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TORUS EMBEDDINGS AND DUALIZING COMPLEXES

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Introduction. Let M be the moduli space of a class of smooth varieties. Then, the best compactification of M will be the moduli space of an extended class of "degenerate" varieties which may have some singularities. The main purpose of this paper is to study what kind of singularities are reasonable for these degenerate varieties without specifying any particular class of the smooth varieties. We would like the singularities to be sufficiently simple so that invariants defined for smooth varieties are generalizable, and that we can study the generically smooth deformation of them. In the case of curves, the theory of the stable curves by Deligne and Mumford [DM] shows that it is reasonable to take only ordinary double points as the singularities. In the higher dimensional cases, however, the degenerate Jacobian varieties of Oda, Seshadri and Ishida [OS], [I1] or more generally, the stable quasiabelian varieties of Namikawa and Nakamura [N1], [N2] show normal crossing singularities to be too restrictive for the degenerate abelian varieties. Looking at many examples of degenerate varieties, we came to take, as the local models of singularities, subschemes, invariant under the torus action, of torus embeddings. Thus they are generalizations of toroidal embeddings by Mumford et al. [TE]. But these are too general, and we must find out good conditions on them. It is meaningful to give the condition for the local models to be Cohen-Macaulay or Gorenstein. In the classification of smooth varieties, the canonical invertible sheaves play an important role. The Serre duality theorem is generalized for Cohen-Macaulay varieties with the canonical invertible sheaves replaced by the dualizing sheaves. They are invertible if the varieties are Gorenstein. The sphericity, which we define later, will be a good condition for the local model to be Gorenstein.

We now explain the content of this paper in more detail.

Let N be a free $\boldsymbol{Z}$-module of rank $r \geqq 0$, and let M be the dual $\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$. Then for a fixed field k, an affine torus embedding of

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dimension r is written as $X_\pi = \operatorname{Spec}(k[M \cap \pi^\vee])$ for a cone $\pi = \boldsymbol{R}_0 a_1 + \cdots + \boldsymbol{R}_0 a_s$ ($a_1, \cdots, a_s \in N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$) of $N_{\boldsymbol{R}}$ where $\boldsymbol{R}_0 = \{c \in \boldsymbol{R}; c \geqq 0\}$ and $\pi^\vee = \{x \in M_{\boldsymbol{R}}; \langle x, a\rangle \geqq 0 \ {}^\forall a \in \pi\}$. The torus $T_N = \operatorname{Spec}(k[M])$ acts on X_π naturally. Set $\Gamma(\pi) = \{\text{the faces of } \pi\}$. Then there exists the following relation between elements of $\Gamma(\pi)$ and closed subschemes of X_π.

$$\begin{array}{ccccc}
\Gamma(\pi) & \simeq & \left\{\begin{array}{l}M\text{-homogeneous quotient}\\ \text{integral domains of}\\ \text{the ring } k[M \cap \pi^\vee]\end{array}\right\} & \simeq & \left\{\begin{array}{l}T_N\text{-invariant irre-}\\ \text{ducible reduced closed}\\ \text{subschemes of } X_\pi\end{array}\right\} \\
\cup\!\!\!\!| & & \cup\!\!\!\!| & & \cup\!\!\!\!| \\
\sigma & \longmapsto & S(\sigma) = k[M \cap \pi^\vee \cap \sigma^\perp] & \longmapsto & V(\sigma) = \operatorname{Spec}(S(\sigma))\,,
\end{array}$$

where $\sigma^\perp = \{x \in M_{\boldsymbol{R}}; \langle x, a\rangle = 0 \ {}^\forall a \in \sigma\}$. Furthermore, we know that $\dim \sigma + \dim S(\sigma) = r$ for every $\sigma \in \Gamma(\pi)$.

DEFINITION. A subset Σ of $\Gamma(\pi)$ is said to be *star closed* if $\Sigma \ni \sigma$, $\Gamma(\pi) \ni \tau$ and $\tau \succ \sigma$ imply $\Sigma \ni \tau$, and it is said to be *locally star closed* if $\Sigma \ni \rho, \sigma$, $\Gamma(\pi) \ni \tau$ and $\rho \succ \tau \succ \sigma$ imply $\Sigma \ni \tau$.

Then we can determine the T_N-invariant closed subschemes of X_π as follows.

$$\begin{array}{ccccc}
\left\{\begin{array}{l}\text{star closed}\\ \text{subsets of}\\ \Gamma(\pi)\end{array}\right\} & \simeq & \left\{\begin{array}{l}\text{reduced } M\text{-homoge-}\\ \text{neous quotient}\\ \text{rings of } k[M \cap \pi^\vee]\end{array}\right\} & \simeq & \left\{\begin{array}{l}T_N\text{-invariant}\\ \text{reduced closed}\\ \text{subschemes of } X_\pi\end{array}\right\} \\
\cup\!\!\!\!| & & \cup\!\!\!\!| & & \cup\!\!\!\!| \\
\Sigma & \longmapsto & S_\Sigma = k\Big[\bigcup_{\sigma \in \Sigma} M \cap \pi^\vee \cap \sigma^\perp\Big] & \longmapsto & Y_\Sigma = \operatorname{Spec}(S_\Sigma)
\end{array}$$

The main purpose of this article is to characterize the properties of the ring S_Σ in terms of combinatorial conditions on the set $\Sigma \subset \Gamma(\pi)$. In the case π is non-singular and $\dim \pi = r$, i.e., when $k[M \cap \pi^\vee]$ is a polynomial ring, Reisner [R1] and Hochster [H2] gave conditions for S_Σ to be Cohen-Macaulay and Gorenstein, respectively. We will generalize their results to an arbitrary π using a completely different method, that of dualizing complexes.

For a face $\sigma \in \Gamma(\pi)$ of dimension d, we set $M^{(\sigma)} = M \cap \sigma^\perp$. Then $M^{(\sigma)}$ is a free $\boldsymbol{Z}$-module of rank $r-d$. We define a free $\boldsymbol{Z}$-module $\boldsymbol{Z}_\sigma$ of rank one by $\boldsymbol{Z}_\sigma = \bigwedge^{r-d} M^{(\sigma)}$. For faces $\sigma, \tau \in \Gamma(\pi)$ with $\tau \succ \sigma$ and $\dim \tau - \dim \sigma = 1$, there exists a natural isomorphism $q_{\tau/\sigma}\colon \boldsymbol{Z}_\sigma \xrightarrow{\sim} \boldsymbol{Z}_\tau$. For a locally star closed subset Φ, we define the complex $C^{\boldsymbol{\cdot}}(\Phi, \boldsymbol{Z})$ as follows. We set $C^i(\Phi, \boldsymbol{Z}) = \bigoplus_{\sigma \in \Phi_i} \boldsymbol{Z}_\sigma$, where $\Phi_i = \{\sigma \in \Phi; \dim \sigma = i\}$, and the coboundary maps are defined naturally by $q_{\tau/\sigma}$'s. Thus we can consider the cohomology groups $H^i(\Phi, k)$ of the complex $C^{\boldsymbol{\cdot}}(\Phi, k) = C^{\boldsymbol{\cdot}}(\Phi, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} k$.

Reisner's result is generalized as follows, where $\Sigma(\rho) = \{\sigma \in \Sigma; \rho \succ \sigma\}$, and $h = \min\{i; \Sigma_i \neq \emptyset\}$, the "height" of Σ.

COROLLARY 3.5. *The ring S_Σ is Cohen-Macaulay if and only if $H^i(\Sigma(\rho), k) = 0$ for every $i \neq h$ and for every $\rho \in \Sigma$.*

DEFINITION. A star closed subset Σ of $\Gamma(\pi)$ is said to be *spherical* if

$$H^i(\Sigma(\rho), k) \simeq \begin{cases} k & \text{if} \quad i = h \\ 0 & \text{if} \quad i \neq h \end{cases}$$

for every ρ in Σ.

The result of Hochster is generalized as follows.

THEOREM 5.10. *The ring S_Σ is Gorenstein if Σ is spherical.*

This theorem is a consequence of Theorem 5.9 which is a generalization of Stanley's characterization of Gorenstein normal semigroup rings [S1].

In §3, we construct the dualizing complex $K^\cdot$ of the ring S_Σ which consists of M-graded S_Σ-modules and coboundary homomorphisms of degree 0. The above theorems are obtained by considering the m-component for each $m \in M$. In a special case the complex $K^\cdot$ appeared in [N2].

In §7 and §8, we study special cases in more detail. In §7, we give a natural one-to-one correspondence between the set of Gorenstein normal semigroup rings of dimension r and the set of convex polytopes of dimension $r-1$. In §8, we show that every normal semigroup ring of dimension 3 which is a complete intersection is of the form $k[x, y, z, w, u]/(xz - w^b u^c, yw - u^a)$ for a triple (a, b, c) of non-negative integers.

The generalization of our theory to general torus embeddings or toroidal embeddings and its relation to that of global duality will be treated in a forthcoming paper.

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Notation.

$\boldsymbol{Z}$: the ring of rational integers

$\boldsymbol{Z}_0 = \{c \in \boldsymbol{Z};\ c \geqq 0\}$
$\boldsymbol{R}$: the field of real numbers
$\boldsymbol{R}_0 = \{t \in \boldsymbol{R};\ t \geqq 0\}$
$A \backslash B = \{x;\ x \in A,\ x \notin B\}$

If A and B are subsets of an additive group,

$$A + B = \{a + b;\ a \in A,\ b \in B\}\ ,$$
$$A - B = \{a - b;\ a \in A,\ b \in B\}\ .$$

1. Cones and their faces. Throughout this paper, we fix a free $\boldsymbol{Z}$-module N of rank $r \geqq 0$. By π we denote a strongly convex rational polyhedral cone in $N_{\boldsymbol{R}}$ $(=N \otimes_{\boldsymbol{Z}} \boldsymbol{R})$, i.e., there exist $a_1, \cdots, a_s$ in N with $\pi = \boldsymbol{R}_0 a_1 + \cdots + \boldsymbol{R}_0 a_s$ and $\pi \cap (-\pi) = \{0\}$. Let M be the dual $\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ of N. Then the dual cone $\pi^\vee = \{x \in M_{\boldsymbol{R}};\ \langle x, a\rangle \geqq 0\ {}^\forall a \in \pi\}$ is a convex rational polyhedral cone of dimension r in $M_{\boldsymbol{R}}$. The following proposition is fundamental. For the proof we refer the reader to [MO].

PROPOSITION 1.1. *The map $\sigma \mapsto \sigma^* = \pi^\vee \cap \sigma^\perp$ gives rise to a bijection*

$$\{\text{the faces of } \pi\} \to \{\text{the faces of } \pi^\vee\}\ ,$$

where $\sigma^\perp = \{x \in M_{\boldsymbol{R}};\ \langle x, a\rangle = 0\ {}^\forall a \in \sigma\}$. Moreover, $\sigma^* \succ \tau^*$ if and only if $\tau \succ \sigma$, and $\dim \sigma + \dim \sigma^* = r$ *for every pair of faces σ, τ of π.*

DEFINITION 1.2. We denote by $\Gamma(\pi)$ the set of the faces of π.

For a face σ of π, we use the following notations.

$$M^{(\sigma)} = M \cap \sigma^\perp$$
$$M_\sigma = M/M^{(\sigma)}$$
$$N_\sigma = N \cap (\sigma + (-\sigma))$$
$$N^{(\sigma)} = N/N_\sigma\ .$$

Clearly $N^{(\sigma)}$ (resp. N_σ) is the dual $\boldsymbol{Z}$-module of $M^{(\sigma)}$ (resp. M_σ) of rank $r - \dim \sigma$ (resp. $\dim \sigma$). For a face ρ of π with $\rho \succ \sigma$, we denote by $\rho^{(\sigma)}$ the image of ρ in the quotient $N^{(\sigma)}_{\boldsymbol{R}} = N^{(\sigma)} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. We see easily that $\rho^{(\sigma)}$ is a face of the cone $\pi^{(\sigma)}$ in $N^{(\sigma)}_{\boldsymbol{R}}$.

