p: \int_0^\infty \left(t^{(\beta-\alpha)/2}e^{-t}\|J^{-\beta}H_t f\|_p\right)^q \frac{dt}{t} < \infty\right\}.$$

From Lemma 4 and Theorem 10 (i), we get

$$B^{\alpha/2}_{q,\beta/2}(\mathfrak{J}_p) = \Lambda(\alpha; p, q) \quad \text{if} \quad 1 < p < \infty\,.$$

THEOREM 14. *Let* $1 \leqq p \leqq \infty$.

(i) *If* $1 \leqq q_1 \leqq q_2 < \infty$ *and* $\beta \leqq \alpha$ *then*

$$\begin{aligned}\mathfrak{T}\Lambda(\alpha; p, 1) &\subset \mathfrak{T}\Lambda(\alpha; p, q_1) \subset \mathfrak{T}\Lambda(\alpha; p, q_2)\\ &\subset \mathfrak{T}\lambda(\alpha; p, \infty) \subset \mathfrak{T}\Lambda(\alpha; p, \infty) \subset \mathfrak{T}\Lambda(\beta; p, 1)\,.\end{aligned}$$

(ii) *If* $1 \leqq p < r \leqq \infty$, $\delta = \rho(1/p - 1/r)$ *and* $1 \leqq q \leqq \infty$ *then* $\mathfrak{T}\Lambda(\alpha; p, q) \subset \mathfrak{T}\Lambda(\alpha-\delta; r, q)$ *and* $\mathfrak{T}\lambda(\alpha; p, \infty) \subset \mathfrak{T}\lambda(\alpha-\delta; r, \infty)$.

In each case the inclusion mapping is continuous.

PROOF. The part (i) follows from Lemma 3 (i) and Lemma 7 (i) and the part (ii) follows from Theorem 4 (iii), Theorem 5 and Proposition 3 (ii) (cf. T.M. Flett [4: Theorem 20]).

THEOREM 15. *Let* α *be real and let* $1 \leqq p, q \leqq \infty$. *Then the space* $\mathfrak{T}\Lambda(\alpha; p, q)$ *is a Banach space with respect to the norm* $\|\cdot\|_{\alpha;p,q}$.

Flett [4: Appendix II] has proved this theorem in the case of n dimensional Euclidean space by using the following series of lemmas and the same results also hold in our case.

LEMMA 11. *Let* α *be real, and let* $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$.

(i) *Let* $u(x, t)$ *be a temperature on* $G \times (0, \infty)$ *such that* $\|u(\cdot, t)\|_p$

is locally integrable on $(0, \infty)$ *and let* $u^s(x, t)$ *be the function given by* $u^s(x, t) = u(x, s+t)$, *where* $s > 0$. *Then* $u^s \in \mathfrak{T}\Lambda(\alpha; p, q) \cap \mathfrak{T}\lambda(\alpha; p, \infty)$, *and for each* $c > 0$, $\|u^s\|_{\alpha;p,q} \leqq C$ *for all* $s \geqq c$. *The same results hold also for the function* $u^{y,s}(x, t)$ *given by* $u^{y,s}(x, t) = u(yx, s+t)$ *where* $y \in G$ *and* $s > 0$.

(ii) *Let* $u \in \mathfrak{T}\Lambda(\alpha; p, q)$. *Then for each* $s > 0$, $u^s \in \mathfrak{T}\Lambda(\alpha; p, q)$ *and* $\|u^s\|_{\alpha;p,q} \leqq \|u\|_{\alpha;p,q}$. *Further,*

(iii) *if* $1 \leqq q < \infty$, *then* $u^s \to u$ *in* $\mathfrak{T}\Lambda(\alpha; p, q)$ *as* $s \to 0$, *and*

(iv) *if* $q = \infty$, *then* $u^s \to u$ *in* $\mathfrak{T}\Lambda(\alpha; p, \infty)$ *as* $s \to 0$ *if and only if* $u \in \mathfrak{T}\lambda(\alpha; p, \infty)$.

LEMMA 12. *Let* α *be real, and let* $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ *and let* $u \in \mathfrak{T}$ *such that* $\|u^s\|_{\alpha;p,q} = O(1)$ *as* $s \to 0$. *Then* $u \in \mathfrak{T}\Lambda(\alpha; p, q)$ *and* $\|u\|_{\alpha;p,q} = \lim_{s\to 0} \|u^s\|_{\alpha;p,q}$.

LEMMA 13. *Let* α *be real and let* $1 \leqq p \leqq \infty$, $f \in L^p$ *and* $u(x, t) = H_t f(x)$. *Then for each* $s > 0$,

$$\|u^s\|_{\alpha;p,1} \leqq \begin{cases} C(1 + s^{-\alpha/2})\|f\|_p & (\alpha > 0) \\ C(1 + \log^+(1/s))\|f\|_p & (\alpha = 0) \\ C\|f\|_p & (\alpha < 0)\,. \end{cases}$$

LEMMA 14. *Let* α *real,* $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, *and let* $\{f_n\}$ *be a sequence of functions converging in* L^p *to a function* f. *Let* $u_n(x, t) = H_t f_n(x)$ *and* $u(x, t) = H_t f(x)$. *Then for each* $s > 0$, u_n^s *converges to* u^s *in* $\mathfrak{T}\Lambda(\alpha; p, q)$ *as* $n \to \infty$.

4. Besov spaces in terms of the Poisson semigroup.

DEFINITION 3. For any $\alpha > 0$ and $f \in L^p$, $1 \leqq p \leqq \infty$, we define $J^\alpha f$ by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} H_t f(x) dt\,.$$

This integral exists for almost all $x \in G$, and $\|J^\alpha f\|_p \leqq \|f\|_p$, $1 \leqq p \leqq \infty$. It follows easily that $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha, \beta > 0$. If $u = H_t f$, $J^\alpha u$ in this definition coincides with that of Definition 1. Moreover it follows that $J^\alpha H_t f = H_t J^\alpha f$. For each $\alpha > 0$ we define $G_\alpha(x)$ by

$$G_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} h_t(x) dt\,.$$

This integral has following properties:

(a) For each $\alpha > 0$, $G_\alpha \in L^1$.

(b) For $\alpha > 0$, $\beta > 0$, $G_\alpha * G_\beta = G_{\alpha+\beta}$.

(c) For each $\alpha > 0$ and an l-tuple $\gamma = (\gamma_1, \cdots, \gamma_l)$ of nonnegative integers,

$$D^\gamma G_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} D^\gamma h_t(x) dt$$

for all $x \neq e$.

(d) For each $\alpha > 0$ and $f \in L^p$, $1 \leqq p \leqq \infty$, $J^\alpha f(x) = G_\alpha * f(x)$.

We define the operator P_t by

$$P_t = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\lambda}}{\sqrt{\lambda}} H_{t^2/4\lambda} d\lambda \quad (t > 0) .$$

Then for each $t > 0$ we have $P_t f = p_t * f$ if $f \in L^p$, $1 \leqq p \leqq \infty$, where

$$p_t(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\lambda}}{\sqrt{\lambda}} h_{t^2/4\lambda}(x) d\lambda$$

(say $p_t(x) = p(x, t)$).

The operator P_t and its kernel p_t satisfy most of properties in Theorem 1, Theorem 2 and Theorem 3 except the following properties:

(a) If $f \in L^p$, $1 \leqq p \leqq \infty$, $P_t f$ satisfies the Laplace equation, that is,

$$\frac{\partial^2}{\partial t^2}(P_t f)(x) = \mathfrak{J}(P_t f)(x) .$$

(b) $p(rs, rt) = r^{-\rho} p(x, t)$ for all $r > 0$.

Further important properties for P_t and p_t are listed below:

(c) For $t > 0$ and $s > 0$, $p_t * h_s = h_s * p_t$.

(d) If $f \in L^p$, $1 \leqq p \leqq \infty$ and $\alpha > 0$, $P_t J^\alpha f(x) = J^\alpha P_t f(x)$ and $p_t * G_\alpha = G_\alpha * p_t$ for each $t > 0$. These properties imply a following lemma analogous to Theorem 4.

LEMMA 15. *Let $p_t(x) = p(x, t)$ be the kernel function of the semigroup $\{P_t\}_{t \geqq 0}$. Then*

(i) *for all $t > 0$*

$$|p(x, t)| \leqq \begin{cases} C|x|^{-\rho} & \text{if } |x| \geqq t \\ Ct^{-\rho} & \text{if } |x| \leqq t . \end{cases}$$

If $\alpha = (\alpha_1, \cdots, \alpha_l)$ is an l-tuple of nonnegative integers and k is a nonnegative integer then

$$\left| \frac{\partial^k}{\partial t^k} D^\alpha p(x, t) \right| \leqq \begin{cases} C|x|^{-(\rho+|\alpha|+k)} & \text{if } |x| \geqq t \\ Ct^{-(\rho+|\alpha|+k)} & \text{if } |x| \leqq t . \end{cases}$$

Also, for all $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha p_t\right\|_p \leqq Ct^{-(|\alpha|+k+\rho(1-1/p))} \quad (1 \leqq p \leqq \infty)\,.$$

Further,

(ii) *let* $u(x, t) = P_t f(x)$, $f \in L^p$ $(1 \leqq p \leqq \infty)$. *Then for all* $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot\,, t)\right\|_p \leqq Ct^{-(|\alpha|+k)}\|f\|_p\,.$$

Also, if $1 \leqq p < r \leqq \infty$ *and* $\delta = \rho(1/p - 1/r)$, *then for all* $t > 0$,

$$\left\|\frac{\partial^k}{\partial t^k} D^\alpha u(\cdot\,, t)\right\|_r \leqq Ct^{-(|\alpha|+k+\delta)}\|f\|_p\,.$$

(iii) *For each* $t > 0$ *the functions* $x \mapsto u(x, t)$ *and* $x \mapsto (\partial^k/\partial t^k)u(x, t)$ *are uniformly continuous on* G, *and the functions* $t \mapsto \|u(\cdot\,, t)\|_p$ *and* $t \mapsto \|(\partial^k/\partial t^k)u(\cdot\,, t)\|_p$ *are decreasing on* $(0, \infty)$ *if* $1 \leqq p \leqq \infty$ *and are continuous on* $(0, \infty)$ *if* $1 \leqq p < \infty$.

LEMMA 16. *Let* $1 \leqq p \leqq \infty$, *and let* $u(x, t)$ *be a harmonic function on* $G \times (0, \infty)$, *that is,* $u(x, t)$ *is a solution of the Laplace equation* $\partial^2 u/\partial t^2 = \mathfrak{J} u$ *on* $G \times (0, \infty)$, *such that for each* $t_0 > 0$, $u(x, t)$ *is bounded for* $t \geqq t_0$ *and* $\|u(\cdot\,, t)\|_p$ *exists for each* $t > 0$.

(i) *Let* $\beta > 0$ *and* $1 \leqq q \leqq \infty$. *If* $\int_0^\infty t^{q\beta-1}\|u(\cdot\,, t)\|_p^q dt < \infty$, *then for* $t > 0$,

$$\|u(\cdot\,, t)\|_p \leqq Ct^{-\beta}\left\{\int_0^\infty t^{q\beta-1}\|u(\cdot\,, t)\|_p^q dt\right\}^{1/q}\,.$$

Moreover, if $q \leqq r \leqq \infty$, *then*

$$\left\{\int_0^\infty t^{r\beta-1}\|u(\cdot\,, t)\|_p^r dt\right\}^{1/r} \leqq C\left\{\int_0^\infty t^{q\beta-1}\|u(\cdot\,, t)\|_p^q dt\right\}^{1/q}\,.$$

(ii) *Let* k *be a nonnegative integer and let* β *be real. If* $\int_0^\infty t^{q\beta-1}\|u(\cdot\,, t)\|_p^q dt < \infty$, *then*

$$\left\{\int_0^\infty t^{q(\beta+k)-1}\left\|\frac{\partial^k}{\partial t^k} u(\cdot\,, t)\right\|_p^q dt\right\}^{1/q} \leqq C\left\{\int_0^\infty t^{q\beta-1}\|u(\cdot\,, t)\|_p^q dt\right\}^{1/q}\,.$$

(iii) *Let* α *be a positive number and let* k *be a nonnegative integer. If*

$$\int_0^\infty \left(t^{\alpha+k}\left\|\frac{\partial^k}{\partial t^k} u(\cdot\,, t)\right\|_p\right)^q \frac{dt}{t} < \infty\,,$$

then

$$\left(\int_0^\infty (t^\alpha \|u(\cdot, t)\|_p)^q \frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty \left(t^{\alpha+k} \left\|\frac{\partial^k}{\partial t^k} u(\cdot, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} .$$

Further, $\|(\partial^k/\partial t^k)u(\cdot, t)\|_p = o(t^{-(\alpha+k)})$ *as* $t \to 0$ *if and only if* $\|u(\cdot, t)\|_p = o(t^{-\alpha})$ *as* $t \to 0$. (*In the case* $q = \infty$ *we consider these inequalities under usual modification using* sup *notation instead of integral notation.*)

PROOF. Part (i) follows from Lemma 9 and part (ii) follows from Lemma 15 (ii). To prove (iii), we note that

$$u(x, t) = \int_t^\infty \frac{\partial u}{\partial t}(x, s) ds ,$$

so that

$$\|u(\cdot, t)\|_p \leqq \int_t^\infty \left\|\frac{\partial u}{\partial t}(\cdot, s)\right\|_p ds .$$

For $1 \leqq q < \infty$, using Lemma 5, we get

$$\begin{aligned} \left(\int_0^\infty (t^\alpha \|u(\cdot, t)\|_p)^q \frac{dt}{t}\right)^{1/q} &\leqq \left(\int_0^\infty \left(t^\alpha \int_t^\infty \left\|\frac{\partial u}{\partial t}(\cdot, s)\right\|_p ds\right)^q \frac{dt}{t}\right)^{1/q} \\ &\leqq \left(\frac{1}{\alpha}\right)\left(\int_0^\infty t^{\alpha+1}\left\|\frac{\partial u}{\partial t}(\cdot, t)\right\|_p\right)^q \frac{dt}{t}\Bigg)^{1/q} . \end{aligned}$$

By induction, part (iii) follows if $1 \leqq q < \infty$. For $q = \infty$ it is easy to prove. The last part of (iii) is also easy to prove.