PROPOSITION 1.3. *Let σ be a face of π. Then $\sigma^* = \pi^\vee \cap \sigma^\perp \subset M^{(\sigma)}_{\boldsymbol{R}}$ is the dual cone of $\pi^{(\sigma)} \subset N^{(\sigma)}_{\boldsymbol{R}}$, and $\rho^* = (\rho^{(\sigma)})^*$ for every $\rho \in \Gamma(\pi)$ with $\rho \succ \sigma$, where $(\rho^{(\sigma)})^* = (\pi^{(\sigma)})^\vee \cap (\rho^{(\sigma)})^\perp$ in $M^{(\sigma)}_{\boldsymbol{R}}$. In particular, the map*

$$\begin{array}{ccc} \{\rho \in \Gamma(\pi);\ \rho \succ \sigma\} & \to & \Gamma(\pi^{(\sigma)}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \rho & \longmapsto & \rho^{(\sigma)} \end{array}$$

is bijective.

PROOF. Let x be an element of $M_R^{(\sigma)}$. Then, $x \in \pi^{(\sigma)\vee} \cap \rho^{(\sigma)\perp} \Leftrightarrow \langle x, a\rangle \geqq 0 \ \forall a \in \pi^{(\sigma)}$ and $x \in (\rho^{(\sigma)})^\perp \Leftrightarrow \langle x, a\rangle \geqq 0 \ \forall a \in \pi$ and $x \in \rho^\perp \Leftrightarrow x \in \pi^\vee \cap \rho^\perp$. Hence we have $(\rho^{(\sigma)})^* = \rho^*$. In particular, we have $(\pi^{(\sigma)})^\vee = \pi^\vee \cap \sigma^\perp$ if we set $\rho = \sigma$. By Proposition 1.1, there are one-to-one correspondences

$$\{\rho \in \Gamma(\pi); \rho \succ \sigma\} \xleftrightarrow{1:1} \{\text{the faces of } \sigma^*(=(\pi^{(\sigma)})^\vee)\} \xleftrightarrow{1:1} \Gamma(\pi^{(\sigma)}) .$$

Hence the bijectivity is clear. q.e.d.

For every face σ of π of dimension d, we denote $Z_\sigma = \bigwedge^{r-d} M^{(\sigma)}$ which is non-canonically isomorphic to Z. Let σ be a one-codimensional face of τ. Then $\tau^{(\sigma)}$ is a one-dimensional cone, and $N^{(\sigma)} \cap \tau^{(\sigma)}$ is isomorphic to the semigroup Z_0. Let a be the generator of the semigroup $N^{(\sigma)} \cap \tau^{(\sigma)}$. We get an exact sequence

$$0 \longrightarrow M^{(\tau)} \longrightarrow M^{(\sigma)} \xrightarrow{\langle\ ,\, a\rangle} Z \longrightarrow 0$$

of Z-modules. By this exact sequence, we have a natural isomorphism

$$q_{\tau/\sigma}\colon Z_\sigma \xrightarrow{\sim} Z_\tau ,$$

i.e., if $\dim \sigma = p$, then for any $m_1 \in M^{(\sigma)}$ and $m_2, \cdots, m_p \in M^{(\tau)}$, the element $m_1 \wedge m_2 \wedge \cdots \wedge m_p$ is sent to $\langle m_1, a\rangle m_2 \wedge \cdots \wedge m_p$.

LEMMA 1.4. *Let σ and ρ be faces of π with $\dim \rho - \dim \sigma = 2$ and $\rho \succ \sigma$. Then there are exactly two faces τ_1, τ_2 of π with $\dim \rho - \dim \tau_i = 1$ and $\rho \succ \tau_i \succ \sigma$, $i = 1, 2$. Furthermore, we have $q_{\rho/\tau_1} \circ q_{\tau_1/\sigma} + q_{\rho/\tau_2} \circ q_{\tau_2/\sigma} = 0$.*

PROOF. In view of Proposition 1.3, we may assume $\sigma = \{0\}$ and $\dim \rho = 2$ by replacing π by $\pi^{(\sigma)}$. Then this lemma is obvious. q.e.d.

Let Σ be a subset of $\Gamma(\pi)$. Then we set, for each $i \in Z$,

$$C^i(\Sigma, Z) = \bigoplus_{\sigma \in \Sigma_i} Z_\sigma$$

where $\Sigma_i = \{\sigma \in \Sigma; \dim \sigma = i\}$. We define a homomorphism

$$\delta^i\colon C^i(\Sigma, Z) \to C^{i+1}(\Sigma, Z) ,$$

for each i, as follows: For $\sigma \in \Sigma_i$ and $\tau \in \Sigma_{i+1}$, its (σ, τ)-component is the isomorphism $q_{\tau/\sigma}$ if $\tau \succ \sigma$ and the zero map otherwise.

DEFINITION 1.5. Let Φ be a subset of $\Gamma(\pi)$. Then for a subset Σ of Φ, we say

$$\Sigma \text{ is} \begin{cases} \textit{star closed} \text{ in } \Phi \text{ if } \Sigma \ni \sigma,\ \Phi \ni \tau \text{ and } \tau \succ \sigma \text{ imply } \Sigma \ni \tau \\ \textit{star open} \text{ in } \Phi \text{ if } \Sigma \ni \sigma,\ \Phi \ni \tau \text{ and } \sigma \succ \tau \text{ imply } \Sigma \ni \tau \\ \textit{locally star closed} \text{ in } \Phi \text{ if } \Sigma \ni \rho, \sigma,\ \Phi \ni \tau \text{ and } \rho \succ \tau \succ \sigma \text{ imply } \Sigma \ni \tau . \end{cases}$$

PROPOSITION 1.6. *If Σ is a locally star closed subset of $\Gamma(\pi)$. Then $\delta^{i+1}\cdot\delta^i\colon C^i(\Sigma, \boldsymbol{Z})\to C^{i+2}(\Sigma, \boldsymbol{Z})$ is the zero map for every i, i.e., the sequence $C^{\cdot}(\Sigma, \boldsymbol{Z}) = (\cdots\to 0\to C^0(\Sigma, \boldsymbol{Z})\to C^1(\Sigma, \boldsymbol{Z})\to\cdots\to C^r(\Sigma, \boldsymbol{Z})\to 0\to\cdots)$ is a finite complex of free $\boldsymbol{Z}$-modules.*

PROOF. It is sufficient to show that the (σ, ρ)-component of $\delta^{i+1}\circ\delta^i$ is zero for every pair (σ, ρ) with $\sigma\in\Sigma_i$ and $\rho\in\Sigma_{i+2}$. It is clearly the case, if σ is not a face of ρ. Hence we may assume $\rho\succ\sigma$. Then τ_1, τ_2 in Lemma 1.4 is in Σ since Σ is locally star closed. Hence the (σ, ρ)-component of $\delta^{i+1}\circ\delta^i$ is equal to $q_{\rho/\tau_1}\circ q_{\tau_1/\sigma} + q_{\rho/\tau_2}\circ q_{\tau_2/\sigma}$, which is the zero map by Lemma 1.4. q.e.d.

Hence we can define the cohomology group $H^i(\Sigma, \boldsymbol{Z})$ $(i\in\boldsymbol{Z})$ for every locally star closed subset Σ of $\Gamma(\pi)$. Let Φ be a locally star closed subset of $\Gamma(\pi)$ and let Σ be a star closed subset (resp. star open subset) of Φ. Then there exists a natural homomorphism $C^{\cdot}(\Sigma, \boldsymbol{Z})\to C^{\cdot}(\Phi, \boldsymbol{Z})$ (resp. $C^{\cdot}(\Phi, \boldsymbol{Z})\to C^{\cdot}(\Sigma, \boldsymbol{Z})$) of complexes.

DEFINITION 1.7. A locally star closed subset Φ of $\Gamma(\pi)$ is said to be *homologically trivial* if $H^i(\Phi, \boldsymbol{Z}) = 0$ for every $i\in\boldsymbol{Z}$.

PROPOSITION 1.8. *Let Σ be a locally star closed subset of $\Gamma(\pi)$. If a subset Σ' of Σ is star closed in Σ or equivalently $\Sigma'' = \Sigma\backslash\Sigma'$ is star open in Σ, then there exists a cohomology exact sequence*

$$0\to H^0(\Sigma', \boldsymbol{Z})\to H^0(\Sigma, \boldsymbol{Z})\to H^0(\Sigma'', \boldsymbol{Z})\to H^1(\Sigma', \boldsymbol{Z})\to\cdots$$
$$\cdots\to H^p(\Sigma', \boldsymbol{Z})\to H^p(\Sigma, \boldsymbol{Z})\to H^p(\Sigma'', \boldsymbol{Z})\to H^{p+1}(\Sigma', \boldsymbol{Z})\to\cdots.$$

In particular, if any two of Σ, Σ', Σ'' are homologically trivial, so is the other.

PROOF. Since there is a short exact sequence

$$0\to C^{\cdot}(\Sigma', \boldsymbol{Z})\to C^{\cdot}(\Sigma, \boldsymbol{Z})\to C^{\cdot}(\Sigma'', \boldsymbol{Z})\to 0$$

of complexes, the assertion is well known. q.e.d.

Let Σ' and Σ'' be locally star closed subsets of $\Gamma(\pi)$ such that they are star closed in the union $\Sigma'\cup\Sigma''$ and the intersection $\Sigma'\cap\Sigma''$ is star closed in both of them. Then clearly $\Sigma'\cup\Sigma''$ and $\Sigma'\cap\Sigma''$ are locally star closed in $\Gamma(\pi)$, and we have a short exact sequence

$$0\to C^{\cdot}(\Sigma'\cap\Sigma'', \boldsymbol{Z})\to C^{\cdot}(\Sigma', \boldsymbol{Z})\oplus C^{\cdot}(\Sigma'', \boldsymbol{Z})\to C^{\cdot}(\Sigma'\cup\Sigma'', \boldsymbol{Z})\to 0\,.$$

Hence we have the following.

PROPOSITION 1.9 (Mayer-Vietoris). *In the above situation, we have an exact sequence*

$$\cdots \to H^p(\Sigma' \cap \Sigma'', \boldsymbol{Z}) \to H^p(\Sigma', \boldsymbol{Z}) \oplus H^p(\Sigma'', \boldsymbol{Z}) \to H^p(\Sigma' \cup \Sigma'', \boldsymbol{Z})$$
$$\to H^{p+1}(\Sigma' \cap \Sigma'', \boldsymbol{Z}) \to H^{p+1}(\Sigma', \boldsymbol{Z}) \oplus H^{p+1}(\Sigma'', \boldsymbol{Z}) \to \cdots .$$

2. Homologically trivial subsets of $\Gamma(\pi)$. In this section we aim to prove the homological triviality of certain subsets of $\Gamma(\pi)$ which will reappear in §4 in connection with the dualizing complex.

DEFINITION 2.1. For an element x of $M_{\boldsymbol{R}}$, we define the star open subset $\Gamma(\pi)_x$ of $\Gamma(\pi)$ by $\Gamma(\pi)_x = \{\sigma \in \Gamma(\pi); \langle x, \sigma\rangle \subset \boldsymbol{R}_0\}$ and $\Gamma_1(\pi)_x = \{\gamma \in \Gamma(\pi)_x; \dim \gamma = 1\}$.

Since a polyhedral cone is generated by its 1-dimensional faces, it is easy to see that $\Gamma(\pi)_x = \{\sigma \in \Gamma(\pi)$; every 1-dimensional face of σ is in $\Gamma_1(\pi)_x\}$. Hence $\Gamma(\pi)_x$ is uniquely determined by $\Gamma_1(\pi)_x$.

LEMMA 2.2 *If the cardinality* $^\sharp\Gamma_1(\pi)_x$ *is not less than* 2, *then there exist* $\gamma \in \Gamma_1(\pi)_x$ *and* $y, z \in N_{\boldsymbol{R}}$ *such that* $\gamma \subset y^\perp$, $\Gamma(\pi)_y = \Gamma(\pi)_x$ *and* $\Gamma_1(\pi)_z = \Gamma_1(\pi)_x \backslash \{\gamma\}$.