DEFINITION 4. Let $\alpha > 0$. We define the space $\tilde{\Lambda}(\alpha; p, q)$, where k is a positive integer with $k > \alpha$, and $1 \leqq p \leqq \infty, 1 \leqq q \leqq \infty$ to be the space of $f \in L^p$ for which

$$\int_0^\infty \left(t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} P_t f\right\|_p\right)^q \frac{dt}{t} < \infty \qquad (1 \leqq q < \infty)$$

or

$$\sup_{t>0}\left\{t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} P_t f\right\|_p\right\} < \infty \qquad (q = \infty) ,$$

equipped with the norm

$$|||f|||_{\alpha;p,q} = \|f\|_p + \left(\int_0^\infty \left(t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} P_t f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q}$$

or

$$|||f|||_{\alpha;p,\infty} = \|f\|_p + \sup_{t>0}\left\{t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} P_t f\right\|_p\right\} .$$

We denote by $\tilde{\lambda}(\alpha; p, \infty)$ the subspace of those $f \in \tilde{\Lambda}(\alpha; p, \infty)$ for which

$$\left\|\frac{\partial^k}{\partial t^k} P_t f\right\|_p = o(t^{-(k-\alpha)}) \quad \text{as} \quad t \to 0 .$$

Lemma 16 implies that the definition of the space $\tilde{\Lambda}(\alpha; p, q)$ and $\tilde{\Lambda}(\alpha; p, \infty)$ is independent of the choice of the integer k.

We obtain the following theorem similar to Theorem 12.

THEOREM 16. *Let* $1 \leqq p \leqq \infty$.

(i) *For* $0 < \alpha < 1$,

$$\tilde{\Lambda}(\alpha; p, q) = \left\{f \in L^p: \int_G (|y|^{-\alpha}\|f(xy) - f(x)\|_p)^q |y|^{-\rho} dy < \infty\right\} \quad (1 \leqq q < \infty)$$

and

$$\tilde{\Lambda}(\alpha; p, \infty) = \{f \in L^p: \sup_{|y|>0} \{|y|^{-\alpha}\|f(xy) - f(x)\|_p\} < \infty\} .$$

Moreover the norm $|||\cdot|||_{\alpha;p,q}$ *and* $|||\cdot|||_{\alpha;p,\infty}$ *are equivalent to*

$$\|f\|_p + \left(\int_G (|y|^{-\alpha}\| f(xy) - f(x)\|_p)^q |y|^{-\rho} dy\right)^{1/q}$$

and $\|f\|_p + \sup_{|y|>0}\{|y|^{-\alpha}\|f(xy) - f(x)\|_p\}$ *respectively, and*

$$\tilde{\lambda}(\alpha; p, \infty) = \{f \in L^p: \|f(xy) - f(x)\|_p = o(|y|^{\alpha}) \quad \text{as} \quad |y| \to 0\} .$$

(ii) *For* $0 < \alpha < 2$,

$$\tilde{\Lambda}(\alpha; p, q) = \left\{f \in L^p: \int_G (|y|^{-\alpha}\| f(xy) + f(xy^{-1}) - 2f(x)\|_p)^q |y|^{-\rho} dy < \infty\right\} \quad (1 \leqq q < \infty)$$

and

$$\tilde{\Lambda}(\alpha; p, \infty) = \{f \in L^p: \sup_{|y|>0} \{|y|^{-\alpha}\| f(xy) + f(xy^{-1}) - 2f(x)\|_p\} < \infty\} .$$

Moreover the norm $|||\cdot|||_{\alpha;p,q}$ *and* $|||\cdot|||_{\alpha;p,\infty}$ *are equivalent to*

$$\|f\|_p + \left(\int_G (|y|^{-\alpha}\|f(xy) + f(xy^{-1}) - 2f(x)\|_p)^q |y|^{-\rho} dy \right)^{1/q}$$

and $\|f\|_p + \sup_{|y|>0}\{|y|^{-\alpha}\|f(xy) + f(xy^{-1}) - 2f(x)\|_p\}$ *respectively, and*

$$\tilde{\lambda}(\alpha; p, \infty) = \{f \in L^p: \| f(xy) + f(xy^{-1}) - 2f(x)\|_p = o(|y|^{\alpha}) \text{ as } |y| \to 0\} .$$

THEOREM 17. *Let* $\beta > 0, \alpha > 0$ *and let* $1 \leqq p \leqq \infty, 1 \leqq q \leqq \infty$. *Then* J^β *is a linear homeomorphism of* $\tilde{\Lambda}(\alpha; p, q)$ *onto* $\tilde{\Lambda}(\alpha + \beta; p, q)$ *of* $\tilde{\lambda}(\alpha; p, \infty)$ *onto* $\tilde{\lambda}(\alpha + \beta; p, \infty)$.

To prove this theorem we need the following lemma:

LEMMA 17. *Suppose that* $u(x, t)$ *is harmonic on* $G \times (0, \infty)$, *which* *for each $c > 0$, $u(x, t)$ is bounded in $t \geqq c$ and $\|u(\cdot, t)\|_p$ exists for each $t > 0$. Given $D > 0$, $\alpha > 0$, $t_0 > 0$ and an integer $k > \alpha$ such that*

$$\left(\int_0^\infty \left(t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} u(\cdot, t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqq D < \infty \quad (1 \leqq q < \infty)$$

or

$$\sup_{t>0} \left\{t^{k-\alpha}\left\|\frac{\partial^k}{\partial t^k} u(\cdot, t)\right\|_p\right\} \leqq D < \infty \quad (q = \infty)$$

and $\|u(\cdot, t)\|_p \leqq D$ for all $t \geqq t_0$. Then there exists a function $f \in \tilde{\Lambda}(\alpha; p, q)$ such that

(a) $u(x, t) = P_t f(x)$,

(b) $(\partial u/\partial t)(\cdot, t)_p = o(t^{-1})$ *as* $t \to 0$

(c) $|||f|||_{\alpha; p, q} \leqq CD$.

This follows from Lemma 15 and Lemma 16 (cf. M. H. Taibleson [22: Lemma 5]).

We return to the proof of Theorem 17 (cf. M. H. Taibleson [22: Theorem 5]). First we shall show $G_\beta \in \tilde{\Lambda}(\beta; 1, \infty)$ for all $\beta > 0$. Suppose $0 < \beta < 1$. For $x \neq e$ we see that using Theorem 4 (i),

$$(1) \qquad \begin{aligned} |G_\beta(x)| &= C\left|\int_0^\infty t^{\beta/2-1}e^{-t}h_t(x)dt\right| \leqq C\int_0^{|x|^2} t^{\beta/2-1}e^{-t}|x|^{-\rho}dt \\ &\quad + C\int_{|x|^2}^\infty t^{-(\rho-\beta)/2-1}e^{-t}dt \leqq C|x|^{-\rho}\int_0^{|x|^2} t^{\beta/2-1}dt + C\int_{|x|^2}^\infty t^{-(\rho-\beta)/2-1}dt \\ &= C|x|^{-\rho+\beta} \end{aligned}$$

and

$$(2) \quad \begin{aligned} |X_jG_\beta| &\leqq C\int_0^\infty t^{\beta/2-1}e^{-t}|X_jh_t|dt = C\int_0^{|x|^2} t^{\beta/2-1}e^{-t}|X_jh_t|dt \\ &\quad + C\int_{|x|^2}^\infty t^{\beta/2-1}e^{-t}|X_jh_t|dt \leqq C\int_0^{|x|^2} t^{\beta/2-1}e^{-t}|x|^{-(\rho+1)}dt \\ &\quad + C\int_{|x|^2}^\infty t^{-(\rho-\beta+1)/2-1}e^{-t}dt \leqq C|x|^{-(\rho+1)+\beta}, \quad j = 1, \cdots, n. \end{aligned}$$

We write

$$\begin{aligned} \int_G |G_\beta(xy) - G_\beta(x)|dx &= \int_{|x|\leqq 2|y|} |G_\beta(xy) - G_\beta(x)|dx \\ &\quad + \int_{|x|>2|y|} |G_\beta(xy) - G_\beta(x)|dx. \end{aligned}$$

The first integral can be estimated by using (1). Then

$$\begin{aligned} \int_{|x|\leqq 2|y|} |G_\beta(xy) - G_\beta(x)|dx &\leqq \int_{|x|\leqq 2|y|} (|G_\beta(xy)| + |G_\beta(x)|)dx \\ &\leqq 2\int_{|x|\leqq 3C|y|} |G_\beta(x)|dx \leqq C\int_{|x|\leqq 3C|y|} |x|^{-\rho+\beta}dx \leqq C|y|^\beta. \end{aligned}$$

Next to estimate the second integral we use (2) and Lemma 10. Then

$$\int_{|x|<2|y|} |G_\beta(xy) - G_\beta(x)|dx \leqq C|y| \sum_j \int_{|x|>2|y|} |X_j G_\beta(x)|dx$$
$$\leqq C|y| \int_{|x|>2|y|} |x|^{-(\rho+1)+\beta} dx = C|y|^\beta .$$

Therefore, $\|G_\beta(xy) - G_\beta(x)\|_1 \leqq C|y|^\beta$, so that by Theorem 15, $G_\beta \in \tilde{\Lambda}(\beta; 1, \infty)$, that is, $\|(\partial/\partial t)P_t G_\beta\|_1 \leqq Ct^{\beta-1}$. To pass to the general case $\beta > 0$, let k be a positive integer with $k = [\beta] + 1$. We write $\beta = \beta_1 + \cdots + \beta_k$, $0 < \beta_i < 1$. We observe that

$$P_t G_\beta = P_t(G_{\beta_1} * G_{\beta_2} * \cdots * G_{\beta_k}) = (p_{t/k} * G_{\beta_1}) * (p_{t/k} * G_{\beta_2}) * \cdots * (p_{t/k} * G_{\beta_k}) .$$

Consequently,

$$\left\|\frac{\partial^k}{\partial t^k} P_t G_\beta\right\|_1 \leqq \left\|\frac{\partial}{\partial t} P_{t/k} G_{\beta_1}\right\|_1 \left\|\frac{\partial}{\partial t} P_{t/k} G_{\beta_2}\right\|_1 \cdots \left\|\frac{\partial}{\partial t} P_{t/k} G_{\beta_k}\right\|_1$$
$$\leqq C\left(\frac{t}{k}\right)^{\beta_1 - 1} \cdots \left(\frac{t}{k}\right)^{\beta_k - 1} = Ct^{\beta - k} ,$$

so that $G_\beta \in \tilde{\Lambda}(\beta; 1, \infty)$ for all $\beta > 0$.

Since it follows easily from Proposition 1 (iii) that J^β is one-to-one, we shall show that the image of $\tilde{\Lambda}(\alpha; p, q)$ under J^β lies in $\tilde{\Lambda}(\alpha + \beta; p, q)$. To prove this it is enough to see that if k_1 and k_2 are positive integers with $k_1 > \alpha$, $k_2 > \beta$, then

$$\left(\int_0^\infty \left(t^{k_1+k_2-(\alpha+\beta)} \left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} P_t J^\beta f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty \left(t^{k_1-\alpha} \left\|\frac{\partial^{k_1}}{\partial t^{k_1}} P_t f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q}$$

for $f \in \tilde{\Lambda}(\alpha; p, q)$. Since $\|(\partial^{k_2}/\partial t^{k_2}) P_{t/2} G_\beta\|_1 \leqq Ct^{\beta - k_2}$, we get

$$\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} P_t J^\beta f\right\|_p = \left\|\frac{\partial^{k_2}}{\partial t^{k_2}} P_{t/2} G_\beta * \frac{\partial^{k_1}}{\partial t^{k_1}} P_{t/2} f\right\|_p$$
$$\leqq \left\|\frac{\partial^{k_2}}{\partial t^{k_2}} P_{t/2} G_\beta\right\|_1 \left\|\frac{\partial^{k_1}}{\partial t^{k_1}} P_{t/2} f\right\|_p \leqq Ct^{\beta - k_2} \left\|\frac{\partial^{k_1}}{\partial t^{k_1}} P_{t/2} f\right\|_p .$$

Hence,

$$\left(\int_0^\infty \left(t^{k_1+k_2-(\alpha+\beta)} \left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} P_t J^\beta f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty \left(t^{k_1-\alpha} \left\|\frac{\partial^{k_1}}{\partial t^{k_1}} P_{t/2} f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leqq C\left(\int_0^\infty \left(t^{k_1-\alpha} \left\|\frac{\partial^{k_1}}{\partial t^{k_1}} P_t f\right\|_p\right)^q \frac{dt}{t}\right)^{1/q},$$

so that $J^\beta \tilde{\Lambda}(\alpha; p, q) \subset \tilde{\Lambda}(\alpha + \beta; p, q)$. The corresponding inequality also holds for $q = \infty$. To see that J^β is onto, let $f \in \tilde{\Lambda}(\alpha + \beta: p, q)$ and $u(x, t) = P_t f(x)$, and let $0 < \beta < 2$. Then for $(x, t) \in G \times (0, \infty)$,

$$(3)\qquad J^{\beta}J^{2-\beta}\Big(1+\frac{\partial^2}{\partial t^2}\Big)u(x,t)=J^2\Big(1+\frac{\partial^2}{\partial t^2}\Big)u(x,t)$$
$$=\int_0^\infty e^{-s}H_s\Big(1+\frac{\partial^2}{\partial t^2}\Big)u(x,t)ds=\int_0^\infty e^{-s}\Big(1+\frac{\partial^2}{\partial t^2}\Big)p_t*(h_s*f)(x)ds$$
$$=\int_0^\infty e^{-s}(1+\mathfrak{J})H_su(x,t)ds=u(x,t)\,.$$

Put $g(x,t)=J^{2-\beta}(1+\partial^2/\partial t^2)u(x,t)$. Then using Lemma 15 (ii),

$$(4)\qquad \|g(x,t)\|_p\leqq\Big\|\Big(1+\frac{\partial^2}{\partial t^2}\Big)u(x,t)\Big\|_p\leqq C(1+t^{-2})\|f\|_p$$
$$\leqq C\,|||f|||_{\alpha+\beta;p,q}\quad\text{if}\quad t\geqq1\,.$$

Now, let k_1, k_2 be two positive integers with $k_1>\alpha$, $k_2>\beta$. Since $G_{2-\beta}\in\tilde{\Lambda}(2-\beta;1,\infty)$,

$$\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}g(x,t)\Big\|_p=\Big\|\int_0^\infty s^{(2-\beta)/2-1}e^{-s}\Big(\Big(1+\frac{\partial^2}{\partial t^2}\Big)p_{t/2}*h_s\Big)*\Big(\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}p_{t/2}*f\Big)ds\Big\|_p$$
$$=\Big\|\Big(\Big(1+\frac{\partial^2}{\partial t^2}\Big)P_{t/2}G_{2-\beta}\Big)*\Big(\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}P_{t/2}f\Big)\Big\|_p\leqq C(1+t^{-\beta})\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u(x,t/2)\Big\|_p\,.$$

Hence,

$$\Big(\int_0^\infty\Big(t^{k_1+k_2-\alpha}\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}g(x,t)\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leqq C\Big(\int_0^\infty\Big(t^{k_1+k_2-\alpha}\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u\Big(x,\frac{t}{2}\Big)\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$+C\Big(\int_0^\infty\Big(t^{k_1+k_2-(\alpha+\beta)}\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u\Big(x,\frac{t}{2}\Big)\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}\,.$$

The first integral is divided into the integrals over $(0,1)$ and $(1,\infty)$, say I_1 and I_2 respectively. Then

$$I_1^{1/q}=\Big(\int_0^1\Big(t^{\beta}\cdot t^{k_1+k_2-(\alpha+\beta)}\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u\Big(x,\frac{t}{2}\Big)\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leqq C\Big(\int_0^1\Big(t^{k_1+k_2-(\alpha+\beta)}\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u\Big(x,\frac{t}{2}\Big)\Big\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}\leqq C\|f\|_{\alpha+\beta;p,q}\,.$$

Since

$$\Big\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}u\Big(x,\frac{t}{2}\Big)\Big\|_p\leqq Ct^{-(k_1+k_2)}\|f\|_p\,,$$

from Lemma 15 (ii),

$$I_2^{1/q}\leqq C\Big(\int_1^\infty t^{k_1+k_2-\alpha}t^{-(k_1+k_2)}\|f\|_p\Big)^q\frac{dt}{t}\Big)^{1/q}=C\|f\|_p\leqq C\|f\|_{\alpha+\beta;p,q}\,.$$

Hence

$$(5)\qquad \left(\int_0^\infty \left(t^{k_1+k_2-\alpha}\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}\, g(x,t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqq C\|f\|_{\alpha+\beta;p,q}\ .$$

On the other hand, by Lemma 15 (ii),

$$(6)\quad \|g(x,t)\|_\infty \leqq \int_0^\infty s^{(2-\beta)/2-1}e^{-s}\left\|\left(1+\frac{\partial^2}{\partial t^2}\right)P_t f\right\|_\infty ds \leqq C(t^{-\rho/p}+t^{-2-\rho/p})\|f\|_p$$

(4), (5) and (6) imply the hypothesis of Lemma 17 are valid, and so there exists a function $g \in \tilde{\Lambda}(\alpha; p, q)$ $(1 \leqq q < \infty)$ such that $g(x, t) = P_t g(x)$. (3) gives

$$P_t f(x) = u(x,t) = J^\beta g(x,t) = J^\beta P_t g(x) = P_t J^\beta g(x)\ ,$$

so that $f(x) = J^\beta g(x)$, that is, J^β is onto if $0 < \beta < 2$ and $1 \leqq q < \infty$. To pass the general case $\beta > 0$, let $k = [\beta/2] + 1$ and put $g_k(x, t) = J^{2k-\beta}(1 + \partial^2/\partial t^2)^k u(x, t)$. Then

$$J^\beta g_k(x,t) = J^{2k}\left(1+\frac{\partial^2}{\partial t^2}\right)^k u(x,t) = u(x,t)$$

and

$$\|g_k(x,t)\|_p \leqq C(1+t^{-2})^k\|f\|_p \leqq C\|f\|_{\alpha+\beta;p,q} \quad \text{if} \quad t \geqq 1\ .$$

We write $\beta = \beta_1 + \cdots + \beta_k$ where $0 < \beta_i < 2$. Then

$$\begin{aligned}\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} g_k(x,t)\right\|_p &\leqq \left\|\left(1+\frac{\partial^2}{\partial t^2}\right)P_{t/(k+1)}G_{2-\beta_1}\right\|_1 \cdots \\ &\quad \cdots \left\|\left(1+\frac{\partial^2}{\partial t^2}\right)P_{t/(k+1)}G_{2-\beta_k}\right\|_1 \left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} P_{t/(k+1)}f\right\|_p \\ &\leqq C(1+t^{-\beta_1})\cdots(1+t^{-\beta_k})\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}\, u(x, t/(k+1))\right\|_p \\ &\leqq C(1+t^{-\beta})\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}}\, u(x, t/(k+1))\right\|_p\ .\end{aligned}$$

Hence

$$\left(\int_0^\infty \left(t^{k_1+k_2-\alpha}\left\|\frac{\partial^{k_1+k_2}}{\partial t^{k_1+k_2}} g_k(x,t)\right\|_p\right)^q \frac{dt}{t}\right)^{1/q} \leqq C\|f\|_{\alpha+\beta;p,q}\ .$$

From Lemma 17, there exists a function $g \in \tilde{\Lambda}(\alpha; p, q)$ $(1 \leqq q < \infty)$ such that $g_k(x, t) = P_t g(x)$, so that $J^\beta g(x) = f(x)$. In the case $q = \infty$ the corresponding result also follows. Moreover the result for $\tilde{\lambda}(\alpha; p, \infty)$ is verified easily from the above paragraph.

THEOREM 18. *Let* $\alpha > 0$, *and let* $1 \leqq p \leqq \infty, 1 \leqq q \leqq \infty$. *Then*

$\Lambda(\alpha; p, q) = \tilde{\Lambda}(\alpha; p, q)$ *and* $\lambda(\alpha; p, \infty) = \tilde{\lambda}(\alpha; p, \infty)$.

PROOF. This theorem is an immediate consequence of Theorem 7, Theorem 12, Theorem 16 and Theorem 17.

5. Sobolev spaces. We summarize the fundamental properties of the operator $\mathfrak{J}_p^\alpha$ derived from the general theory of fractional powers of operators. We shall refer to the comprehensive treatment of this subject in the papers of Folland [5] and Komatsu [10], [11]. Suppose that $1 < p < \infty$, $\operatorname{Re}\alpha > 0$ and $k = [\operatorname{Re}\alpha] + 1$. The operator $\mathfrak{J}_p^\alpha$ is defined by

$$\mathfrak{J}_p^\alpha f = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(k-\alpha)} \int_\varepsilon^\infty t^{k-\alpha-1} \mathfrak{J}^k H_t f\, dt$$

on the domain $D(\mathfrak{J}_p^\alpha)$ of all $f \in L^p$ such that the limit exists in the L^p-norm. The operator $\mathfrak{J}_p^{-\alpha}$ is defined by

$$\mathfrak{J}_p^{-\alpha} f = \lim_{\eta\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^\eta t^{\alpha-1} H_t f\, dt$$

on the domain $D(\mathfrak{J}_p^{-\alpha})$ of all $f \in L^p$ such that the limit exists in the L^p-norm. If $\operatorname{Re}\alpha > 0$ and $k = [\operatorname{Re}\alpha] + 1$, we define $(1 + \mathfrak{J}_p)^\alpha$ by

$$(1 + \mathfrak{J}_p)^\alpha f = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(k-\alpha)} \int_\varepsilon^\infty t^{k-\alpha-1} e^{-t} (1 + \mathfrak{J})^k H_t f\, dt$$

on the domain $D((1 + \mathfrak{J}_p)^\alpha)$ of all $f \in L^p$ such that the limit exists in the L^p-norm. Also, we define $(1 + \mathfrak{J}_p)^{-\alpha}$ by

$$(1 + \mathfrak{J}_p)^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} H_t f\, dt$$

on L^p. The function $(1 + \mathfrak{J}_p)^{-\alpha} f$ of this definition coincides with the function $J^{2\alpha} f$ of Definition 3 for $f \in L^p$. To consider the case $\operatorname{Re}\alpha = 0$, let $\mathfrak{J}_2 = \int_0^\infty \lambda\, dE(\lambda)$ be the spectral resolution of the operator $\mathfrak{J}_2$. Then for $\operatorname{Re}\alpha \neq 0$,

$$\mathfrak{J}_2^\alpha = \int_0^\infty \lambda^\alpha dE(\lambda), \quad (1 + \mathfrak{J}_2)^\alpha = \int_0^\infty (1 + \lambda)^\alpha dE(\lambda) .$$

These make sense even if $\operatorname{Re}\alpha = 0$ and can be extended to other values of p. The fractional powers $\mathfrak{J}_p^\alpha$ $(\alpha \in \boldsymbol{C})$ of $\mathfrak{J}_p$ $(1 < p < \infty)$ defineda bove coincide with the fractional powers in the sense of Komatsu. We shall list the fundamental properties of fractional powers $\mathfrak{J}_p^\alpha$ $(\alpha \in \boldsymbol{C})$ of $\mathfrak{J}_p$ $(1 < p < \infty)$:

(a) $\mathfrak{J}_p^\alpha$ is a closed operator on L^p.