PROOF. Let $\{\gamma_1, \cdots, \gamma_s\}$ be the set of 1-dimensional faces of π and let $\Gamma_1(\pi)_x = \{\gamma_1, \cdots, \gamma_p\}$ for some $2 \leqq p \leqq s$. We take $a_i \in N$ for $1 \leqq i \leqq s$ such that $\gamma_i = \boldsymbol{R}_0 a_i$. Then $\langle x, a_i\rangle \geqq 0$ for $i = 1, \cdots, p$ and $\langle x, a_i\rangle < 0$ for $i = p+1, \cdots, s$. Furthermore, by adding to x an element of sufficiently small norm in the interior $\operatorname{int}\pi^\vee$ of $\pi^\vee$, we may assume $\langle x, a_i\rangle > 0$ for every $i = 1, \cdots, p$. Let w be an element of $\operatorname{int}\pi^\vee$, then since $\langle w, a_i\rangle > 0$ for every i, there exists a positive number t_i with $\langle x - t_i w, a_i\rangle = 0$ for each $1 \leqq i \leqq p$. If $t_i = t_j$ for some $1 \leqq i < j \leqq p$, then $\langle x, a_i\rangle - \langle t_i w, a_i\rangle = 0$, $\langle x, a_j\rangle - \langle t_i w, a_j\rangle = 0$ and w satisfies the linear equation $\langle x, a_i\rangle\langle w, a_j\rangle - \langle x, a_j\rangle\langle w, a_i\rangle = 0$, which is non-trivial since a_i and a_j are linearly independent. Hence if we take w which satisfies this equation for no $1 \leqq i < j \leqq p$, then $t_1, \cdots, t_p$ are mutually distinct. Since $\operatorname{int}\pi^\vee$ is an open subset of $M_{\boldsymbol{R}}$, this is possible. By renumbering $\gamma_1, \cdots, \gamma_p$, if necessary, we may assume $t_1 > t_2 > \cdots > t_p$. Set $y = x - t_p w$ and $z = x - t_0 w$ for some $t_p < t_0 < t_{p-1}$. Then $\langle y, a_i\rangle$ and $\langle z, a_i\rangle$ are positive (resp. negative) for $1 \leqq i \leqq p-1$ (resp. $p+1 \leqq i \leqq s$), and $\langle y, a_p\rangle = 0$, $\langle z, a_p\rangle < 0$. Hence $\Gamma_1(\pi)_y = \{\gamma_1, \cdots, \gamma_p\}$, $\gamma_p \in y^\perp$ and $\Gamma_1(\pi)_z = \{\gamma_1, \cdots, \gamma_{p-1}\}$. q.e.d.

For the subset $\{a_1, \cdots, a_s\}$ in the proof of the above proposition and for an element x in $M_{\boldsymbol{R}}$, we have $\Gamma_1(\pi)_x = \{\gamma_i; \langle x, a_i\rangle \geqq 0\}$. Since $\operatorname{int}\pi^\vee = \{y \in M_{\boldsymbol{R}}; \langle y, a_i\rangle > 0, i = 1, \cdots, s\}$, the set $\Gamma_1(\pi)_x$ is empty if and only if x is in $-\operatorname{int}\pi^\vee$.

PROPOSITION 2.3. *$\Gamma(\pi)_x$ is homologically trivial if x is outside $-\operatorname{int}\pi^\vee$.*

PROOF. We prove this proposition by double induction on $^\sharp\Gamma_1(\pi)_x$ and $\dim\pi$. If $\Gamma_1(\pi)_x=\{\gamma\}$, then

$$C^\cdot(\Gamma(\pi)_x, \boldsymbol{Z}) = (0 \longrightarrow \boldsymbol{Z}_{\{0\}} \xrightarrow{q_{\gamma/\{0\}}} \boldsymbol{Z}_\gamma \longrightarrow 0)$$

and $\Gamma_1(\pi)$ is homologically trivial. If $\dim\pi=1$, then $^\sharp\Gamma_1(\pi)_x=1$ and $\Gamma(\pi)_x$ is homologically trivial. We assume $^\sharp\Gamma_1(\pi)_x \geqq 2$ and $\dim\pi \geqq 2$. Then by Lemma 2.2, there exist $\gamma\in\Gamma_1(\pi)_x$ and $y, z\in M_{\boldsymbol{R}}$ such that $\Gamma(\pi)_y=\Gamma(\pi)_x$ and $\Gamma_1(\pi)_z=\Gamma_1(\pi)_x\backslash\{\gamma\}$. Since $\Gamma(\pi)_z$ is star open in $\Gamma(\pi)_x$, and $\Gamma(\pi)_z$ is homologically trivial by the induction assumption, it is sufficient to prove that $\Gamma(\pi)_y\backslash\Gamma(\pi)_z$ is homologically trivial in view of Proposition 1.8. $\Gamma(\pi)_y\backslash\Gamma(\pi)_z=\{\sigma\in\Gamma(\pi);\ \langle y, \sigma\rangle\subset\boldsymbol{R}_0 \text{ and } \sigma\succ\gamma\}\simeq\Gamma(\pi^{(\gamma)})_y$. Since $^\sharp\Gamma_1(\pi)_y\geqq 2$, there exists an $a\in\pi^{(\gamma)}\backslash\{0\}$ with $\langle y, a\rangle\geqq 0$. Hence $y\notin-\operatorname{int}(\pi^{(\gamma)})^\vee$ and we are done again by the induction assumption. q.e.d.

If we take x in π, $\Gamma(\pi)_x$ is equal to $\Gamma(\pi)$. Hence the above proposition implies, in particular, that $\Gamma(\pi)$ itself is homologically trivial if $\dim\pi>0$.

DEFINITION 2.4. For a star closed subset Σ of $\Gamma(\pi)$ and for an element m in M, we define the locally star closed subset $\Sigma^{(m)}$ of Σ by

$$\Sigma^{(m)}=\{\rho\in\Sigma;\ m\in-\rho^\vee \text{ and } m^\perp\cap\rho\in\Sigma\}\ .$$

REMARK 2.5. For $m\in M\cap\pi^\vee$, it is easy to see that $\Sigma^{(m)}=\Sigma(\eta)$, where $\eta=\pi\cap m^\perp\in\Gamma(\pi)$ and $\Sigma(\eta)=\{\rho\in\Sigma;\ \eta\succ\rho\}$.

PROPOSITION 2.6. *If $m\notin M\cap\pi^\vee$, then $\Sigma^{(m)}$ is homologically trivial for any star closed subset Σ of $\Gamma(\pi)$.*

PROOF. If $\Sigma=\Gamma(\pi)$, then $\Sigma^{(m)}$ is equal to $\Gamma(\pi)_{-m}$, since $m^\perp\cap\rho$ is in $\Gamma(\rho)\subset\Gamma(\pi)$ for every $m\in-\rho^\vee$. Hence $\Sigma^{(m)}$ is homologically trivial by Proposition 2.3. We prove this proposition by induction on the cardinality of $\Gamma(\pi)\backslash\Sigma$. Let η be a face of the highest dimension in $\Gamma(\pi)\backslash\Sigma$, and let $\Sigma'=\{\eta\}\cup\Sigma$. Then Σ' is a star closed subset of $\Gamma(\pi)$ and $\Sigma'^{(m)}$ is homologically trivial by the induction assumption. It is clear that $\Sigma^{(m)}$ is a star closed subset of $\Sigma'^{(m)}$, and hence it is sufficient to prove the homological triviality of $\Sigma'^{(m)}\backslash\Sigma^{(m)}$. If $m^\perp$ does not contain η, then $\Sigma'^{(m)}=\Sigma^{(m)}$ and there is nothing to prove. Assume $m^\perp\supset\eta$. Then, $\Sigma'^{(m)}\backslash\Sigma^{(m)}=\{\rho\in\Sigma';\ m\in-\rho^\vee \text{ and } m^\perp\cap\rho=\eta\}\simeq\{\rho^{(\eta)}\in\Gamma(\pi^{(\eta)});\ m\in-(\rho^{(\eta)})^\vee \text{ and } m^\perp\cap\rho^{(\eta)}=\{0\}\}=\{\rho^{(\eta)}\in\Gamma(\pi^{(\eta)});\ \langle-m, \rho^{(\eta)}\backslash\{0\}\rangle\subset(\boldsymbol{R}_0\backslash\{0\})\}=\Gamma(\pi^{(\eta)})_{-m-x}$, where x is an element of $\operatorname{int}(\pi^{(\eta)})^\vee$ of sufficiently small norm. Since $m\in M^{(\eta)}\backslash(\pi^{(\eta)})^\vee$, the point $-m-x$ is not in $-(\pi^{(\eta)})^\vee = -(\pi^\vee \cap \eta^\perp)$ and $\Gamma(\pi^{(\eta)})_{-m-x}$ is homologically trivial by Proposition 2.3. Thus $\Sigma'^{(m)}\backslash\Sigma^{(m)}$ is homologically trivial. q.e.d.

3. Homogeneous quotients of semigroup rings. We fix a field k of an arbitrary characteristic from this section on. Let N, M and π be as in §1. We denote $k_\sigma = k \otimes_{\mathbf{Z}} \mathbf{Z}_\sigma$ for every $\sigma \in \Gamma(\pi)$, and we use the same symbol $q_{\tau/\sigma}$ for the isomorphism $1_k \otimes q_{\tau/\sigma}: k_\sigma \xrightarrow{\sim} k_\tau$. We denote by $k[M]$ the k-vector space with the basis $\{e(m)\}_{m\in M}$ which has the k-algebra structure defined by $e(m)e(m') = e(m+m')$ for every pair (m, m') of elements in M. For a subsemigroup $\mathscr{S}$ of M with $0 \in \mathscr{S}$, we denote by $k[\mathscr{S}]$ the k-subalgebra $\bigoplus_{m\in\mathscr{S}} ke(m)$ of $k[M]$. If a subset $\mathscr{I} \subset \mathscr{S}$ is an ideal of $\mathscr{S}$, i.e., $m \in \mathscr{I}$ and $m' \in \mathscr{S}$ imply $m + m' \in \mathscr{I}$, then $k[\mathscr{I}] = \bigoplus_{m\in\mathscr{I}} ke(m)$ is an ideal of $k[\mathscr{S}]$. In order to simplify the notation, we denote by $k[\mathscr{S}\backslash\mathscr{I}]$ the quotient ring of $k[\mathscr{S}]$ with respect to the ideal $k[\mathscr{I}]$, identify it with the k-vector subspace $\bigoplus_{m\in\mathscr{S}\backslash\mathscr{I}} ke(m)$ of $k[M]$. Note that, in this ring, the multiplication $e(m)e(m')$ for m and m' in $\mathscr{S}\backslash\mathscr{I}$ is equal to $e(m+m')$ if $m + m' \notin \mathscr{I}$ and 0 if $m + m' \in \mathscr{I}$. These rings and ideals have the structure of M-graded objects. We take $M \cap \pi^\vee$ as such a semigroup $\mathscr{S}$ of M, and study the semigroup ring $S = k[M \cap \pi^\vee]$. Then $X_\pi = \operatorname{Spec}(S)$ is a torus embedding, i.e., the torus $T_N = \operatorname{Spec}(k[M])$ is an open subset of X_π and acts on X_π. According to the remark of [MO, (5, 3)], the map

$$\begin{array}{ccc} \Gamma(\pi) & \to & \left\{\begin{array}{l} T_N\text{-invariant irreducible reduced} \\ \text{closed subschemes of } X_\pi \end{array}\right\} \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \sigma & \longmapsto & V(\sigma) = \operatorname{Spec}(k[M \cap \pi^\vee \cap \sigma^\perp]) \end{array}$$

is bijective. Note that $k[M \cap \pi^\vee \cap \sigma^\perp]$ is thought of as the quotient of S by the M-homogeneous prime ideal $P(\sigma) = k[M \cap (\pi^\vee\backslash\sigma^\perp)]$. For a star closed subset Σ of $\Gamma(\pi)$, define the ideal $J(\Sigma) = \bigcap_{\sigma\in\Sigma} P(\sigma)$. Then $J(\Sigma)$ is an M-homogeneous semiprime ideal of S. Conversely, it is clear that every M-homogeneous semiprime ideal of S is equal to $J(\Sigma)$ for a star closed subset Σ of $\Gamma(\pi)$. Thus we have a bijection

$$\begin{array}{ccc} \left\{\begin{array}{l} \text{star closed subsets} \\ \text{of } \Gamma(\pi) \end{array}\right\} & \to & \left\{\begin{array}{l} T_N\text{-invariant reduced closed} \\ \text{subschemes of } X_\pi \end{array}\right\} \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \Sigma & \longmapsto & Y_\Sigma = \operatorname{Spec}(S/J(\Sigma))\,. \end{array}$$

For a non-empty star closed subset Σ, we call $\min\{\dim\sigma; \sigma \in \Sigma\}$ the *height* of Σ and denote it by $\operatorname{ht}\Sigma$. Since $\operatorname{ht} P(\sigma) = \dim\sigma$, the height of Σ is equal to that of the ideal $J(\Sigma)$ or the codimension of Y_Σ in X_π.

We now fix a non-empty star closed subset Σ of $\Gamma(\pi)$. Our main purpose is to find the condition for the ring $S_\Sigma = S/J(\Sigma)$ to be Gorenstein or Cohen-Macaulay.

Since $J(\Sigma) \subset P(\sigma)$ for every σ in Σ, the ring $S/P(\sigma)$ is an S_Σ-module. If σ is a face of τ, then $P(\sigma)$ is contained in $P(\tau)$. When $\dim\tau - \dim\sigma = 1$, we denote by $Q_{\tau/\sigma}$ the homomorphism $S/P(\sigma) \otimes_k k_\sigma \to S/P(\tau) \otimes_k k_\tau$ defined by the tensor product of the quotient map and $q_{\tau/\sigma}$. Define the S_Σ-module K^i to be the direct sum $\bigoplus_{\sigma\in\Sigma_i} S/P(\sigma)\otimes_k k_\sigma$ for every $i = 0, \cdots, r = \operatorname{rank} N$, and define the coboundary map $\delta^i\colon K^i \to K^{i+1}$ by

$$\delta^i((f_\sigma)_{\sigma\in\Sigma_i}) = (\textstyle\sum_{\sigma\in\Sigma_i,\tau\succ\sigma} Q_{\tau/\sigma}(f_\sigma))_{\tau\in\Sigma_{i+1}}\,.$$

Note that all our rings and modules are naturally M-graded. Moreover, for every i with $0 \leqq i \leqq r-1$, δ^i is a homomorphism of M-graded S_Σ-modules of degree 0. Hence we can consider the m-component $K_m^\cdot$ of the sequence $K^\cdot = (\cdots \to 0 \to K^0 \xrightarrow{\delta^0} K^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{r-1}} K^r \to 0 \to \cdots)$ for every m in M. Recall that, for $\rho \in \Gamma(\pi)$, we denote $\Sigma(\rho) = \{\sigma \in \Sigma;\, \rho \succ \sigma\}$. It is clear that $\Sigma(\rho)$ is a locally star closed subset of $\Gamma(\pi)$ and is empty if $\rho \notin \Sigma$.

PROPOSITION 3.1. *The m-component $K_m^\cdot$ of $K^\cdot$ is the 0-complex if $m \notin M\cap\pi^\vee$. If $m \in M\cap\pi^\vee$, then there exists a natural isomorphism $K_m^\cdot \xrightarrow{\sim} C^\cdot(\Sigma(\rho), k)$ where $\rho = \pi\cap m^\perp$. In particular, $K_m^\cdot$ is the 0-complex if $\rho\notin\Sigma$. Let m' be an element of $M\cap\pi^\vee$ and let $\eta = \pi\cap(m+m')^\perp$. Then the diagram*

$$\begin{tikzcd}
K_m^\cdot \arrow[r, "\sim"] \arrow[d, "e(m')\times"] & C^\cdot(\Sigma(\rho), k) \arrow[d] \\
K_{m+m'}^\cdot \arrow[r, "\sim"] & C^\cdot(\Sigma(\eta), k)
\end{tikzcd}$$

commutes, where $e(m')\times\colon K_m^\cdot \to K_{m+m'}^\cdot$ is the multiplication by the homogeneous element $e(m')\in S$, and $C^\cdot(\Sigma(\rho), k)\to C^\cdot(\Sigma(\eta), k)$ is the natural homomorphism corresponding to the star open inclusion $\Sigma(\eta)\hookrightarrow\Sigma(\rho)$.

PROOF. If $m\notin M\cap\pi^\vee$, then $k[M\cap\pi^\vee\cap\sigma^\perp]_m = 0$ for every $\sigma\in\Gamma(\pi)$, and the first assertion is obvious. For m in $M\cap\pi^\vee$, the m-component of $k[M\cap\pi^\vee\cap\sigma^\perp]$ is equal to $ke(m)$ if $m^\perp\supset\sigma$ and zero otherwise. Since $\rho = \pi\cap m^\perp$ is a face of π, we have the natural isomorphism

$$\begin{array}{ccc}
K_m^i = \bigoplus_{\sigma\in\Sigma(\rho)_i} ke(m)\otimes_k k_\sigma & \xrightarrow{\sim} & C^i(\Sigma(\rho), k) = \bigoplus_{\sigma\in\Sigma(\rho)_i} k_\sigma\,. \\
\cup & & \cup \\
(e(m)\otimes a_\sigma)_{\sigma\in\Sigma(\rho)_i} & \longmapsto & (a_\sigma)_{\sigma\in\Sigma(\rho)_i}\,.
\end{array}$$

Hence the sequences $K_m^\cdot$ and $C^\cdot(\Sigma(\rho), k)$ are isomorphic since the coboundary maps are defined by $q_{\tau/\sigma}$'s. If $\rho \notin \Sigma$, then $\Sigma(\rho)$ is empty and $K_m^\cdot$ is the 0-complex. Clearly $\eta = \pi \cap (m + m')^\perp$ is a face of ρ, and hence $\Sigma(\eta)$ is a star open subset of $\Sigma(\rho)$. Since $e(m')e(m)$ is $e(m + m')$ in $k[M \cap \pi^\vee \cap \sigma^\perp]$ if $\sigma \in \Sigma(\eta)$ and is zero otherwise, we get a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\sigma \in \Sigma(\rho)_i} ke(m) \otimes_k k_\sigma & \longrightarrow & \bigoplus_{\sigma \in \Sigma(\rho)_i} k_\sigma \\ \big\downarrow{\scriptstyle e(m')\times} & & \big\downarrow{\scriptstyle p_i} \\ \bigoplus_{\sigma \in \Sigma(\eta)_i} ke(m + m') \otimes_k k_\sigma & \longrightarrow & \bigoplus_{\sigma \in \Sigma(\eta)_i} k_\sigma , \end{array}$$

for every i, where $p_i((a_\sigma)_{\sigma \in \Sigma(\rho)_i}) = (a_\sigma)_{\sigma \in \Sigma(\eta)_i}$, for every $(a_\sigma)_{\sigma \in \Sigma(\rho)_i}$. Hence we are done. q.e.d.

COROLLARY 3.2. *The sequence $K^\cdot$ is a complex.*

PROOF. By Propositions 1.6 and 3.1, $K^\cdot$ for each m in M is a complex. Hence $K^\cdot$ is also a complex.

The following is our main theorem.

THEOREM 3.3. *$K^\cdot$ is the dualizing complex of the ring S_Σ.*

Recall that, for a noetherian ring A with Spec(A) connected, the *dualizing complex* $R^\cdot$ of A is determined uniquely up to quasi-isomorphism, dimension shift and the tensor product of projective modules of rank one, where a homomorphism $R_1^\cdot \to R_2^\cdot$ is a quasi-isomorphism if the induced homomorphism $H^i(R_1^\cdot) \to H^i(R_2^\cdot)$ is an isomorphism for every $i \in \boldsymbol{Z}$. For the detail, see [RD, Ch. V]. The important fact is that if the dualizing complex $R^\cdot$ exists, then

A : Cohen-Macaulay $\Leftrightarrow \exists d,\ H^i(R^\cdot) = 0\ \forall i \neq d$,

A : Gorenstein $\Leftrightarrow \exists d,\ H^i(R^\cdot) = 0\ \forall i \neq d$ and $H^d(R^\cdot)$ is a projective A-module of rank one.

When A is Cohen-Macaulay, $H^d(R^\cdot)$ is usually called a *dualizing module* or a *canonical module* of A.

When $\dim \pi$ is equal to $r = \operatorname{rank} N$, then $\pi^\vee = \{0\}$ and $P(\pi) = \bigoplus_{m \in M \cap (\pi^\vee \setminus \{0\})} ke(m)$ is a maximal ideal of S. We denote the maximal ideal $P(\pi)/J(\Sigma)$ of S_Σ by $\mathfrak{m}$. Theorem 3.3 is a rather easy consequence of the following proposition, which we prove in the next section.

PROPOSITION 3.4. *If $\dim \pi = r$, then the hyperextension groups are*

$$\operatorname{Exp}^i_{S_\Sigma}(S_\Sigma/\mathfrak{m}, K^\cdot) \simeq \begin{cases} S_\Sigma/\mathfrak{m} & \text{if } i = r \\ 0 & \text{if } i \neq r . \end{cases}$$

REMARK. Note that $K^{\cdot}$ corresponds to the sequence

$$\cdots \to 0 \to \bigoplus_{\alpha \in \Sigma_0} O_{V(\alpha)} \otimes_k k_\alpha \to \bigoplus_{\beta \in \Sigma_1} O_{V(\beta)} \otimes_k k_\beta \to \cdots \to O_{V(\pi)} \otimes_k k_\pi \to 0 \to \cdots ,$$

if we consider them as a sheaf and complex of sheaves on $Y_\Sigma = \operatorname{Spec}(S_\Sigma)$. For a face ρ of π in Σ, the restriction of $K^{\cdot}$ to the affine open set $X_\rho = \operatorname{Spec}(k[M \cap \rho^\vee])$ coincides with the complex defined similarly for ρ and $\Sigma(\rho) = \{\sigma \in \Sigma; \rho \succ \sigma\}$ as we defined $K^{\cdot}$ for π and Σ.

PROOF OF THEOREM 3.3. We prove this theorem by induction on r. If $r = 0$, then S_Σ and $K^{\cdot}$ are both equal to k. Hence the assertion is obvious. If $\dim \pi$ is less than r, then X_π is the product of a torus T' of dimension $r - \dim \pi$ and the torus embedding X'_π of dimension $\dim \pi$ defined by the pair (π, N_π). Hence Y_Σ is also a product $T' \times Y'_\Sigma$. We are done by the induction assumption, since $K^{\cdot}$ is isomorphic to the pull-back of that of Y'_Σ for the projection $Y_\Sigma \to Y'_\Sigma$ and we can apply [RD, Ch. 5, Theorem 8.3]. Thus we may assume $\dim \pi = r$. For a proper face ρ of π the restriction $K^{\cdot}|_{X_\rho}$ is the dualizing complex of $Y_\Sigma \cap X_\rho$ in view of the above remark and what we have seen above. Since $\bigcup_{\rho \in \Gamma(\pi), \rho \neq \pi} X_\rho = X_\pi \setminus V(\pi)$, it is sufficient to prove that $K^{\cdot}$ is a dualizing complex at the unique T_N-invariant point $V(\pi)$ of X_π. Since $\mathfrak{m} = P(\pi)/I(\Sigma)$ is the ideal of $V(\pi)$ in Y_Σ, we are done by [RD, Ch. 5, Proposition 3.4] and our Proposition 3.4. q.e.d.

COROLLARY 3.5. *S_Σ is Cohen-Macaulay if and only if $H^i(\Sigma(\rho), k) = 0$ for every $i \neq h$ and for every $\rho \in \Sigma$, where $h = \operatorname{ht} \Sigma$.*

PROOF. If S_Σ is Cohen-Macaulay, there exists an integer d and $H^i(K^{\cdot}) = 0$ for every $i \neq d$. By Proposition 3.1, this is equivalent to $H^i(\Sigma(\rho), k) = 0$ for every $\rho \in \Sigma$ and every $i \neq d$. For an element $\rho \in \Sigma_h$, we have $\Sigma(\rho) = \{\rho\}$ and $H^h(\Sigma(\rho), k) \simeq k$. Hence we have $d = h$. The converse is obvious. q.e.d.

DEFINITION 3.6. We call a star closed subset Σ of $\Gamma(\pi)$ *Cohen-Macaulay* if Σ satisfies the condition of Corollary 3.5.

This definition depends on the fixed field k. Since $C^{\cdot}(\Sigma, k) = C^{\cdot}(\Sigma, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} k$, it actually depends on its characteristic (cf. Reisner [R1, §1, Remark 3]).

COROLLARY 3.7. *If Σ is Cohen-Macaulay, then* $\operatorname{Ker}[K^h \xrightarrow{\delta^h} K^{h+1}]$ *is a dualizing module of the ring S_Σ.*

PROOF. Since $\Sigma_{h-1} = \varnothing$, we have $K^{h-1} = 0$ and the assertion is obvious. q.e.d.