(b) If k is a positive integer, $D(\mathfrak{J}_p^k)$ is defined inductively to be the set of all $f \in D(\mathfrak{J}_p^{k-1})$ such that $\mathfrak{J}_p^{k-1}f \in D(\mathfrak{J}_p)$, and $\mathfrak{J}_p^k f = \mathfrak{J}_p \mathfrak{J}_p^{k-1} f$.

(c) If $f \in D(\mathfrak{J}_p^\beta) \cap D(\mathfrak{J}_p^{\alpha+\beta})$ then $\mathfrak{J}_p^\beta f \in D(\mathfrak{J}_p^\alpha)$ and $\mathfrak{J}_p^\alpha \mathfrak{J}_p^\beta f = \mathfrak{J}_p^{\alpha+\beta} f$. Moreover, $\mathfrak{J}_p^{\alpha+\beta}$ is the smallest closed extension of $\mathfrak{J}_p^\alpha \mathfrak{J}_p^\beta$. In particular $\mathfrak{J}_p^{-\alpha} = (\mathfrak{J}_p^\alpha)^{-1}$.

(d) $\mathfrak{J}_p^\alpha$ is the dual operator of $\mathfrak{J}_{p'}^\alpha$, where $1/p + 1/p' = 1$.

(e) If $f \in D(\mathfrak{J}_p^\alpha) \cap L^q$ then $f \in D(\mathfrak{J}_q^\alpha)$ if and only if $\mathfrak{J}_p^\alpha f \in L^q$, in which case $\mathfrak{J}_p^\alpha f = \mathfrak{J}_q^\alpha f$.

(f) If $\operatorname{Re}\alpha < \operatorname{Re}\beta$ and $f \in D(\mathfrak{J}_p^\alpha) \cap D(\mathfrak{J}_p^\beta)$, then $f \in D(\mathfrak{J}_p^\gamma)$ whenever $\operatorname{Re}\alpha \leqq \operatorname{Re}\gamma \leqq \operatorname{Re}\beta$, and $\|\mathfrak{J}_p^\gamma f\|_p \leqq C\|\mathfrak{J}_p^\alpha f\|_p^\theta \|\mathfrak{J}_p^\beta f\|_p^{1-\theta}$ where $\theta = \operatorname{Re}(\gamma-\alpha)/\operatorname{Re}(\beta-\alpha)$. Moreover, $\mathfrak{J}_p^\gamma f$ is an analytic L^p valued function of γ on the strip $\operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}\beta$ and is continuous on $\operatorname{Re}\alpha \leqq \operatorname{Re}\gamma \leqq \operatorname{Re}\beta$.

(g) If $\operatorname{Re}\alpha = 0$, $\|\mathfrak{J}_p^\alpha f\|_p \leqq C|\Gamma(1-\alpha)|^{-1}\|f\|_p$ for $f \in L^p$.

The same results from (a) to (g) also hold for the operator $(1 + \mathfrak{J}_p)^\alpha$ $(\alpha \in \boldsymbol{C})$.

(h) If $\operatorname{Re}\alpha > 0$, then $D(\mathfrak{J}_p^\alpha) = D((1 + \mathfrak{J}_p)^\alpha) = R((1 + \mathfrak{J}_p)^{-\alpha})$ where $R((1 + \mathfrak{J}_p)^{-\alpha})$ is the range of the operator $(1 + \mathfrak{J}_p)^{-\alpha}$.

(i) For any $f \in L^p$ and $\operatorname{Re}\alpha > 0$, $H_t f \in D(\mathfrak{J}_p^\alpha)$ and $e^{-t}H_t f \in D((1+\mathfrak{J}_p)^\alpha)$. Moreover, $H_t \mathfrak{J}_p^\alpha f = \mathfrak{J}_p^\alpha H_t f$ for $f \in D(\mathfrak{J}_p^\alpha)$. Henceforth we use sometimes the properties (a)$\sim$(i) in the proofs below without mention. Now we shall give a definition of Sobolev spaces L_α^p.

DEFINITION 5. Let α be a real and let $1 \leqq p \leqq \infty$. We define the space L_α^p to be the space of all forms $J^\alpha u$ where $u(x, t) = H_t f(x)$, $f \in L^p$, equipped with the norm $\|J^\alpha u\|_{\alpha;p} = \|f\|_p$. This space L_α^p is obviously a Banach space. If $\alpha > 0$, then we shall identify L_α^p with the space $\{J^\alpha f\colon f \in L^p\}$ under the canonical isomorphism. Folland [5] has defined the Sobolev space S_α^p as $S_\alpha^p = D(\mathfrak{J}_p^{\alpha/2})$ where $\alpha \geqq 0$ and $1 < p < \infty$. This definition coincides with our definition when $\alpha \geqq 0$ and $1 < p < \infty$ from the property (h) of $\mathfrak{J}_p^\alpha$ mentioned previously.

PROPOSITION 4. (i) *Let $1 \leqq p \leqq \infty$ and let α, β real. Then J^β is an isometrical isomorphism of L_α^p onto $L_{\alpha+\beta}^p$.*

(ii) *If $\alpha \leqq \beta$, and $1 \leqq p \leqq \infty$ then $L_\beta^p \subset L_\alpha^p$ and the inclusion mapping is continuous.*

(iii) *If $1 < p < q < \infty$ and $\beta = \alpha - \rho(1/p - 1/q)$, then $L_\alpha^p \subset L_\beta^q$ and the inclusion mapping is continuous.*

(iv) *If k is a positive integer and $1 < p < \infty$, then $L_k^p = \{f \in L^p\colon D^\alpha f \in L^p$ for $|\alpha| \leqq k\}$ and the norm in L_k^p is equivalent to $\sum_{|\alpha| \leqq k} \|D^\alpha f\|_p$ where $D^\alpha f$ is a derivative of f in the sense of distributions.*

PROOF. Parts (i) and (ii) are obvious. To prove (iii), let $u(x, t) = H_t f(x)$, $f \in L^p$. From Lemma 6 (i) taking $r = (p + q)/2$ and $\delta = \rho(1/p - 1/q)$,

$$\left\{\int_0^\infty t^{q\delta/2-1}e^{-t}\,\|u(\cdot,\,t)\|_r^q dt\,\right\}^{1/q} \leqq C\|f\|_p < \infty\,.$$

Hence

$$\left\{\int_0^\infty t^{q(\alpha-\beta-\rho(1/r-1/q))/2}e^{-t}\|u(\cdot,\,t)\|_r^q dt\,\right\}^{1/q}\,.$$

From Lemma 6 (ii), there exists a function $g \in L^q$ such that $J^{\alpha-\beta}u(x, t) = H_t g(x)$ and $\|g\|_q \leqq C\|f\|_p$, so that $J^\alpha u = J^\beta(J^{\alpha-\beta}u) = J^\beta H_t g$, that is, $L_\alpha^p \subset L_\beta^q$ and the inclusion mapping is continuous. Part (iv) is that of Corollary (4.13) in Folland's paper [5].

THEOREM 19. (i) *Assume that* $1 < p \leqq q \leqq \infty$, $1 < p < r < \infty$ *and* $\delta = \rho(1/p - 1/r)$. *Then* $L_\alpha^p \subset \mathfrak{T}\Lambda(\alpha - \delta; r, q)$.

(ii) *Assume that* $1 < p < r < \infty$, $1 < q \leqq r$, $\delta = \rho(1/p - 1/r)$. *Then* $\mathfrak{T}\Lambda(\alpha + \delta; p, q) \subset L_\alpha^r$. *Further, these inclusion mappings are continuous.*

PROOF. Let $u \in L_\alpha^p$, then there exists a function $f \in L^p$ such that $J^{-\alpha}u(x, t) = H_t f(x)$. Let $\beta > \alpha - \delta$. Then from Lemma 3 and Lemma 6,

$$\left\{\int_0^\infty t^{q(\beta-\alpha+\delta)/2-1}e^{-t}\|J^{-\beta}u(\cdot,\,t)\|_r^q dt\right\}^{1/q} = \left\{\int_0^\infty t^{q(\delta+\beta-\alpha)/2-1}e^{-t}\|J^{-(\beta-\alpha)}J^{-\alpha}u(\cdot,\,t)\|_r^q dt\right\}^{1/q}$$
$$= \left\{\int_0^\infty t^{q(\delta+\beta-\alpha)/2-1}e^{-t}\|J^{-(\beta-\alpha)}H_t f\|_r^q dt\right\}^{1/q} \leqq C\left\{\int_0^\infty t^{q\delta/2-1}e^{-t}\|H_t f\|_r^q dt\right\}^{1/q} \leqq C\|f\|_p\,,$$

so that $u \in \mathfrak{T}\Lambda(\alpha - \delta; r, q)$. (For $q = \infty$, the corresponding inequality also follows similarly.) This completes the proof of (i).

Let $u \in \mathfrak{T}\Lambda(\alpha + \delta; p, q)$ and let $\beta > \alpha + \delta$. Then, using Lemma 6,

$$\|J^{-\alpha}u(\cdot,\,s)\|_r = \|J^{\beta-\alpha}(J^{-\beta}u(\cdot,\,s))\|_r \leqq C\left\{\int_0^\infty t^{q(\beta-\alpha-\delta)/2-1}e^{-t}\|J^{-\beta}u(\cdot,\,t)\|_p^q dt\,\right\}^{1/q}$$

and there exists a function $f \in L^r$ such that $J^{-\alpha}u(x, t) = H_t f$. Hence $u \in L_\alpha^p$. This completes the proof of (ii).

We shall give an alternative definition of the Sobolev space L_α^p when $\alpha > 0$.

DEFINITION 6. Let $\alpha > 0$, and let $1 < p < \infty$, and k be a positive integer with $k > \alpha/2 + 1$. Then we denote by $\mathfrak{L}_\alpha^p$ the space of those $f \in L^p$ such that $\left\|\left(\int_0^\infty |t^{k-\alpha/2}\mathfrak{F}^k H_t f(x)|^2 t^{-1}dt\right)^{1/2}\right\|_p < \infty$, equipped with the norm

$$|||f|||_{\alpha;p} = \|f\|_p + \left\|\left(\int_0^\infty |t^{k-\alpha/2}\mathfrak{F}^k H_t f(x)|^2 \frac{dt}{t}\right)^{1/2}\right\|_p\,.$$

LEMMA 18. *Let* $\alpha > 0$, $\beta \geqq 0$, $\gamma \geqq 1$ *and let* $1 \leqq p \leqq q \leqq \infty$. *If* $u(x, t) = H_t f(x)$, $f \in L^r$, $1 < r < \infty$, *such that* $\int_0^\infty |t^{\alpha+\gamma}\mathfrak{F}^{\beta+\gamma}u(x, t)|^p t^{-1}dt < \infty$, *then*

$$\left(\int_0^\infty |t^\alpha \mathfrak{J}^\beta u(x, t)|^q \frac{dt}{t}\right)^{1/q} \leqq C\left(\int_0^\infty |t^{\alpha+\gamma}\mathfrak{J}^{\beta+\gamma}u(x, t)|^p \frac{dt}{t}\right)^{1/q} .$$

(*We have the above inequality under usual modification in the case* $q = \infty$ *or* $p = \infty$. *We note that* $\mathfrak{J}^\alpha u$ *is well-defined for all* $\alpha > 0$.)

PROOF. We see that

$$\mathfrak{J}^\beta u(x, s) = \mathfrak{J}^{-\gamma}\mathfrak{J}^{\beta+\gamma}u(x, s) = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}\mathfrak{J}^{\beta+\gamma}u(x, s+t)dt .$$

Hence

$$|\mathfrak{J}^\beta u(x, s)| \leqq C\int_0^\infty t^{\gamma-1}|\mathfrak{J}^{\beta+\gamma}u(x, s+t)|dt \leqq C\int_s^\infty t^{\gamma-1}|\mathfrak{J}^{\beta+\gamma}u(x, t)|dt ,$$

so that

$$\left(\int_0^\infty s^\alpha \mathfrak{J}^\beta u(x, s)^q \frac{ds}{s}\right)^{1/q} \leqq C\left(\int_0^\infty \left(s^\alpha \int_s^\infty t^{\gamma-1}|\mathfrak{J}^{\beta+\gamma}u(x, t)|dt\right)^q \frac{ds}{s}\right)^{1/q}$$

$$= C\left(\int_0^\infty\left(\int_s^\infty s^\alpha t^{-\alpha}|t^{\alpha+\gamma}\mathfrak{J}^{\beta+\gamma}u(x, t)|\frac{dt}{t}\right)^q \frac{ds}{s}\right)^{1/q} .$$

By Lemma 8, we get

$$\left(\int_0^\infty |s^\alpha \mathfrak{J}^\beta u(x, s)|^q \frac{ds}{s}\right)^{1/q} \leqq C\left(\int_0^\infty |t^{\alpha+\gamma}\mathfrak{J}^{\beta+\gamma}u(x, t)|^p \frac{dt}{t}\right)^{1/p} .$$

The corresponding inequality for $q = \infty$ or $p = \infty$ also holds. This completes the proof of Lemma 18.

X will denote Banach spaces, and $L^p(M, d\mu; X)$ the space of X-valued L^p-functions on a measure space $(M, d\mu)$, equipped with the norm $\left(\int_M \|f\|^p d\mu\right)^{1/p}$ where $\|\cdot\|$ is a norm in X. If $I = (0, \infty)$, the space $L_*^p(I, X)$ means the space $L^p(I, t^{-1}dt; X)$ and $L_*^p(I)$ means the space $L^p(I, t^{-1}dt; \boldsymbol{R})$.

THEOREM 20. *Let* $\alpha > 0$. *Then*

(i) $\Lambda(\alpha; p, p) \subset \mathfrak{L}_\alpha^p \subset \Lambda(\alpha; p, 2)$ $(1 < p \leqq 2)$

(ii) $\Lambda(\alpha; p, 2) \subset \mathfrak{L}_\alpha^p \subset \Lambda(\alpha; p, p)$ $(2 \leqq p < \infty)$.

In particular,

$$\Lambda(\alpha; p, 1) \subset \mathfrak{L}_\alpha^p \subset \Lambda(\alpha; p, \infty) \quad (1 < p < \infty) .$$

These inclusion mappings are all continuous.

PROOF. To prove (i), let $1 < p \leqq 2$ and let f be any element in $\Lambda(\alpha; p, p)$, and let k be an integer as in the definition of $\mathfrak{L}_\alpha^p$. Put $u(x, t) = H_t f(x)$. From Theorem 13, we have $t^{k+1-\alpha/2}\mathfrak{J}^{k+1}u \in L_*^p(I; L^p) = L^p(G; L_*^p(I))$.

By Lemma 18, $t^{k-\alpha/2}\mathfrak{J}^k u \in L^p(G; L^2_*(I))$. Therefore $f \in \mathfrak{L}^p_\alpha$.

On the other hand, let f be any element in $\mathfrak{L}^p_\alpha$, that is, $t^{k-\alpha/2}\mathfrak{J}^k u \in L^p(G; L^2_*(I))$. Then, by use of Minkowski's inequality, $t^{k-\alpha/2}\mathfrak{J}^k u \in L^2_*(I; L^p)$. Thus, by Theorem 13, $f \in \Lambda(\alpha; p, 2)$.

Next we shall prove (ii). Suppose $2 \leqq p < \infty$. Let $f \in \Lambda(\alpha; p, 2)$, that is, $t^{k-\alpha/2}\mathfrak{J}^k u \in L^2_*(I; L^p)$ (Theorem 13). Using Minkowski's inequality, we get $t^{k-\alpha/2}\mathfrak{J}^k u \in L^p(G; L^2_*(I))$. Hence $f \in \mathfrak{L}^p_\alpha$. On the other hand, let $f \in \mathfrak{L}^p_\alpha$, that is, $t^{k-\alpha/2}\mathfrak{J}^k u \in L^p(G; L^2_*(I))$. Then by Lemma 18,

$$t^{k-1-\alpha/2}\mathfrak{J}^{k-1} u \in L^p(G; L^p_*(I)) = L^p_*(I; L^p) .$$

Thus $f \in \Lambda(\alpha; p, p)$. Further, from Theorem 14, $\Lambda(\alpha; p, 1) \subset \mathfrak{L}^p_\alpha \subset \Lambda(\alpha; p, \infty)$.

THEOREM 21. *Let $\alpha > 0$, and let $1 < p < \infty$, then $L^p_\alpha = \mathfrak{L}^p_\alpha$ and the norm $\|\cdot\|_{\alpha;p}$ is equivalent to $|||\cdot|||_{\alpha;p}$.*

REMARK. This theorem implies that the definition of $\mathfrak{L}^p_\alpha$ is independent of the choice of an integer $k > \alpha/2 + 1$.

To prove the above theorem we need the following lemma:

LEMMA 19. *Let $1 < p < \infty$. If $f \in L^p$, we define a g-function $g_k(f)$ for an integer $k \geqq 1$ by*

$$g_k(f)(x) = \left(\int_0^\infty |t^k\mathfrak{J}^k H_t f(x)|^2 \frac{dt}{t}\right)^{1/2} .$$

Then $C\|f\|_p \leqq \|g_k(f)\|_p \leqq C\|f\|_p$.

PROOF. To see that $C\|f\|_p \leqq \|g_1(f)\|_p$, it suffices to show that $E_0 = 0$ in Stein's book [21: Chapter V, Section 6, Corollary 2]. Suppose $f \in E_0(L^2)$, that is, $H_t f = f$ for all $t > 0$. By Theorem 4 (ii), $\|f\|_\infty = \|H_t f\|_\infty \leqq Ct^{-\rho/4}\|f\|_2$ for $t > 0$. Taking $t \to \infty$, we get $\|f\|_\infty = 0$. Hence $f = 0$, that is, $E_0 = 0$. From [21: Chapter V, Section 6, Corollary 1], $\|g_k(f)\|_p \leqq C\|f\|_p$. By Lemma 18, $g_k(f) \geqq g_1(f)$ on G. Therefore,

$$C\|f\|_p \geqq \|g_k(f)\|_p \geqq \|g_1(f)\|_p \geqq C\|f\|_p .$$

Now we return to Theorem 20. Let $f \in \mathfrak{L}^p_\alpha$ and set $u(x, t) = H_t f(x)$. Since $u \in D(\mathfrak{J}^\beta_p)$ for all $\beta > 0$, $\mathfrak{J}^{\alpha/2}u$ belongs to L^p. Hence we see from Lemma 18 and Lemma 19 that

$$\begin{aligned}\|\mathfrak{J}^{\alpha/2}u(x, s)\|_p &\leqq C\|g_1(\mathfrak{J}^{\alpha/2}u(x, s))\|_p = C\left\|\left(\int_0^\infty |t\mathfrak{J}H_t\mathfrak{J}^{\alpha/2}u(x, s)|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \\ &= C\left\|\left(\int_0^\infty |t\mathfrak{J}^{1+\alpha/2}u(x, s+t)|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \leqq C\left\|\left(\int_0^\infty |t\mathfrak{J}^{1+\alpha/2}u(x, t)|^2 t^{-1}dt\right)^{1/2}\right\|_p \\ &\leqq C\left\|\left(\int_0^\infty |t^{k-\alpha/2}\mathfrak{J}^k u(x, t)|^2 t^{-1}dt\right)^{1/2}\right\|_p .\end{aligned}$$

Hence, $||\mathfrak{J}^{\alpha/2}u(x, s)||_p$ is uniformly bounded with respect to $s > 0$. By Theorem 3, there exists a function $g \in L^p$ such that $\mathfrak{J}^{\alpha/2}u(x, s) = H_s g(x)$. By Theorem 2, $u(x, s)$ and $H_s g(x)$ converge to f and g, respectively in the L^p-norm. Since $\mathfrak{J}_p^{\alpha/2}$ is a closed operator, $\mathfrak{J}_p^{\alpha/2} f = g$, that is, $f \in D(\mathfrak{J}_p^{\alpha/2}) = L_\alpha^p$. Conversely, let $f \in D(\mathfrak{J}_p^{\alpha/2}) = L_\alpha^p$. Then $\mathfrak{J}_p^{\alpha/2} f \in L^p$ and $\mathfrak{J}_p^{\alpha/2} H_t f = H_t \mathfrak{J}_p^{\alpha/2} f$. From Lemma 18 and Lemma 19, we have

$$\left\|\left(\int_0^\infty |t^{k-\alpha/2}\mathfrak{J}^k H_t f(x)|^2 \frac{dt}{t}\right)^{1/2}\right\|_p = \left(\int_0^\infty |t^{k-\alpha/2}\mathfrak{J}^{k-\alpha/2} H_t \mathfrak{J}_p^{\alpha/2} f(x)|^2 \frac{dt}{t}\right)^{1/2}\Bigg\|_p$$

$$\leqq C\left\|\left(\int_0^\infty |t^{k+1}\mathfrak{J}^{k+1} H_t \mathfrak{J}_p^{\alpha/2} f(x)|^2 \frac{dt}{t}\right)^{1/2}\right\|_p \leqq C||g_{k+1}(\mathfrak{J}_p^{\alpha/2} f)||_p \leqq C||\mathfrak{J}_p^{\alpha/2} f||_p < \infty\ .$$

Therefore, $f \in \mathfrak{L}_\alpha^p$.

COROLLARY. *If α is real and $1 < p < \infty$, then*

(i) $\mathfrak{T}\Lambda(\alpha; p, p) \subset L_\alpha^p \subset \mathfrak{T}\Lambda(\alpha; p, 2)$ $(1 < p \leqq 2)$

(ii) $\mathfrak{T}\Lambda(\alpha; p, 2) \subset L_\alpha^p \subset \mathfrak{T}\Lambda(\alpha; p, p)$ $(2 \leqq p < \infty)$.

In particular,

$$\mathfrak{T}\Lambda(\alpha; p, 1) \subset L_\alpha^p \subset \mathfrak{T}\Lambda(\alpha; p, \infty) \quad (1 < p < \infty)\ .$$

These inclusion mappings are all continuous.

This corollary is immediate from Theorem 7, Proposition 4 (i) and Theorem 20.

6. Interpolation theorems for Besov spaces and Sobolev spaces. Now we shall discuss the interpolation space of Besov spaces and Sobolev spaces. First we recall the definition of interpolation spaces of real and complex methods (example, see [1]). Let (X_0, X_1) be an interpolation couple of Banach spaces and let ξ_0 and ξ_1 be real numbers with $\xi_0\xi_1 < 0$, and let $1 \leqq p_0 \leqq \infty$, $1 \leqq p_1 \leqq \infty$. We denote by $W(p_0, \xi_0, X_0; p_1, \xi_1, X_1)$ the space of those functions $u(t)$ such that $t^{\xi_0}u(t) \in L_*^{p_0}(I; X_0)$, $t^{\xi_1}u(t) \in L_*^{p_1}(I; X_1)$, equipped with the norm

$$||t^{\xi_0}u(t)||_{L_*^{p_0}(I;X_0)} + ||t^{\xi_1}u(t)||_{L_*^{p_1}(I;X_1)}\ .$$

We denote by $(X_0, X_1)_{\theta,p}$ the space of all forms $a = \int_0^\infty u(t)t^{-1}dt$ taking $u \in W(p_0, \xi_0, X_0; p_1, \xi_1, X_1)$ where $\theta = \xi_0/(\xi_0 - \xi_1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, equipped with the norm

$$a \mapsto \inf\Big\{||t^{\xi_0}u(t)||_{L_*^{p_0}(I;X_0)} + ||t^{\xi_1}u(t)||_{L_*^{p_1}(I;X_1)}:$$

$$u \in W(p_0, \xi_0, X_0; p_1, \xi_1, X_1) \text{ such that } a = \int_0^\infty u(t)\,\frac{dt}{t}\Big\}\ .$$

Then we have

$$(X_0, X_1)_{\theta,p} = \{a \in X_0 + X_1\colon \text{ for all } t > 0,\ a = u(t) + v(t) \text{ with } t_\jmath^{\xi_0} u(t) \in L_*^{p_0}(I; X_0),\ t^{\xi_1} v(t) \in L_*^{p_1}(I; X_1)\}\,.$$

We denote by $\mathfrak{F}(X_0, X_1)$ the space of all functions $f(\xi)$, $\xi = s + it$ defined in the strip $0 \leqq s \leqq 1$ of the ξ-plane, with values in $X_0 + X_1$ continuous and bounded with respect to the norm of $X_0 + X_1$ in $0 \leqq s \leqq 1$ and analytic in $0 < s < 1$, and such that $f(it) \in X_0$ is X_0-continuous and tends to zero as $|t| \to \infty$, $f(1 + it) \in X_1$ is X_1-continuous and tends to zero as $|t| \to \infty$, equipped with the norm

$$\|f\|_{\mathfrak{F}} = \max[\sup_t \|f(it)\|_{X_0}\,,\quad \sup_t \|f(1 + it)\|_{X_1}]\,.$$

Given $0 < \theta < 1$, the interpolation space $[X_0, X_1]_\theta$ is defined by

$$[X_0, X_1]_\theta = \{x\colon\ x = f(\theta);\ f \in \mathfrak{F}(X_0, X_1)\}$$

equipped with the norm

$$\|x\|_\theta = \inf\{\|f\|_{\mathfrak{F}}\colon\ f(\theta) = x\}\,.$$

We defined the space $L_*^{p,\sigma}(I; X)$ by $L^p(I, t^{-\sigma p-1}dt; X)$ $(1 \leqq p \leqq \infty)$ where X is a Banach space. To prove interpolation theorems we need the following lemmas:

Lemma 20. (i) *Let σ, τ be real numbers with $\sigma \neq \tau$ and let $0 < \theta < 1$. Put $\mu = (1 - \theta)\sigma + \theta\tau$. Then*

$$(L_*^{\infty,\sigma}(I; X),\ L_*^{\infty,\tau}(I; X))_{\theta,q} \subset L_*^{q,\mu}(I; X) \subset (L_*^{1,\sigma}(I; X),\ L_*^{1,\tau}(I; X))_{\theta,q}$$

with continuous inclusion mappings.