PROPOSITION 3.8. *The ring S_Σ is Gorenstein if and only if Σ is Cohen-Macaulay and there exists an M-graded isomorphism $S_\Sigma \xrightarrow{\sim} H^h(K^\cdot)$ with $h = \operatorname{ht} \Sigma$, where the degree of the isomorphism may not be zero.*

PROOF. The "if" part is obvious since $K^\cdot$ is the dualizing complex of the ring S_Σ. Assume S_Σ is Gorenstein. Then Σ is Cohen-Macaulay and $H^h(K^\cdot)$ is a free S_Σ-module generated by a homogeneous element by Proposition 8 of Appendix. Thus the assertion is proved. q.e.d.

We will give a more precise condition for the Gorensteinness of the ring S_Σ in §5, which is a generalization of the results of Hochster [H2] and Stanley [S1].

4. **Proof of Proposition** 3.4. We need some elementary facts on M-graded rings and modules. We list them here without proof, since they can be proved as in the $\boldsymbol{Z}$-graded case or in the non-graded case. Some of them were proved by Goto and Watanabe [GW1], [GW2].

Let A be an M-graded noetherian ring with $A_0 \simeq k$.

DEFINITION 4.1. For M-graded A-modules E, F, and for an element m of M, we denote by $\underline{\mathrm{Hom}}_A^m(E, F)$ the set of A-homomorphisms of degree m from E to F. We denote $\underline{\mathrm{Hom}}_A(E, F) = \bigoplus_{m \in M} \underline{\mathrm{Hom}}_A^m(E, F)$.

LEMMA 4.2. *The natural homomorphism*

$$\underline{\mathrm{Hom}}_A(E, F) \to \mathrm{Hom}_A(E, F)$$

is an isomorphism if E is an A-module of finite type.

For an M-graded A-module E, we denote $E^* = \underline{\mathrm{Hom}}_k(E, k)$, where k $(\simeq A_0)$ is considered as an M-graded ring concentrated at $0 \in M$.

LEMMA 4.3. *$\underline{\mathrm{Hom}}_k(\cdot, k)$ is an exact functor for the category of M-graded k-modules.*

For M-graded A-module E and F, the tensor product $E \otimes_A F$ has the structure of an M-graded A-module such that for homogeneous elements $x \in E$ and $y \in F$, we have $\deg(x \otimes y) = \deg x + \deg y$.

LEMMA 4.4. *There exists a natural isomorphism*

$$\underline{\mathrm{Hom}}_A(E, \underline{\mathrm{Hom}}_k(F, G)) \to \underline{\mathrm{Hom}}_k(E \underset{A}{\otimes} F, G)$$

for every pair of M-graded A-modules E, F, and for every M-graded k-module G.

Recall that, for a ring A and for complexes $E^\cdot$, $F^\cdot$ of A-modules bounded above and below, respectively, the hyperextension group

$\mathrm{Ext}^i_A(E^{\cdot}, F^{\cdot})$ depends only on the quasi-isomorphism classes of $E^{\cdot}$ and $F^{\cdot}$. Furthermore, if either $E^{\cdot}$ is a complex of projective A-modules or $F^{\cdot}$ is a complex of injective A-modules, $\mathrm{Ext}^i_A(E^{\cdot}, F^{\cdot})$ is equal to the i-th cohomology group of the complex $\mathrm{Hom}^{\cdot}_A(E^{\cdot}, F^{\cdot})$. (Cf. [RD, Ch. 1 §6].) In order to calculate $\mathrm{Ext}^i_{S_\Sigma}(S_\Sigma/\mathfrak{m}, K^{\cdot})$, we replace $K^{\cdot}$ by a complex $I^{\cdot}$ which is quasi-isomorphic to $K^{\cdot}$ and plays the role of an injective object as far as M-graded modules are concerned.

For a ρ in Σ, let B_ρ be the set of homogeneous elements of $S_\Sigma\backslash P(\rho)/I(\Sigma)$. Then the localization $B^{-1}S_\Sigma$ is an M-graded ring which can be written as $k[\bigcup_{\sigma\in\Sigma(\rho)} M\cap\rho^\vee\cap\sigma^\perp]$.

LEMMA 4.5. *$\underline{\mathrm{Hom}}_{S_\Sigma}(\cdot, (B_\rho^{-1}S_\Sigma)^*)$ is an exact functor for the category of M-graded S_Σ-modules, where $(B_\rho^{-1}S_\Sigma)^* = \underline{\mathrm{Hom}}_k(B_\rho^{-1}S_\Sigma, k)$.*

PROOF. For an M-graded S_Σ-module F, the S_Σ-module

$$\underline{\mathrm{Hom}}_{S_\Sigma}(F, (B_\rho^{-1}S_\Sigma)^*)$$

is equal to $\underline{\mathrm{Hom}}_k(F\otimes_{S_\Sigma} B_\rho^{-1}S_\Sigma, k)$ by Lemma 4.4. Since the functor $\otimes_{S_\Sigma} B_\rho^{-1}S_\Sigma$ is exact, this lemma follows from Lemma 4.3. q.e.d.

$(B_\rho^{-1}S_\Sigma)^*$ is an S_Σ-module of the form $k[-\bigcup_{\sigma\in\Sigma(\rho)} M\cap\rho^\vee\cap\sigma^\perp]$. Since we see easily that $\pi^\vee\cap(-\bigcup_{\sigma\in\Sigma(\rho)} M\cap\rho^\vee\cap\sigma^\perp)$ is equal to $M\cap\pi^\vee\cap\rho^\perp$, there exists a natural inclusion $S/P(\rho) = k[M\cap\pi^\vee\cap\rho^\perp]\to(B_\rho^{-1}S_\Sigma)^*$. Now we set $I^i = \bigoplus_{\rho\in\Sigma_i}(B_\rho^{-1}S_\Sigma)^*\otimes_k k_\rho$ and define a coboundary map $\tilde{\delta}^i\colon I^i\to I^{i+1}$ by

$$\tilde{\delta}^i((g_\sigma)_{\sigma\in\Sigma_i}) = (\textstyle\sum_{\sigma\in\Sigma_i,\tau>\sigma}\tilde{Q}_{\tau/\sigma}(g_\sigma))_{\tau\in\Sigma_{i+1}},$$

where $\tilde{Q}_{\tau/\sigma}\colon(B_\sigma^{-1}S_\Sigma)^*\otimes_k k_\sigma\to(B_\tau^{-1}S_\Sigma)^*\otimes_k k_\tau$ is the S_Σ-homomorphism defined by the tensor product of the dual of the natural S_Σ-homomorphism $B_\tau^{-1}S_\Sigma\to B_\sigma^{-1}S_\Sigma$ and $q_{\tau/\sigma}$. Then there is a natural inclusion $K^i\hookrightarrow I^i$ which makes the diagram

$$\begin{array}{ccc} K^i & \xrightarrow{\delta^i} & K^{i+1} \\ \cap\downarrow & & \cap\downarrow \\ I^i & \xrightarrow{\tilde{\delta}^i} & I^{i+1} \end{array}$$

commutative for every i.

PROPOSITION 4.6. *The sequence $I^{\cdot}$ is a complex and the homomorphism $K^{\cdot}\to I^{\cdot}$ of complexes is a quasi-isomorphism.*

PROOF. Since I^i ($i = 0, \cdots, r$) are M-graded and $\tilde{\delta}^i$'s are homomorphisms of degree 0, it is enough to check the assertion for m-components

$I_m^\cdot$ and $K_m^\cdot \to I_m^\cdot$ for every $m \in M$. For an m in M, the m-component of $k[-\bigcup_{\sigma\in\Sigma(\rho)} M\cap\rho^\vee\cap\sigma^\perp]$ is isomorphic to k if and only if ρ is in $\Sigma^{(m)}$ (see Definition 2.4). Hence the m-component of I^i is isomorphic to $C^i(\Sigma^{(m)}, k)$. Thus clearly $I_m^\cdot$ is isomorphic to $C^\cdot(\Sigma^{(m)}, k)$, and $I^\cdot$ is a complex. If m is in $M\cap\pi^\vee$, the induced homomorphism $K_m^\cdot \to I_m^\cdot$ is an isomorphism by Remark 2.5 and Proposition 3.1. If m is not in $M\cap\pi^\vee$, $K_m^\cdot$ is the 0-complex by Proposition 3.1. On the other hand, all the cohomology groups of $I_m^\cdot$ vanish since $\Sigma^{(m)}$ is homologically trivial by Proposition 2.6. Hence $K_m^\cdot \to I_m^\cdot$ is a quasi-isomorphism. Thus the homomorphism $K^\cdot \to I^\cdot$ is a quasi-isomorphism. q.e.d.

Let $\cdots \to F^{-2} \to F^{-1} \to F^0 \to S_\Sigma/\mathfrak{m} \to 0$ be an S_Σ-free resolution of $S_\Sigma/\mathfrak{m}$. We can choose this resolution in such a way that F^i's are M-graded free S_Σ-modules of finite type, and the homomorphisms $F^i \to F^{i+1}$ are of degree 0. Then $\mathrm{Ext}^i_{S_\Sigma}(S_\Sigma/\mathfrak{m}, K^\cdot)$ is equal to the i-th cohomology group of the complex $\mathrm{Hom}^\cdot_{S_\Sigma}(F^\cdot, I^\cdot)$ since $I^\cdot$ is quasi-isomorphic to $K^\cdot$ and $F^\cdot$ is projective. In the double complex

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}_{S_\Sigma}(F^{-2}, I^0) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{-2}, I^1) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{-2}, I^2) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}_{S_\Sigma}(F^{-1}, I^0) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{-1}, I^1) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{-1}, I^2) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}_{S_\Sigma}(F^{0}, I^0) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{0}, I^1) & \longrightarrow & \mathrm{Hom}_{S_\Sigma}(F^{0}, I^2) & \longrightarrow
\end{array}$$

every column is exact in view of Lemma 4.5. Hence in one of the associated spectral sequences for the double complex, we have

$$''E_1^{p,q} = \begin{cases} \mathrm{Hom}_{S_\Sigma}(S_\Sigma/\mathfrak{m}, I^q) & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 . \end{cases}$$

By Lemmas 4.2 and 4.4, we have

$$\mathrm{Hom}_{S_\Sigma}(S_\Sigma/\mathfrak{m}, (B_\rho^{-1}S_\Sigma)^*) \simeq \underline{\mathrm{Hom}}_k(S_\Sigma/\mathfrak{m} \otimes_{S_\Sigma} B_\rho^{-1}S_\Sigma, k) .$$

Since $\mathfrak{m}\cap B_\rho^{-1}$ is non-empty for $\rho \neq \pi$, we have $S_\Sigma/\mathfrak{m} \otimes_{S_\Sigma} B_\rho^{-1}S_\Sigma = 0$ for such ρ. By our assumption $\dim\pi = r$, we have $\mathrm{Hom}_{S_\Sigma}(S_\Sigma/\mathfrak{m}, I^q) = 0$ for $q < r$. Thus we have

$$''E_1^{p,q} \simeq \begin{cases} S_\Sigma/\mathfrak{m} & \text{if } p = 0 \text{ and } q = r \\ 0 & \text{otherwise} . \end{cases}$$

Hence we conclude that

$$H^i(\mathrm{Hom}^{\cdot}_{S_\Sigma}(F^{\cdot}, I^{\cdot})) \simeq \begin{cases} S_\Sigma/\mathfrak{m} & \text{if} \quad i = r \\ 0 & \text{if} \quad i \neq r . \end{cases}$$

Since $\mathrm{Ext}^i_{S_\Sigma}(S_\Sigma/\mathfrak{m}, K^{\cdot}) = H^i(\mathrm{Hom}^{\cdot}_{S_\Sigma}(F^{\cdot}, I^{\cdot}))$, we are done. q.e.d.

5. The condition for the ring S_Σ to be Gorenstein. Recall that for a non-empty star closed subset Σ of $\Gamma(\pi)$, we denote $\mathrm{ht}\,\Sigma = \min\{\dim\sigma; \sigma\in\Sigma\}$ and it is equal to the codimension of Y_Σ in the torus embedding X_π or the height of the ideal $J(\Sigma)$ of $k[M\cap\pi^\vee]$.

LEMMA 5.1. *Let r' be the dimension of π and let Σ be a non-empty star closed subset of $\Gamma(\pi)$. Then if $\mathrm{ht}\,\Sigma < r'$, the cohomology group $H^{r'}(\Sigma, k)$ is equal to zero.*

PROOF. Σ_i is non-empty for every i with $\mathrm{ht}\,\Sigma \leqq i \leqq r'$. Hence $\Sigma_{r'-1}$ is non-empty and the homomorphism $C^{r'-1}(\Sigma, k) \to C^{r'}(\Sigma, k) = k_\pi$ is surjective. q.e.d.

LEMMA 5.2. *Let Σ', Σ'' be non-empty star closed subsets of $\Gamma(\pi)$. If $\mathrm{ht}\,\Sigma'$ and $\mathrm{th}\,\Sigma''$ are less than r' and if $\Sigma'\cap\Sigma'' = \{\pi\}$ then $H^{r'-1}(\Sigma'\cup\Sigma'', k) \neq 0$.*

PROOF. In the Mayer-Vietoris exact sequence

$$H^{r'-1}(\Sigma'\cup\Sigma'', k) \to H^{r'}(\{\pi\}, k) \to H^{r'}(\Sigma', k) \oplus H^{r'}(\Sigma'', k)$$

we have $H^{r'}(\{\pi\}, k) = k_\pi$ and $H^{r'}(\Sigma', k) = H^{r'}(\Sigma'', k) = 0$ by Lemma 5.1. Hence we have $H^{r'-1}(\Sigma'\cup\Sigma'', k) \neq 0$. q.e.d.

Let Σ be a Cohen-Macaulay subset of $\Gamma(\pi)$ with $h = \mathrm{ht}\,\Sigma$. Then Σ_h is the set of the minimal faces of Σ, i.e., Σ is the star closure of Σ_h. Indeed, for $\rho\in\Sigma$ with $\dim\rho > h$, we have $H^{\dim\rho}(\Sigma(\rho), k) = 0$. Hence $\Sigma(\rho) \neq \{\rho\}$ and ρ is not minimal in Σ.

PROPOSITION 5.3. *Let Σ be Cohen-Macaulay with $h = \mathrm{ht}\,\Sigma$. If non-empty subsets Σ' and Σ'' of Σ are the star closures of their subsets Σ'_h and Σ''_h with $\Sigma_h = \Sigma'_h \cup \Sigma''_h$ and $\Sigma'_h \cap \Sigma''_h = \varnothing$, then Σ'_{h+1} and Σ''_{h+1} intersect.*

PROOF. Let ρ be an element of the minimal dimension in $\Sigma'\cap\Sigma''$. Then $\Sigma'(\rho)\cap\Sigma''(\rho)$ is equal to $\{\rho\}$ and $\Sigma(\rho) = \Sigma'(\rho)\cup\Sigma''(\rho)$ is a Cohen-Macaulay subset of $\Gamma(\rho)$. Hence by Lemma 5.2, we have $H^{\dim\rho-1}(\Sigma(\rho), k) \neq 0$. Since $\mathrm{ht}\,\Sigma(\rho) = h$, the dimension of ρ is equal to $h+1$. q.e.d.

PROPOSITION 5.4. *Let Σ be a Cohen-Macaulay subset of $\Gamma(\pi)$ with $\mathrm{ht}\,\Sigma = h < \dim\pi$. Assume $H^h(\Sigma(\rho), k) \simeq k$ for every $\rho\in\Sigma_{h+1}$. Then for* *any non-zero cocycle* $a = (a_\sigma)_{\sigma \in \Sigma_h}$ *in* $C^h(\Sigma, k)$, *the component* $a_\sigma \in k_\sigma$ *is not zero for every* σ *in* Σ_h.

PROOF. Set $\Sigma'_h = \{\sigma \in \Sigma_h; a_\sigma = 0\}$ and $\Sigma''_h = \{\sigma \in \Sigma_h; a_\sigma \neq 0\}$, and let Σ' and Σ'' be their star closures, respectively. If the proposition were false, then Σ' and Σ'' would clearly satisfy the condition of Proposition 5.3. Hence there exist $\rho \in \Sigma_{h+1}$, $\sigma' \in \Sigma'_h$ and $\sigma'' \in \Sigma''_h$ with $\rho \succ \sigma', \sigma''$. Our assumption $H^h(\Sigma(\rho), k) \simeq k$ implies $\Sigma(\rho) = \{\rho, \sigma', \sigma''\}$. Hence the ρ-component of $\delta^h(a) = (\sum_{\sigma \in \Sigma_h, \tau \succ \sigma} q_{\tau/\sigma}(a_\sigma))_{\tau \in \Sigma_{h+1}}$ is equal to $q_{\rho/\sigma''}(a_{\sigma''}) \neq 0$. This is impossible since a is a cocycle. q.e.d.

COROLLARY 5.5. *Under the same condition as in Proposition* 5.4, $H^h(\Sigma, k)$ *is at most* 1-*dimensional.*

PROOF. Let a and b be two non-zero cocycles of $C^h(\Sigma, k)$. Then the above proposition implies $b - ta = 0$ for some t in k. Thus a and b are linearly dependent. q.e.d.

COROLLARY 5.6. *In addition to the condition of Proposition* 5.4, *assume* $H^h(\Sigma, k) \simeq k$. *Then, for every* τ *in* Σ, *the induced homomorphism*

$$H^h(\Sigma, k) \to H^h(\Sigma(\tau), k)$$

is an isomorphism.

PROOF. Since $C^{h-1}(\Sigma, k) = 0$ (resp. $C^{h-1}(\Sigma(\tau), k) = 0$), the cohomology group $H^h(\Sigma, k)$ (resp. $H^h(\Sigma(\tau), k)$) is a submodule of $C^h(\Sigma, k)$ (resp. $C^h(\Sigma(\tau), k)$). Hence by Proposition 5.4, the kernel of this homomorphism is 0. Since $H^h(\Sigma(\tau), k)$ is at most one-dimensional, we are done. q.e.d.

DEFINITION 5.7. A non-empty star closed subset Σ of $\Gamma(\pi)$ with $\operatorname{ht}\Sigma = h$ is said to be *spherical* if

$$H^i(\Sigma(\eta), k) \simeq \begin{cases} k & \text{if} \quad i = h \\ 0 & \text{if} \quad i \neq h \end{cases}$$

for every $\eta \in \Sigma$. We say Σ is *semispherical* if there exists $\rho \in \Sigma$ such that $\Sigma(\rho)$ contains all the minimal elements of Σ and if $\Sigma(\rho)$ is spherical in $\Gamma(\rho)$.

In view of Corollaries 5.5 and 5.6, Σ is spherical if and only if Σ is Cohen-Macaulay, $H^h(\Sigma(\rho), k) \simeq k$ for every $\rho \in \Sigma_{h+1}$, and if $H^h(\Sigma, k) \neq 0$.

PROPOSITION 5.8. *If a star closed subset* $\Sigma \subset \Gamma(\pi)$ *is semispherical with respect to* $\rho \in \Sigma$, *then* $H^i(\Sigma(\eta), k) = 0$ *for every* i *and for every* $\eta \in \Sigma \backslash \Sigma(\rho)$. *In particular,* Σ *is Cohen-Macaulay and* ρ *is unique for* Σ.

PROOF. Since Σ is semispherical with respect to ρ, the intersection $\rho \cap \eta$ is in Σ, and $\Gamma(\eta) \supset \Sigma(\eta)$ is semispherical with respect to $\rho \cap \eta$. Hence it is sufficient to prove the proposition in the case $\eta = \pi$ and $\rho \neq \pi$. Consider the double complex $A^{\cdot\cdot}$ defined by

$$A^{i,j} = \bigoplus_{\lambda \in \Sigma_i, \sigma \in \Sigma(\rho)_j, \lambda > \sigma} k_\lambda \bigotimes_k k_\sigma ,$$

where the coboundary maps are defined naturally by $q_{\tau/\rho}$'s in §1. Fix an isomorphism $k \xrightarrow{\sim} H^h(\Sigma(\rho), k)$ for $h = \operatorname{ht}\Sigma$. Then since $A^{i,j} = \bigoplus_{\lambda \in \Sigma_i} k_\lambda \otimes C^j(\Sigma(\rho \cap \lambda), k)$, the composite homomorphism

$$k \xrightarrow{\sim} H^h(\Sigma(\rho), k) \xrightarrow{\sim} H^h(\Sigma(\rho \cap \lambda), k)$$

induces an exact sequence

$$0 \to C^i(\Sigma, k) \xrightarrow{\phi_i} A^{i,h} \to A^{i,h+1} \to \cdots$$

for every i. Since, obviously, ϕ_i's are commutative with the coboundary maps, we get a homomorphism $C^\cdot(\Sigma, k) \to A^{\cdot h}$ of complexes. Hence $'E_2^{p,q}$ of the spectral sequence for the double complex $A^{\cdot\cdot}$ is as follows.

$$'E_2^{p,q} \simeq \begin{cases} H^p(\Sigma, k) & \text{if } q = h \\ 0 & \text{if } q \neq h . \end{cases}$$

On the other hand, $A^{\cdot j}$ is isomorphic to $\bigoplus_{\sigma \in \Sigma(\rho)_j} C^\cdot(\{\lambda \in \Sigma; \lambda \succ \sigma\}, k)$. Since Σ is star closed, $\{\lambda \in \Sigma; \lambda \succ \sigma\}$ is in bijective correspondence with $\Gamma(\pi^{(\sigma)})$, and hence by the remark before Definition 2.4, $\{\lambda \in \Sigma; \lambda \succ \sigma\}$ is homologically trivial unless $\sigma = \pi$. Thus we know $''E_1^{p,q} = 0$ for every pair (p, q). Hence by the general theory of the spectral sequences, we have $H^p(\Sigma, k) = 0$ for every p. q.e.d.

For a face σ of π, we denote by $\operatorname{rel.int}(\pi^\vee \cap \sigma^\perp)$ the relative interior of the cone $\pi^\vee \cap \sigma^\perp$, i.e., its interior in $M_R^{(\sigma)}$. Let x be a point of $\pi^\vee$. Then x is contained in $\operatorname{rel.int}(\pi^\vee \cap \sigma^\perp)$ if and only if $\sigma = \pi \cap x^\perp$. Hence $\pi^\vee$ is decomposed into the disjoint union $\coprod_{\sigma \in \Gamma(\pi)} \operatorname{rel.int}(\pi^\vee \cap \sigma^\perp)$.

THEOREM 5.9. *Let Σ be a star closed subset of $\Gamma(\pi)$. Then S_Σ is Gorenstein if and only if Σ is semispherical with respect to an element $\rho \in \Sigma$ and if there exists $m_0 \in M$ with*

$$M \cap \Big(\coprod_{\sigma \in \Sigma(\rho)} \operatorname{rel.int}(\pi^\vee \cap \sigma^\perp)\Big) = m_0 + M \cap \Big(\bigcup_{\sigma \in \Sigma} (\pi^\vee \cap \sigma^\perp)\Big) .$$

PROOF. Assume S_Σ is Gorenstein. Then by Proposition 3.8, $H^h(K^\cdot)$ is an M-graded free S_Σ-module of rank one generated by a homogeneous element u. Let m_0 be the degree of u. Clearly m_0 is in $M \cap \pi^\vee$. Let $u = (a_\sigma)_{\sigma \in \Sigma_h} \in \bigoplus_{\sigma \in \Sigma_h} k[M \cap \pi^\vee \cap \sigma^\perp] \otimes_k k_\sigma$. Assume $a_\eta = 0$ for an η in Σ_h.