(ii) *Assume that X, X_0 and X_1 are Banach spaces and that α_0, α_1 real numbers, $1 \leqq p_0 < \infty$, $1 \leqq p_1 < \infty$, $0 < \theta < 1$. Then*

$$[L^{p_0}(G; X_0),\ L^{p_1}(G; X_1)]_\theta = L^p(G; [X_0, X_1]_\theta)\,,$$
$$(L_*^{p_0,\alpha_0}(I, X),\ L_*^{p_1,\alpha_1}(I; X))_{\theta,p} = L_*^{p,\alpha}(I; X)\,,$$

and

$$[L_*^{p_0,\alpha_0}(I, X),\ L_*^{p_1,\alpha_1}(I; X)]_\theta = L_*^{p,\alpha}(I; X)$$

where $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$.

Part (i) is Lemma 7.4 of Muramatsu's paper [18], and for (ii), example see J. Bergh and J. Löfström [1] or Calderón [2].

Lemma 21. (i) *Let k be a positive integer and $0 < \theta < 1$, $1 \leqq q \leqq \infty$, $1 < p < \infty$. Then $(L_{2k}^p, L^p)_{\theta,q} = \Lambda(2\theta k; p, q)$.*

(ii) *If σ and τ are positive numbers and $1 \leqq \xi, \eta \leqq \infty$, $1 < p < \infty$,* *then* $[\Lambda(\sigma; p, \xi), \Lambda(\tau; p, \eta)]_\theta = \Lambda(\mu; p, \zeta)$ *where* $1/\zeta = (1-\theta)/\xi + \theta/\eta$ *and* $\mu = (1-\theta)\sigma + \theta\tau$.

PROOF. Komatsu [11: Theorem 3.1] implies (i) and Grisvard [6: Theorem 4.1] implies (ii).

LEMMA 22. *Let* k *be a positive number and* $0 < \theta < 1$, $1 \leqq q \leqq \infty$, $1 < p < \infty$. *Then*

(i) *The mapping*

$$f \mapsto v(\lambda) \frac{(2k-1)!}{[(k-1)!]^2} \lambda^k \mathfrak{J}^k (\lambda + \mathfrak{J})^{-2k} f$$

is linear and continuous from $\Lambda(2k\theta; p, q)$ *onto* $W(q, -k\theta, L^p_{2k}; q, k(1-\theta), L^p)$ *and* $f = \int_0^\infty v(\lambda)(d\lambda/\lambda)$.

(ii) *The mapping*

$$u \mapsto f = \frac{(2k-1)!}{[(k-1)!]^2} \int_0^\infty \lambda^{k-1} \mathfrak{J}^k (\lambda + \mathfrak{J})^{-2k} u(\lambda) d\lambda$$

is linear and continuous from $L^{q,k\theta}_*(I; L^p_{2k}) + L^{q,-k(1-\theta)}_*(I; L^p)$ *onto* $\Lambda(2k\theta; p, q)$.

PROOF. This follows from Grisvard [6: Proposition 3.1 and Proposition 3.2] using Lemma 21 (i).

PROPOSITION 5. *Let* σ, τ *be real with* $\sigma \neq \tau$ *and* $1 \leqq p \leqq \infty$, $1 \leqq \xi, \eta < \infty$ *and* $0 < \theta < 1$. *Put* $\mu = (1-\theta)\sigma + \theta\tau$ *and* $1/\zeta = (1-\theta)/\xi + \theta/\eta$. *Then*

(i) $(\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta))_{\theta,\zeta} \subset \mathfrak{T}\Lambda(\mu; p, \zeta)$,

(ii) $[\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta)]_\theta \subset \mathfrak{T}\Lambda(\mu; p, \zeta)$.

PROOF. By Theorem 10 (ii) the mapping $T: f \mapsto (\partial^k/\partial t^k) H_t f$ is linear and continuous from $\Lambda(\alpha; p, q)$ into $L^{q,k-\alpha/2}_*(I; L^p)$ for an integer $k > \alpha/2$. By Theorem 7, we may assume $\sigma, \tau > 0$. By interpolation, for a fixed integer $k > \max(\sigma/2, \tau/2)$,

$$Tf \in (L^{\xi,k-\sigma/2}_*(I; L^p), L^{\eta,k-\tau/2}_*(I; L^p))_{\theta,\zeta}$$

for all $f \in (\Lambda(\sigma; p, \xi), \Lambda(\tau; p, \eta))_{\theta,\zeta}$. By Lemma 20 (ii), $Tf \in L^{\zeta,k-\mu/2}_*(I; L^p)$. This implies that $f \in \Lambda(\mu; p, \zeta)$, that is,

$$(\Lambda(\sigma; p, \xi), \Lambda(\tau; p, \eta))_{\theta,\zeta} \subset \Lambda(\mu; p, \zeta) .$$

Similarly, $[\Lambda(\sigma; p, \xi), \Lambda(\tau; p, \eta)]_\theta \subset \Lambda(\mu; p, \zeta)$.

If $1 < p < \infty$, we have a result stronger than the above proposition.

THEOREM 22. *Let* σ *and* τ *be real numbers with* $\sigma \neq \tau$ *and let*

$0<\theta<1,\ \ 1\leqq \xi, \eta, q\leqq\infty,\ \ 1<p<\infty$. *Set* $\mu=(1-\theta)\sigma+\theta\tau$; *then*

(i) $(\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta))_{\theta,q}=\mathfrak{T}\Lambda(\mu; p, q)$,

(ii) $[\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta)]_\theta=\mathfrak{T}\Lambda(\mu; p, \zeta)$ *where* $1/\zeta=(1-\theta)/\xi+\theta/\eta$.

PROOF. Let $0<\sigma, \tau<1$ with $\sigma\neq\tau$, and let k be a positive integer $1\leqq q\leqq\infty$, $1<p<\infty$ and $0<\theta<1$. If $f\in(\Lambda(2k\sigma; p, \infty), \Lambda(2k\tau; p, \infty))_{\theta,q}\subset\Lambda(2k\sigma; p, \infty)+\Lambda(2k\tau; p, \infty)$, then by Lemma 22 (i), there exists a mapping T_1 from $\Lambda(2k\sigma; p, \infty)$ onto $W(\infty; -k\sigma, L^p_{2k}; \infty, k(1-\sigma), L^p)$ and also from $\Lambda(2k\tau; p, \infty)$ onto $W(\infty, -k\tau, L^p_{2k}; \infty, k(1-\tau), L^p)$ such that $f=\int_0^\infty u(\lambda)\lambda^{-1}d\lambda$ with $T_1f=u(\lambda)$. Put $W_1=W(\infty, -k\sigma, L^p_{2k}; \infty, k(1-\sigma), L^p)$ and $W_2=W(\infty, -k\tau, L^p_{2k}; \infty, k(1-\tau), L^p)$. By interpolation, we have $u(\lambda)\in(W_1, W_2)_{\theta,q}$. Hence,

$$u(\lambda)\in(L^{\infty,k\sigma}_*(I; L^p_{2k}), L^{\infty,k\tau}_*(I; L^p_{2k}))_{\theta,q}$$

and

$$u(\lambda)\in(L^{\infty,-k(1-\sigma)}_*(I; L^p), L^{\infty,-k(1-\tau)}_*(I; L^p))_{\theta,q}\ .$$

From Lemma 20 (i),

$$u(\lambda)\in L^{q,k\mu}_*(I; L^p_{2k})\quad\text{and}\quad u(\lambda)\in L^{q,-k(1-\mu)}_*(I; L^p)\ ,$$

so that $u(\lambda)\in W(q, -k\mu, L^p_{2k}; q, k(1-\mu), L^p)$. Since $f=\int_0^\infty u(\lambda)\lambda^{-1}d\lambda$, we get, using Lemma 21 (i), $f\in(L^p_{2k}, L^p)_{\mu,q}=\Lambda(2\mu k; p, q)$. Hence

$$(\Lambda(2k\sigma; p, \infty), \Lambda(2k\tau; p, \infty))_{\theta,q}\subset\Lambda(2\mu k; p, q)$$

for a positive integer k and $0<\sigma, \tau<1$, $\sigma\neq\tau$. On the other hand, if $f\in\Lambda(2\mu k, p, q)=(L^p_{2k}, L^p)_{\mu,q}$, then we write $f=v_0(\lambda)+v_1(\lambda)$ where $v_0(\lambda)\in L^{q,k\mu}_*(I; L^p_{2k})$ and $v_1(\lambda)\in L^{q,-k(1-\mu)}_*(I; L^p)$. From Lemma 20 (i),

$$v_0(\lambda)\in(L^{1,k\sigma}_*(I; L^p_{2k}), L^{1,k\tau}_*(I; L^p_{2k}))_{\theta,q}$$

and

$$v_1(\lambda)\in(L^{1,-k(1-\sigma)}_*(I; L^p), L^{1,-k(1-\tau)}_*(I; L^p))_{\theta,q}\ .$$

We put

$$L_1=L^{1,k\sigma}_*(I; L^p_{2k})+L^{1,-k(1-\sigma)}_*(I; L^p)$$

and

$$L_2=L^{1,k\tau}_*(I; L^p_{2k})+L^{1,-k(1-\tau)}_*(I; L^p)\ .$$

Hence, $f=v_0(\lambda)+v_1(\lambda)=u(\lambda)\in(L_1, L_2)_{\theta,q}$. By Lemma 22 (ii), there is a mapping T_2 from L_1 onto $\Lambda(2k\sigma; p, 1)$ and from L_2 onto $\Lambda(2k\tau; p, 1)$. By interpolation,

$$T_2u\in(\Lambda(2k\sigma; p, 1), \Lambda(2k\tau; p, 1))_{\theta,q}$$

and, using Lemma 22 (i),

$$T_2 u = \frac{(2k-1)!}{((k-1)!)^2}\int_0^\infty \lambda^{k-1}\mathfrak{J}^k(\lambda+\mathfrak{J})^{-2k}u(\lambda)d\lambda$$
$$= \frac{(2k-1)!}{((k-1)!)^2}\int_0^\infty \lambda^{k-1}\mathfrak{J}^k(\lambda+\mathfrak{J})^{-2k}f d\lambda = f\,,$$

so that $f \in (\Lambda(2k\sigma; p, 1), \Lambda(2k\tau; p, 1))_{\theta,q}$. Hence, $\Lambda(2k\mu; p, q) \subset (\Lambda(2k\sigma; p, 1), \Lambda(2k\tau; p, 1))_{\theta,q}$. Moreover, from Theorem 14,

$$\Lambda(2k\mu; p, q) \subset (\Lambda(2k\sigma; p, 1), \Lambda(2k\tau; p, 1))_{\theta,q} \subset (\Lambda(2k\sigma; p, \xi), \Lambda(2k\tau; p, \eta))_{\theta,q}$$
$$\subset (\Lambda(2k\sigma; p, \infty), \Lambda(2k\tau; p, \infty))_{\theta,q} \subset \Lambda(2k\mu; p, q)\,,$$

so that $\Lambda(2k\mu; p, q) = (\Lambda(2k\sigma; p, \xi), \Lambda(2k\tau; p, \eta))_{\theta,q}$ when $0 < \sigma, \tau, \theta < 1$, $\sigma \neq \tau$, $1 \leqq q \leqq \infty$ and $1 < p < \infty$. By Theorem 7, it follows that

$$(\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta))_{\theta,q} = \mathfrak{T}\Lambda(\mu; p, q)$$

for any real numbers σ, τ. From Theorem 7 and Lemma 21 (ii) it follows that

$$[\mathfrak{T}\Lambda(\sigma; p, \xi), \mathfrak{T}\Lambda(\tau; p, \eta)]_\theta = \mathfrak{T}\Lambda(\mu: p, \zeta)$$

for any real σ, τ, where $1/\zeta = (1-\theta)/\xi + \theta/\eta$. This completes the proof of the theorem.

PROPOSITION 6. *Let σ, τ and α be real numbers and let $0 < \theta < 1$, $1 < p, q < \infty$. Put $1/r = (1-\theta)/p + \theta/q$ and $\mu = (1-\theta)\sigma + \theta\tau$. Then*
(i) $[L^p_\sigma, L^q_\tau]_\theta \subset L^r_\mu$
(ii) $L^p_\mu = [L^p_\sigma, L^p_\tau]_\theta$
(iii) $L^r_\alpha = [L^p_\alpha, L^q_\alpha]_\theta$.
The above inclusion mappings are all continuous.

PROOF. From Proposition 4, we may assume that σ and τ are positive numbers. Let k be a positive integer with $k > \max(\sigma/2+1, \tau/2+1)$. By Theorem 21, the operator $\mathfrak{J}^k$ is a linear bounded operator from L^p_σ into $L^p(G; L^{2,k-\sigma/2}_*)$ and from L^q_τ into $L^q(G; L^{2,k-\tau/2}_*)$. By interpolation, the operator $\mathfrak{J}^k$ is a linear and continuous mapping from $[L^p_\sigma, L^q_\tau]_\theta$ into $[L^p(G; L^{2,k-\sigma/2}_*), L^q(G; L^{2,k-\tau/2}_*)]_\theta$. Since by Lemma 20 (ii), $[L^p(G; L^{2,k-\sigma/2}_*), L^q(G; L^{2,k-\tau/2}_*)]_\theta = L^r(G; L^{2,k-\mu/2}_*)$ and $[L^p_\sigma, L^q_\tau]_\theta \subset [L^p, L^q]_\theta = L^r$, we have $[L^p_\sigma, L^q_\tau]_\theta \subset L^r_\mu$. In order to prove that $L^p_\mu \subset [L^p_\sigma, L^p_\tau]_\theta$, it is enough from Proposition 4 to show that $L^p_{\sigma\theta} \subset [L^p_\sigma, L^p]_\theta$ when $\sigma > 0$, $0 < \theta < 1$. Let $u \in L^p_{\sigma\theta}$, that is, $u(x, t) = J^{\sigma\theta}H_t f(x)$ with $f \in L^p$. We put $g(z) = A(z)J^{\sigma z}H_t f$, where $A(z) = \Gamma(1+z\sigma)$. By the fundamental property (f) of fractional powers $\mathfrak{J}^\alpha_p$, the mapping $z \mapsto g(z)$ is analytic for $0 < \operatorname{Re} z < 1$ and continuous for $0 \leqq \operatorname{Re} z \leqq 1$. We see that for a real s,

$$\|g(is)\|_p = |A(is)|\ \|J^{i\sigma s}H_t f\|_p = |A(is)|\ \|(1+\mathfrak{I})^{-i\sigma s/2}H_t f\|_p$$
$$\leqq |\Gamma(1+is\sigma)|\left|\Gamma\left(1+i\frac{\sigma s}{2}\right)\right|^{-1}\|f\|_p \to 0 \quad \text{as} \quad |s| \to \infty .$$

Hence, $g(is) \in L^p$ and $\|g(is)\|_p \to 0$ as $|s| \to \infty$. On the other hand,

$$g(1+is) = A(1+is)J^{\sigma+i\sigma s}H_t f = A(1+is)J^{\sigma}H_t(J^{i\sigma s}f) \in L^p_\sigma ,$$

and

$$\|g(1+is)\|_{\sigma,p} = |A(1+is)|\ \|J^{i\sigma s}f\|_p$$
$$\leqq |\Gamma(1+\sigma+i\sigma s)|\left|\Gamma\left(1+i\frac{\sigma s}{2}\right)\right|^{-1}\|f\|_p \to 0 \quad \text{as} \quad |s| \to \infty .$$

Hence $g(\theta) = A(\theta)J^{\sigma\theta}H_t f = A(\theta)u(x, t) \in [L^p_\sigma, L^p]_\theta$. This implies that $L^p_{\sigma\theta} \subset [L^p_\sigma, L^p]_\theta$. Therefore (ii) follows from (i). (iii) is verified easily. The proof of the proposition is complete.

REMARK. Part (ii) is essentially due to Folland [5: Theorem (4.7)] when $\sigma \geqq 0$ and $\tau \geqq 0$.

THEOREM 23. *Let α and β be real numbers and let $0 < \theta < 1$, $1 < p < \infty$ and $1 \leqq q \leqq \infty$. Put $\mu = (1-\theta)\alpha + \theta\beta$. Then*

$$(L^p_\alpha, L^p_\beta)_{\theta,q} = \mathfrak{T}\Lambda(\mu; p, q) .$$

PROOF. From Corollary of Theorem 21 and Theorem 22 we obtain

$$\mathfrak{T}\Lambda(\mu; p, q) = (\mathfrak{T}\Lambda(\alpha; p, 1), \mathfrak{T}\Lambda(\beta; p, 1))_{\theta,q} \subset (L^p_\alpha, L^p_\beta)_{\theta,q}$$
$$\subset (\mathfrak{T}\Lambda(\alpha; p, \infty), \mathfrak{T}\Lambda(\beta; p, \infty))_{\theta,q} = \mathfrak{T}\Lambda(\mu; p, q) .$$

Thus, $\mathfrak{T}\Lambda(\mu; p, q) = (L^p_\alpha, L^p_\beta)_{\theta,q}$.

7. The duals of the Sobolev spaces and Besov spaces. If B is a Banach space, then we denote by B' the dual space of B. The following theorem is verified easily from the duality theory of interpolation spaces. (Example; see J. Bergh and J. Löfström [1] and Taibleson [23].)

THEOREM 24. (i) *Let α be real and let $1 < p < \infty$ and $1/p + 1/p' = 1$. Then,*

$$(L^p_\alpha)' = L^{p'}_{-\alpha} .$$

(ii) *Let α be real and let $1 < p < \infty$, $1 < q < \infty$. Put $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then*

$$\mathfrak{T}\Lambda(\alpha; p, q)' = \mathfrak{T}\Lambda(-\alpha; p', q') .$$

Flett [4] has shown that the duals of $\mathfrak{T}\Lambda(\alpha; p, 1)$ and $\mathfrak{T}\lambda(\alpha; p, \infty)$ are $\mathfrak{T}\Lambda(-\alpha; p', \infty)$ and $\mathfrak{T}\Lambda(-\alpha; p', 1)$ respectively after a long series of lemmas in a n-dimensional Euclidean space. But some of these lemmas are not applicable to our case. Thus we shall prove the above duality theorem under appropriate modifications in our case.

LEMMA 23. *Let $1 \leqq p \leqq \infty$ and $1/p + 1/p' = 1$, and let u, v be temperatures on $G \times (0, \infty)$ such that $\|u(\cdot, t)\|_p$ and $\|v(\cdot, t)\|_{p'}$ are locally integrable. Then for all positive numbers s_1, s_2, t_1, t_2 such that $s_1 + s_2 = t_1 + t_2$ we have*

$$\int_G u(x, s_1)v(x, s_2)dx = \int_G u(x, t_1)v(x, t_2)dx .$$

PROOF (cf. Flett [4: Lemma 15]). Obvious from Fubini's theorem.

LEMMA 24. *Let $0 < \alpha < 2$, $1 \leqq p < \infty$, $1/p + 1/p' = 1$. For all $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$ and all $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$ let*

$$\langle u, v\rangle = \int_0^\infty\int_G e^{-t}u\left(y, \frac{t}{2}\right)J^{-2}v\left(y, \frac{t}{2}\right)dydt .$$

Then, (i) $|\langle u, v\rangle| \leqq C\|u\|_{-\alpha;p',\infty}\|v\|_{\alpha;p,1}$

(ii) $\langle u, v\rangle = \lim_{s\to 0}\int_G u(y, s)g(y)dy$, *where* $v(x, t) = H_t g(x)$, $g \in L^p$ (cf. *Theorem* 9).

(iii) *If $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$ and $\langle u, v\rangle = 0$ for all $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$ then $u = 0$.*

(iv) *If $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$ and $\langle u, v\rangle = 0$ for all $u \in \mathfrak{T}\lambda(-\alpha; p', \infty)$ then $v = 0$.*

(v) *If $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$, $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$ and $s > 0$ then $\langle u^s, v\rangle = \langle u, v^s\rangle$, where u^s is as in Lemma 11.*

(vi) *If $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$ and v is a temperature on $G \times (0, \infty)$ such that $t \mapsto \|v(\cdot, t)\|_p$ is decreasing on $(0, \infty)$ then for all positive s, t,*

$$\langle u, v^{s+t}\rangle = \int_G u(y, s)v(y, t)dy .$$

PROOF (cf. Flett [4: Lemma 27]). By Hölder's inequality, part (i) is verified easily. Part (ii) follows immediately from Lemma 11 (iii), Lemma 23 and part (i). Parts (iii), (iv), (v) and (vi) follow easily from Lemma 23.

LEMMA 25. *Let α be real, and let $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$. Suppose that F is a continuous linear functional on either $\mathfrak{T}\Lambda(\alpha; p, q)$ or $\mathfrak{T}\lambda(\alpha; p, \infty)$ and let $h^{y,s}(x, t) = h(y^{-1}x, s + t)$ and $u(y, s) = F(h^{y,s})$ where $(y, s) \in G \times (0, \infty)$. Then $u(y, s)$ is uniformly continuous with respect to y for each $s > 0$ and is bounded on $s \geqq c$ where $c > 0$.*

PROOF (cf. Flett [4: Lemma 25]). First, note that $h^{y,s} \in \mathfrak{T}\Lambda(\alpha; p, q) \cap$

$\mathfrak{T}\lambda(\alpha; p, \infty)$ by Lemma 11 (i). Put

$$(1) \qquad h_y^s(x) = h(x^{-1}y, s) .$$

Then we have, for $y_1, y_2 \in G$ and $s > 0$,

$$|F(h^{y_1,s}) - F(h^{y_2,s})| = |F(h^{y_1,s} - h^{y_2,s})| \leqq ||F||\, ||h^{y_1,s} - h^{y_2,s}||_{\alpha;p,q} .$$

Hence it suffices to prove that $h^{y,s}$ is uniformly continuous for each $s > 0$ and is bounded on $s \geqq c > 0$ with respect to the norm $||\cdot||_{\alpha;p,q}$. By Theorem 14, we may assume $q = \infty$. Using Theorem 8 (ii) and Theorem 4 (i),

$$\begin{aligned} &||h^{y_1,s} - h^{y_2,s}||_{\alpha;p,\infty} \\ &\quad\leqq C\Big[\sup_{t\geqq 1/2}||h^{y_1,s}(x, t) - h^{y_2,s}(x, t)||_p + \sup_{0<t\leqq 1}\Big\{t^{k-\alpha/2}\Big|\Big|\frac{\partial^k}{\partial t^k}\Big(h^{y_1,s}(x, t) - h^{y_2,s}(x, t)\Big)\Big|\Big|_p\Big\}\Big] \\ &\quad\leqq C\Big[\sup_{t\geqq 1/2}||h_t*(h_{y_1}^s - h_{y_2}^s)||_p + \sup_{0<t\leqq 1}\Big\{t^{k-\alpha/2}\Big|\Big|\frac{\partial^k}{\partial t^k}h_{t+s/2}*\Big(h_{y_1}^{s/2} - h_{y_2}^{s/2}\Big)\Big|\Big|_p\Big\}\Big] \\ &\quad\leqq C\{||h_{y_1}^s - h_{y_2}^s||_p + (s/2)^{-\alpha/2}||h_{y_1}^{s/2} - h_{y_2}^{s/2}||_p\} , \end{aligned}$$

where k is a nonnegative integer with $k > \alpha/2$ and $s > 0$, $y_1, y_2 \in G$.