Take an m in $M \cap \text{rel. int}(\pi^\vee \cap \eta^\perp)$. Then $m + m_0$ is not in $M \cap \pi^\vee \cap \sigma^\perp$ for every $\sigma \in \Sigma_h$ except for η. Hence $e(m)u = (e(m)a_\sigma)_{\sigma \in \Sigma_h} = 0$. This is a contradiction, since u is a generator of the free module $H^h(K^\cdot)$. Hence $a_\sigma \neq 0$ for every $\sigma \in \Sigma_h$. In particular $m_0 \in M \cap \pi^\vee \cap \sigma^\perp$ for every $\sigma \in \Sigma_h$ and $\rho = \pi \cap m_0^\perp$ contains every σ in Σ_h. Since Σ is Cohen-Macaulay, Σ_h is the set of minimal faces of Σ. We have to show that $\Sigma(\rho)$ is spherical in $\Gamma(\rho)$. It is Cohen-Macaulay by Corollary 3.5 and Proposition 3.8. Let τ be an element of $\Sigma(\rho)$. Take an element m_1 in $M \cap (\text{rel. int}(\pi^\vee \cap \tau^\perp))$. Then, since $\pi^\vee \cap \tau^\perp \supset \pi^\vee \cap \rho^\perp$, the element $m_2 = m_0 + m_1$ is also in $M \cap (\text{rel. int}(\pi^\vee \cap \tau^\perp))$. Then $H^h(K^\cdot_{m_2}) = (S_\Sigma u)_{m_2} = ke(m_1)u \simeq k$. Since $K^\cdot_{m_2} \simeq C^\cdot(\Sigma(\tau), k)$ by Proposition 3.1, we have $H^h(\Sigma(\tau), k) \simeq k$. Hence $\Sigma(\rho)$ is spherical in $\Gamma(\rho)$, and Σ is semispherical with respect to ρ. By Propositions 3.1 and 5.8, we have

$$\{m \in M;\, H^h(K^\cdot)_m \neq 0\} = M \cap \left(\coprod_{\sigma \in \Sigma(\rho)} \text{rel. int}(\pi^\vee \cap \sigma^\perp)\right).$$

Since $H^h(K^\cdot) = S_\Sigma u$, the equality of the theorem holds. Conversely, assume Σ is semispherical with respect to $\rho \in \Sigma$ and satisfies the equality of the theorem for an $m_0 \in M$. Since $\langle m_0, a\rangle \leqq \langle m, a\rangle$ for every a in π and for every m in $m_0 + M \cap (\bigcup_{\sigma \in \Sigma}(\pi^\vee \cap \sigma^\perp))$, the cone $\pi \cap m_0^\perp$ is the maximal element ρ of $\Sigma(\rho)$. Let u be a non-zero homogeneous element of $H^h(K^\cdot)$ of degree m_0. We have to show that $H^h(K^\cdot) = S_\Sigma u$. Let m be an element of $\bigcup_{\sigma \in \Sigma}(M \cap \pi^\vee \cap \sigma^\perp)$. Then by the equality in the theorem, $m_0 + m$ is in $\text{rel. int}(\pi^\vee \cap \tau^\perp)$ for an element $\tau \in \Sigma(\rho)$. By Proposition 3.1, we get a commutative diagram

$$\begin{array}{ccc}
H^h(\Sigma(\rho), k) & \longrightarrow & H^h(\Sigma(\tau), k) \\
\wr\| & & \wr\| \\
H^h(K^\cdot_{m_0}) & \xrightarrow{e(m)\times} & H^h(K^\cdot_{m_0+m}) \\
\cup & & \cup \\
u & \longmapsto & e(m)u\,.
\end{array}$$

Since $\Sigma(\rho)$ is spherical in $\Gamma(\rho)$, $H^h(\Sigma(\rho), k)$ and $H^h(\Sigma(\tau), k)$ are 1-dimensional and $e(m)u$ is not zero by Corollaries 5.5 and 5.6. Hence $H^h(K^\cdot_{m_0+m}) = ke(m)u$. For $m \in M \cap \pi^\vee$ which is not in $m_0 + M \cap (\bigcup_{\sigma \in \Sigma}(\pi^\vee \cap \sigma^\perp))$, the face $\tau = \pi \cap m^\perp$ is not in $\Sigma(\rho)$ by the equality in the theorem. Hence $H^h(K^\cdot_m) \simeq H^h(\Sigma(\tau), k)$ is equal to zero by Proposition 5.8. Thus we have $H^h(K^\cdot) = k[\bigcup_{\sigma \in \Sigma} M \cap \pi^\vee \cap \sigma^\perp]u = S_\Sigma u$, and S_Σ is Gorenstein by Proposition 3.8. q.e.d.

The following theorem implies that S_Σ is Gorenstein if Σ is spherical.

THEOREM 5.10. *A star closed subset $\Sigma \subset \Gamma(\pi)$ is spherical if and only if S_Σ is Gorenstein and π is the unique element of $\Gamma(\pi)$ which contains all the minimal faces of Σ.*

PROOF. If S_Σ is Gorenstein, then by Theorem 5.9, Σ is semispherical with respect to an element $\rho \in \Sigma$. Since ρ must contain all the minimal faces of Σ, we necessary have $\rho = \pi$ and we are done. Assume Σ is spherical. Since we always have

$$\coprod_{\eta \in \Sigma} \text{rel. int}\,(\pi^\vee \cap \eta^\perp) = \bigcup_{\eta \in \Sigma} \Big(\coprod_{\sigma \in \Sigma(\eta)} \text{rel. int}\,(\pi^\vee \cap \sigma^\perp) \Big) = \bigcup_{\eta \in \Sigma} (\pi^\vee \cap \eta^\perp)\,,$$

the equality of Theorem 5.9 holds for $m_0 = 0$. Hence S_Σ is Gorenstein. If $\Sigma(\rho)$ contains Σ_h ($h = \text{ht}\,\Sigma$) for an element $\rho \in \Sigma$, then Σ is semispherical with respect to ρ. Hence we have $\rho = \pi$ by Proposition 5.8.
q.e.d.

COROLLARY 5.11 (Stanley [S1]). *The ring $k[M \cap \pi^\vee]$ is Gorenstein if and only if $M \cap (\text{int}\,\pi^\vee) = m_0 + M \cap \pi^\vee$ for an element $m_0 \in M$.*

PROOF. This corollary is an easy consequence of Theorem 5.9, since $k[M \cap \pi^\vee] = S_\Sigma$ for $\Sigma = \Gamma(\pi)$ and $\Gamma(\pi)$ is semispherical with respect to $\rho = \{0\}$. q.e.d.

Now, consider the case π is non-singular, i.e., for a $\boldsymbol{Z}$-basis $\{a_1, \cdots, a_r\}$ of N and for an integer $0 \leqq d \leqq r$, we have $\pi = \boldsymbol{R}_0 a_1 + \cdots + \boldsymbol{R}_0 a_d$. Let $\{m_1, \cdots, m_r\} \subset M$ be the basis of M dual to $\{a_1, \cdots, a_r\}$.

LEMMA 5.12. *In the above situation, let Σ be a star closed subset of $\Gamma(\pi)$. If an element $\rho = \boldsymbol{R}_0 a_1 + \cdots + \boldsymbol{R}_0 a_p$ of Σ, for a p with $0 \leqq p \leqq d$, contains all the minimal faces of Σ. Then we have*

$$M \cap \Big(\coprod_{\eta \in \Sigma(\rho)} \text{rel. int}\,(\pi^\vee \cap \eta^\perp) \Big) = \sum_{i=p+1}^{d} m_i + M \cap \Big(\bigcup_{\eta \in \Sigma} (\pi^\vee \cap \eta^\perp) \Big)\,.$$

PROOF. Let σ be a minimal face of Σ. Then we have

$$\textstyle\coprod_{\eta \in \Sigma(\rho), \eta > \sigma} \text{rel. int}\,(\pi^\vee \cap \eta^\perp) = \{x \in \pi^\vee \cap \sigma^\perp ; x^\perp \cap \pi \in \Sigma(\rho)\}\,.$$

Then we see easily that

$$\textstyle\coprod_{\eta \in \Sigma(\rho), \eta > \sigma} M \cap (\text{rel. int}\,(\pi^\vee \cap \eta^\perp)) = \displaystyle\sum_{i=p+1}^{d} m_i + M \cap (\pi^\vee \cap \sigma^\perp)\,.$$

The lemma is proved by taking the union of these equalities for all the minimal faces σ of Σ, since for every $\eta \in \Sigma$ there exists a minimal face $\sigma \in \Sigma$ with $\eta \succ \sigma$ and $\pi^\vee \cap \eta^\perp \subset \pi^\vee \cap \sigma^\perp$. q.e.d.

The following proposition is a consequence of Theorem 5.9 and Lemma 5.12.

PROPOSITION 5.13. *If π is non-singular, then S_Σ is Gorenstein if and only if Σ is semispherical.*

The assertion of this proposition is stated in [H2, Added in proof].

6. Applications. In this section we give some applications of the results in §3.

For a star closed subset Σ of $\Gamma(\pi)$, we denote by $K_\Sigma^\cdot$ the dualizing complex $K^\cdot$ which we constructed for Σ in §3.

Let Σ be a star closed subset of $\Gamma(\pi)$, and let q be an integer with $\operatorname{ht}\Sigma \leqq q \leqq r' = \dim \pi$. We denote $\Sigma_{[q} = \{\sigma \in \Sigma; \dim \sigma \geqq q\}$. Clearly $\Sigma_{[q}$ is a star closed subset of $\Gamma(\pi)$.

PROPOSITION 6.1. *If Σ is Cohen-Macaulay, then $\Sigma_{[q}$ is also a Cohen-Macaulay subset of $\Gamma(\pi)$ of height q.*

PROOF. By the definition of $K_\Sigma^\cdot$, it is clear that

$$K_{\Sigma_{[q}}^\cdot = (\cdots \to 0 \to K_\Sigma^q \to K_\Sigma^{q+1} \to \cdots \to K_\Sigma^r \to 0 \to \cdots) .$$

Hence we have $H^i(K_{\Sigma_{[q}}^\cdot) = 0$ for every $i \neq q$. q.e.d.

Recall that π is an arbitrary strongly convex rational polyhedral cone. Hence $k[M \cap \pi^\vee]$ is a normal semigroup ring.

PROPOSITION 6.2. (1) *The normal semigroup ring $k[M \cap \pi^\vee]$ is Cohen-Macaulay.* (2) *The ideal $k[M \cap (\operatorname{int} \pi^\vee)]$ is a dualizing module of the ring $k[M \cap \pi^\vee]$.*

PROOF. (1) We apply Corollary 3.5 to $\Sigma = \Gamma(\pi)$. Then, for any $\rho \in \Gamma(\pi)$, the star open subset $\Sigma(\rho)$ is equal to $\Gamma(\rho)$, and this is homologically trivial by the remark after Proposition 2.3 if $\rho \neq \{0\}$. It is clear that Σ satisfies the condition of Corollary 3.5 with $h = 0$. Thus $S_\Sigma = k[M \cap \pi^\vee]$ is Cohen-Macaulay. (2) Since $\operatorname{ht}\Sigma = 0$ and $K_\Sigma^0 = k[M \cap \pi^\vee] \otimes_k k_{\{0\}}$, this ideal is isomorphic to $\operatorname{Ker}[K_\Sigma^0 \xrightarrow{\delta^0} K_\Sigma^1] = H^0(K_\Sigma^\cdot)$. q.e.d.

(1) of this proposition was first proved by Hochster [H1], and (2) is stated in [TE, Ch. 1, Theorems 9 and 14].

COROLLARY 6.3. *For every integer q with $0 \leqq q \leqq \dim \pi$, the star closed subset $\Sigma = \Gamma(\pi)_{[q}$ is Cohen-Macaulay.*

PROOF. This follows from Propositions 6.1 and 6.2.

This corollary means that the union of orbits of codimension $\geqq q$ of an affine torus embedding X_π is Cohen-Macaulay for every q with $0 \leqq q \leqq \dim \pi$. For $q = 1$, we have the following better result.

PROPOSITION 6.4. *If* $\Sigma = \Gamma(\pi)_{[1}$ $(=\Gamma(\pi)\backslash\{\{0\}\})$, *then* Σ *is spherical. In particular, the ring* S_Σ *is Gorenstein.*

PROOF. For a ρ in Σ, the subset $\Sigma(\rho)$ is equal to $\Gamma(\rho)\backslash\{\{0\}\}$. Since $\Gamma(\rho)$ is homologically trivial by the remark after Proposition 2.3, we have

$$H^i(\Sigma(\rho), k) \simeq \begin{cases} k & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1 . \end{cases}$$

Hence Σ is spherical, and S_Σ is Gorenstein by Theorem 5.10. q.e.d.

REMARK 6.5. This proposition means that for any torus embedding X with the torus T, the reduced subscheme $X\backslash T$ is Gorenstein. Furthermore, since a noetherian local ring R is Gorenstein if and only if its completion $\hat{R}$ is, it follows that the boundary of any toroidal embedding is Gorenstein. For the definition of toroidal embeddings, see [TE, Ch. 2, §1].

7. Gorenstein semigroup rings. In this section, we aim at classifying Gorenstein semigroup rings. In §7 and §8, by a semigroup ring, we always mean the ring $k[M \cap \pi^\vee]$ for a strongly convex rational polyhedral cone $\pi \subset N_R$ *of dimension* $r = \operatorname{rank} N$. Hence this is equivalent to classifying Gorenstein affine T_N-embeddings with a T_N-invariant point, where T_N is the torus $\operatorname{Spec}(k[M])$. For the Gorensteinness of semigroup rings, Stanley obtained the fundamental result which we generalized in §5. The translation of Stanley's characterization in terms of $\pi^\vee$ into the dual condition in terms of π enables us to classify Gorenstein semigroup rings.