The function $y \mapsto h_y^s$ is uniformly continuous with respect to the norm $||\cdot||_p$ when $1 \leqq p < \infty$ by Corollary (ii) of Theorem 5. Thus, for each $s > 0$ the function $y \mapsto h^{y,s}$ is uniformly continuous with respect to the norm $||\cdot||_{\alpha;p,\infty}$. Further, by Theorem 4 (i),

$$||h^{y,s}||_{\alpha;p,\infty} \leqq C(||h_y^s||_p + (s/2)^{-\alpha/2}||h_y^{s/2}||_p) \leqq C(s^{-\rho(1-1/p)/2} + (s/2)^{-(\alpha+\rho(1-1/p))/2}) .$$

This implies that $||h^{y,s}||_{\alpha;p,\infty}$ is bounded on $s \geqq c > 0$. This completes the proof of the lemma.

LEMMA 26. *Let α be real and let $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$. Suppose that F is a continuous linear functional on either $\mathfrak{T}\Lambda(\alpha; p, q)$ or $\mathfrak{T}\lambda(\alpha; p, \infty)$ and let $u(y, s) = F(h^{y,s})$. Then for each $g \in C_c$ and for each $s > 0$,*

$$\int_G u(y, s)g(y)dy = F(v^s) ,$$

where $v(x, t) = H_t g(x)$ and v^s is as in Lemma 11.

PROOF (cf. Flett [4: Lemma 26]). Let E be the compact support of g, and for each positive integer K let $\{E_i\} = \{E_i^K\}$ be a finite covering of E such that $E_i = B(1/K)y_i$ where $B(1/K) = \{y \in G\colon |y| < 1/K\}$. Let $\{f_i\}$ be a partition of unity, subordinate to the covering $\{E_i\}$. We put

$$S_K(x, t) = \sum_i h(x^{-1}y_i, s + t)\int_{E_i} f_i(y)g(y)dy = \sum_i h(y_i^{-1}x, s + t)\int_{E_i} f_i(y)g(y)dy .$$

Then S_K is a finite linear combination of the functions $h^{y_i,s}$ and therefore belongs to both $\mathfrak{X}\Lambda(\alpha; p, q)$ and $\mathfrak{X}\lambda(\alpha; p, \infty)$ by Lemma 11 (i). To prove that $S_K \to v^s$ in $\mathfrak{X}\Lambda(\alpha; p, q)$ as $K \to \infty$, we put

$$\begin{aligned}\Psi_K(x, t) &= S_K(x, t) - v^s(x, t) = S_K(x, t) - v(x, s + t) \\ &= \sum_i h(y_i^{-1}x, s + t) \int_{E_i} f_i(y)g(y)dy - \int_G \sum_i f_i(y)h(y^{-1}x, s + t)g(y)dy \\ &= \sum_i \int_{E_i} f_i(y)\{h(y_i^{-1}x, s + t) - h(y^{-1}x, s + t)\}g(y)dy\ .\end{aligned}$$

Then

$$||\Psi_K(x, t)||_p \leqq \sum_i \int_{E_i} ||h(y_i^{-1}x, s + t) - h(y^{-1}x, s + t)||_p |g(y)|dy\ .$$

Since $||h(y_i^{-1}x, s + t) - h(y^{-1}x, s + t)||_p \leqq ||h^s_{y_i} - h^s_y||_p$ and the function $y \mapsto h^s_y$ is uniformly continuous when $1 \leqq p < \infty$, where h^s_y is given by (1) in the proof of Lemma 25, we obtain that $||\Psi_K(x, t)||_p$ tends to zero as $K \to \infty$ uniformly in $t > 0$. Since

$$\frac{\partial^k}{\partial t^k}\Psi_K(x, t) = \sum_i \int_{E_i} f_i(y)\left\{\frac{\partial^k}{\partial t^k}h(y_i^{-1}x, s + t) - \frac{\partial^k}{\partial t^k}h(y^{-1}x, s + t)\right\}g(y)dy\ ,$$

the same argument above shows that $||(\partial^k/\partial t^k)\Psi_K(x, t)||_p$ tends to zero as $K \to \infty$ uniformly in $t > 0$, and so by Theorem 8 (ii), $\Psi_K \to 0$ in $\mathfrak{X}\Lambda(\alpha; p, q)$ as $K \to \infty$, that is, $S_K \to v^s$ in $\mathfrak{X}\Lambda(\alpha; p, q)$ as $K \to \infty$. It now follows that

$$\begin{aligned}F(v^s) = \lim_{K\to\infty} F(S_K) &= \lim_{K\to\infty} \sum_i F(h^{y_i,s})\int_{E_i} f_i(y)g(y)dy \\ &= \lim_{K\to\infty} \sum_i u(y_i, s) \int_{E_i} f_i(y)g(y)dy\ .\end{aligned}$$

Moreover, using Lemma 25,

$$\begin{aligned}&\sum_i u(y_i, s)\int_{E_i} f_i(y)g(y)dy - \int_G u(y, s)g(y)dy \\ &\quad = \sum_i \int_{E_i} f_i(y)\{u(y_i, s) - u(y, s)\}g(y)dy \to 0 \quad \text{as} \quad K \to \infty\ .\end{aligned}$$

This completes the proof.

COROLLARY. *Under the hypothesis of Lemma 26,*

(i) *the function* $t \mapsto ||u(\cdot, t)||_{p'}$ *is bounded on* $t \geqq c$ *for each* $c > 0$, *where* $1/p + 1/p' = 1$,

(ii) *for any* $g \in L^p$, $\int_G u(y, s)g(y)dy = F(v^s)$, *where* $v(x, t) = H_t g(x)$, *and*

(iii) *for each* $s > 0$, $u(x, s + t) = H_t u(x, s)$. *Hence, in particular* u *belongs to the space* $\mathfrak{X}$.

PROOF. To prove (i), we see from Theorem 14 and Lemma 13 that

$$(1)\qquad \|u(\cdot, t)\|_{p'} = \sup\left\{\left|\int_G u(y, t)g(y)dy\right| : g \in C_c, \|g\|_p = 1\right\}$$
$$= \sup\{|F(v^s)| : g \in C_c, \|g\|_p = 1\} \leqq \|F\| \sup\{\|v^s\|_{\alpha;p,q} : g \in C_c, \|g\|_p = 1\}$$
$$\leqq C\|F\| \sup\{\|v^s\|_{\alpha;p,1} : g \in C_c, \|g\|_p = 1\} \leqq C\|F\| \quad \text{if} \quad s \geqq c > 0 .$$

Part (ii) follows immediately from Lemma 14. To prove (iii), taking $g(y) = h_x^t(y)$, $x \in G$, $t > 0$ in (ii), we get

$$H_t u(x, s) = \int_G h(y^{-1}x, t)u(y, s)dy = F(h^{x,s+t}) = u(x, s + t) .$$

THEOREM 25. *Let α be real and let $1 \leqq p < \infty$. Then $\mathfrak{T}\Lambda(\alpha; p, 1)' = \mathfrak{T}\Lambda(-\alpha; p', \infty)$. More precisely, if $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$ and F_u is given by $F_u(v) = \langle u, v\rangle$, then the mapping $u \mapsto F_u$ is a linear homeomorphism of $\mathfrak{T}\Lambda(-\alpha; p', \infty)$ onto $\mathfrak{T}\Lambda(\alpha; p, 1)'$.*

PROOF (cf. Flett [4: Theorem 28]). By Theorem 7, we may assume $0 < \alpha < 2$. The mapping $u \mapsto F_u$ is a one-to-one and continuous linear mapping of $\mathfrak{T}\Lambda(-\alpha; p', \infty)$ to $\mathfrak{T}\Lambda(\alpha; p, 1)'$ by Lemma 24 (i), (iii). To prove that this mapping is onto, let $F \in \mathfrak{T}\Lambda(\alpha; p, 1)'$ and $u(y, s) = F(h^{y,s})$. Then from (1) in the proof of Corollary (i) of Lemma 26,

$$\|u(\cdot, s)\|_{p'} \leqq C\|F\|(1 + s^{-\alpha/2}) .$$

Hence, using Theorem 8 (ii),

$$(1)\quad \|u\|_{-\alpha;p',\infty} \leqq C[\sup_{0<t\leqq 1} \{t^{\alpha/2}\|u(\cdot, t)\|_{p'}\} + \sup_{1/2\leqq t} \|u(\cdot, t)\|_{p'}] \leqq C\|F\| < \infty$$

so that by Corollary (iii) of Lemma 26, $u \in \mathfrak{T}\Lambda(-\alpha; p', \infty)$. For any $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$, by Lemma 11 (iii), $v^s \to v$ in $\mathfrak{T}\Lambda(\alpha; p, 1)$ as $s \to 0$, and by Theorem 9, we can find a function $g \in L^p$ such that $v(x, t) = H_t g(x)$. From Corollary (ii) of Lemma 26 and Lemma 24 (ii), we get

$$F(v) = \lim_{s\to 0} F(v^s) = \lim_{s\to 0} \int_G u(y, s)g(y)dy = \langle u, v\rangle .$$

This implies that the mapping $u \mapsto F_u$ is onto. From (1) and Lemma 24 (i), this mapping is a homeomorphism.

COROLLARY. *Let α be real and let $1 \leqq p < \infty$. For $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$, G_v is defined by $G_v(u) = \langle u, v\rangle$. Then, the mapping $v \mapsto G_v$ is a linear homeomorphism of $\mathfrak{T}\Lambda(\alpha; p, 1)$ into $\mathfrak{T}\Lambda(\alpha; p', \infty)'$. Moreover, if H_v is the restriction of G_v to $\mathfrak{T}\lambda(-\alpha; p', \infty)$, then $\|H_v\| = \|G_v\|$.*

PROOF. See Flett [4: Lemma 29 and Lemma 31].

THEOREM 26. *Let α be real, and let $1 < p < \infty$. Then,*

(i) $\mathfrak{T}\Lambda(\alpha; p, 1) = \mathfrak{T}\lambda(-\alpha; p', \infty)'$. *More precisely, if* $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$ *and* H_v *is given by* $H_v(u) = \langle u, v\rangle$, *then the mapping* $v \mapsto H_v$ *is a linear homeomorphism of* $\mathfrak{T}\Lambda(\alpha; p, 1)$ *onto* $\mathfrak{T}\lambda(-\alpha; p', \infty)'$.

(ii) $\mathfrak{T}\lambda(-\alpha; p', \infty)'' = \mathfrak{T}\Lambda(-\alpha; p', \infty)$, *that is, there exists a linear homeomorphism of* $\mathfrak{T}\Lambda(-\alpha; p', \infty)$ *onto* $\mathfrak{T}\lambda(-\alpha; p', \infty)''$. *Moreover, the restriction of this homeomorphism to* $\mathfrak{T}\lambda(-\alpha; p', \infty)$ *is the canonical isometry of* $\mathfrak{T}\lambda(-\alpha; p', \infty)$ *into* $\mathfrak{T}\lambda(-\alpha; p', \infty)''$.

PROOF (cf. Flett [4: Lemma 33]). By Theorem 7, we may assume $0<\alpha<2$. By Corollary of Theorem 25, the mapping $v \mapsto H_v$ is a linear homeomorphism of $\mathfrak{T}\Lambda(\alpha; p, 1)$ into $\mathfrak{T}\lambda(-\alpha; p', \infty)'$. To prove that this mapping is onto, let $H \in \mathfrak{T}\lambda(-\alpha; p', \infty)'$ and $v(y, s) = H(h^{y,s})$. For any $u \in \mathfrak{T}\lambda(-\alpha; p', \infty)$, we have from Corollary (ii) of Lemma 26 that

$$\int_G v(y, s)u(y, t)dy = H(u^{s+t}) .$$

By Lemma 24 (vi),

$$\int_G v(y, s)u(y, t)dy = \langle u, v^{s+t}\rangle .$$

Since s and t are arbitrary positive numbers, we have $H(u^s) = \langle u, v^s\rangle$. By Lemma 11 (iv), $u^s \to u$ in $\mathfrak{T}\lambda(-\alpha; p', \infty)$ as $s \to 0$. Therefore,

$$\lim_{s\to 0} |\langle u, v^s\rangle| = \lim_{s\to 0} |H(u^s)| = |H(u)| < \infty .$$

By the principle of uniform boundedness, $\|H_{v^s}\| = O(1)$ as $s\to 0$. By Corollary of Theorem 25, $\|v^s\|_{\alpha;p,1} = O(1)$ as $s \to 0$. By Lemma 12, $v \in \mathfrak{T}\Lambda(\alpha; p, 1)$. Hence, using Lemma 11(ii), $v^s \to v$ in $\mathfrak{T}\Lambda(\alpha; p, 1)$ as $s \to 0$. Thus, by Lemma 24 (i),

$$H(u) = \lim_{s\to 0} \langle u, v^s\rangle = \langle u, v\rangle .$$

This completes the proof of (i). See [4: Lemma 30] for the proof of (ii).

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