PROPOSITION 7.1 (Stanley [S1]). *The semigroup ring* $k[M \cap \pi^\vee]$ *is Gorenstein if and only if there exists an element* m_π *in* M *with* $m_\pi + M \cap \pi^\vee = M \cap (\operatorname{int} \pi^\vee)$. *Furthermore, such an* m_π *is unique for* π.

PROOF. In view of Corollary 5.11, it is sufficient to prove the uniqueness of m_π under the condition $\dim \pi = r$. If two elements m_π and m'_π in M have the property $m_\pi + M \cap \pi^\vee = m'_\pi + M \cap \pi^\vee$, then we have $m_\pi - m'_\pi \in M \cap (\pi^\vee \cap (-\pi^\vee))$. Since $\dim \pi = r$, the intersection $\pi^\vee \cap (-\pi^\vee) = \pi^\perp$ is equal to $\{0\}$. Hence we have $m_\pi = m'_\pi$.

DEFINITION 7.2. An element m in M is said to be *primitive* if $m \neq 0$ and $\boldsymbol{Z}m = M \cap \boldsymbol{R}m$. An affine hyperplane E of N_R is *primitive* if $E = \{a \in N_R; \langle m, a\rangle = 1\}$ for a primitive element m in M.

Let $\mathscr{S}$ be the set of the pairs $\{E, P\}$ of a primitive hyperplane E in $N_{\mathbf{R}}$ and a polytope P (=the convex hull of a finite set of points) of dimension $r-1$ in E with vertices in $N \cap E$.

If the semigroup ring $k[M \cap \pi^{\vee}]$ is Gorenstein, then obviously the element m_{π} in Proposition 7.1 is primitive. We set $E_{\pi} = \{m_{\pi} = 1\} = \{a \in N_{\mathbf{R}}; \langle m_{\pi}, a\rangle = 1\}$. Then $P_{\pi} = E_{\pi} \cap \pi$ is an $(r-1)$-dimensional polytope, since m_{π} is in the interior of $\pi^{\vee}$.

PROPOSITION 7.3. *In the above situation, every vertex of the polytope P_{π} is in $N \cap E_{\pi}$, i.e., $\{E_{\pi}, P_{\pi}\}$ is an element of $\mathscr{S}$.*

PROOF. Let a be a vertex of P_{π}. Then a is the intersection point of a 1-dimensional face γ of π and E_{π}. Let b be the element of N with $\mathbf{Z}_0 b = N \cap \gamma$. It is sufficient to show that $b = a$. The number $q = \langle m_{\pi}, b\rangle$ is a positive integer because m_{π} is in $M \cap (\operatorname{int} \pi^{\vee})$. Consider the hyperplane $\{b = 1\} = \{x \in M_{\mathbf{R}}; \langle x, b\rangle = 1\}$ in $M_{\mathbf{R}}$. Then the intersection $\{b = 1\} \cap (\operatorname{int} \pi^{\vee})$ contains $m_{\pi}/q + \pi^{\vee} \cap \gamma^{\perp}$. Since $\pi^{\vee} \cap \gamma^{\perp}$ is an $(r-1)$-dimensional convex cone in $\{b = 0\}$, its parallel translation $m_{\pi}/q + \pi^{\vee} \cap \gamma^{\perp} \subset \{b = 1\}$ contains an element of $M \cap \{b = 1\}$. Hence there exists an $m \in M$ in $\{b = 1\} \cap (\operatorname{int} \pi^{\vee})$. Since $m \in M \cap (\operatorname{int} \pi^{\vee}) = m_{\pi} + M \cap \pi^{\vee}$, we have $1 = \langle m, b\rangle \geqq \langle m_{\pi}, b\rangle = q > 0$. Hence the integer q is equal to 1, and b is the intersection point of γ and E_{π}. q.e.d.

By the definition of P_{π}, it is clear that $\pi = \mathbf{R}_0 P_{\pi}$. Hence this proposition shows that if $k[M \cap \pi^{\vee}]$ is a Gorenstein semigroup ring, then $\pi = \mathbf{R}_0 P$ for an element $\{E, P\}$ in $\mathscr{S}$. The following proposition shows that the converse is true.

PROPOSITION 7.4. *Let $\{E, P\}$ be an arbitrary element of $\mathscr{S}$. If we set $\pi = \mathbf{R}_0 P$, then $k[M \cap \pi^{\vee}]$ is Gorenstein and $\{E, P\} = \{E_{\pi}, P_{\pi}\}$.*

PROOF. Let $\{a_1, \cdots, a_s\}$ be the set of the vertices of the polytope P, and let m_0 be the primitive element of M with $E = \{m_0 = 1\}$. Then $\pi^{\vee} = \{x \in M_{\mathbf{R}}; \langle x, a_i\rangle \geqq 0, i = 1, \cdots, s\}$, and $\langle m_0, a_i\rangle = 1$ for every $i = 1, \cdots, s$. Hence m_0 is in $\operatorname{int} \pi^{\vee}$, and we have $m_0 + M \cap \pi^{\vee} \subset M \cap (\operatorname{int} \pi^{\vee})$. Let m be an element of $M \cap (\operatorname{int} \pi^{\vee})$. Then $\langle m, a_i\rangle$ is a positive integer for every i, hence we have $m - m_0 \in M \cap \pi^{\vee}$. Thus $m_0 + M \cap \pi^{\vee} = M \cap (\operatorname{int} \pi^{\vee})$, and $k[M \cap \pi^{\vee}]$ is Gorenstein by Proposition 7.1. By the uniqueness of m_{π}, we have $m_0 = m_{\pi}$, $E = E_{\pi}$ and $P = P_{\pi}$. q.e.d.

By Propositions 7.3 and 7.4, we have the following.

THEOREM 7.5. *The semigroup ring $k[M \cap \pi^{\vee}]$ is Gorenstein if and only if π is equal to $\mathbf{R}_0 P$ for an $\{E, P\}$ in $\mathscr{S}$. Furthermore, such an element $\{E, P\}$ in $\mathscr{S}$ is unique for π.*

Let $GL_Z(N)$ be the automorphism group of the free $\boldsymbol{Z}$-module N. Then each element g in $GL_Z(N)$ induces an automorphism $g^*: M \xrightarrow{\sim} M$ of the dual $\boldsymbol{Z}$-module M of N.

If there exists an isomorphism $\alpha: M \cap \pi_1^\vee \xrightarrow{\sim} M \cap \pi_2^\vee$ of semigroups, then α is extended to an automorphism $\tilde{\alpha}$ of M. Hence there exists $g \in GL_Z(N)$ with $g^* = \tilde{\alpha}$. Clearly we have $g(\pi_2) = \pi_1$. Hence there exists a natural one-to-one correspondence between the set of isomorphism classes of Gorenstein semigroup rings and the quotient of $\mathscr{G}$ by the following equivalence relation: $\{E, P\} \sim \{E', P'\}$ if and only if there exists $g \in GL_Z(N)$ with $g(E) = E'$ and $g(P) = P'$. Note that, by isomorphisms of semigroup rings, we always mean those induced by isomorphisms of semigroups.

We identify N with $\boldsymbol{Z}^r$ by a fixed isomorphism. We also identify M with $(\boldsymbol{Z}^r)^\vee$ by the dual isomorphism. Let $m_0 = (1, 0, \cdots, 0) \in (\boldsymbol{Z}^r)^\vee$. Then $E_0 = \{m_0 = 1\} = \{(1, c_2, \cdots, c_r); c_2, \cdots, c_r \in \boldsymbol{R}\}$ is a primitive hyperplane. We identify E_0 with $\boldsymbol{R}^{r-1}$ by the bijection defined by $(1, c_2, \cdots, c_r) \mapsto (c_2, \cdots, c_r) \in \boldsymbol{R}^{r-1}$.

DEFINITION 7.6. An affine transformation $f: \boldsymbol{R}^{r-1} \to \boldsymbol{R}^{r-1}$ is *$\boldsymbol{Z}$-rational* if it sends $\boldsymbol{Z}^{r-1}$ onto itself. Two polytopes P, P' in $\boldsymbol{R}^{r-1}$ with vertices in $\boldsymbol{Z}^{r-1}$ are *equivalent* if there exists a $\boldsymbol{Z}$-rational affine transformation $f: \boldsymbol{R}^{r-1} \to \boldsymbol{R}^{r-1}$ with $f(P) = P'$.

It is clear that, for every primitive hyperplane E, there exists g in $GL_Z(N)$ with $g(E) = E_0$, and that for every $\boldsymbol{Z}$-rational affine transformation $f: \boldsymbol{R}^{r-1} \to \boldsymbol{R}^{r-1}$, there exists $h \in GL_Z(N)$ with $h(E_0) = E_0$ and $h|_{E_0} = f$. Hence the quotient of $\mathscr{G}$ by $GL_Z(N)$ is in natural one-to-one correspondence with the set of equivalence classes of $(r-1)$-dimensional polytopes in $\boldsymbol{R}^{r-1}$ with vertices in $\boldsymbol{Z}^{r-1}$. Thus we get the following by Theorem 7.5.

THEOREM 7.7. *Let $\mathscr{G}_0$ be the set of $(r-1)$-dimensional polytopes in $\boldsymbol{R}^{r-1}$ with vertices in $\boldsymbol{Z}^{r-1}$. For a polytope P in $\mathscr{G}_0$ with vertices $\{a_1, \cdots, a_q\}$, let π be the convex cone in $\boldsymbol{R}^r = N_{\boldsymbol{R}}$ generated by $\{(1, a_1), \cdots, (1, a_q)\}$. Then $k[M \cap \pi^\vee]$ is a Gorenstein semigroup ring. Conversely, every Gorenstein semigroup ring is isomorphic to such a semigroup ring for a polytope P in $\mathscr{G}_0$. Let $k[M \cap \pi_1^\vee]$ and $k[M \cap \pi_2^\vee]$ be the Gorenstein semigroup rings associated to polytopes P_1 and P_2 in $\mathscr{G}_0$, respectively. Then they are isomorphic if and only if P_1 and P_2 are equivalent, i.e., there exists a $\boldsymbol{Z}$-rational affine transformation $f: \boldsymbol{R}^{r-1} \to \boldsymbol{R}^{r-1}$ with $f(P_1) = P_2$.*

EXAMPLE 7.8. When $r = 2$, then $\mathscr{G}_0$ is the set of intervals $[a, b]$

with $a, b \in \boldsymbol{Z}$ and $a < b$. By the above theorem, two intervals $[a_1, b_1]$ and $[a_2, b_2]$ define isomorphic semigroup rings if and only if their lengths $b_1 - a_1$ and $b_2 - a_2$ are equal. For the interval $[0, n]$ with a positive integer n, the associated cone π in $\boldsymbol{R}^2$ is generated by $(1, 0)$ and $(1, n)$. Hence the dual cone $\pi^\vee$ is $\{(x_1, x_2) \in \boldsymbol{R}^{\vee 2}; x_1 \geqq 0, x_1 + nx_2 \geqq 0\}$. (See Figure 1.) It is easy to see that the semigroup $M \cap \pi^\vee = (\boldsymbol{Z}^2)^\vee \cap \pi^\vee$ is generated

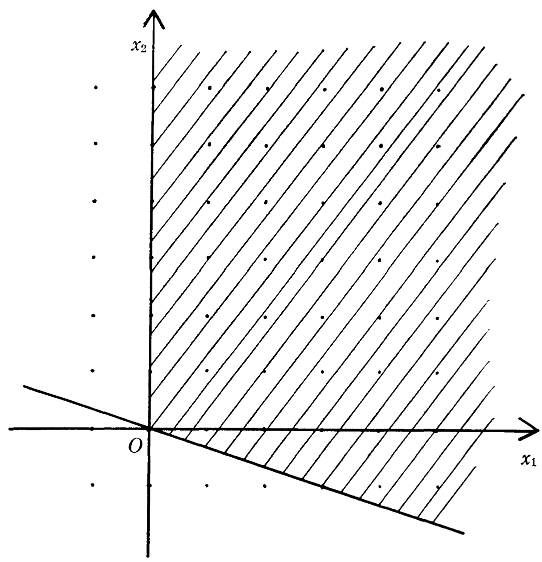

FIGURE 1. The region of $\pi^\vee$ when $n = 3$

by three elements $(0, 1)$, $(n, -1)$ and $(1, 0)$ if $n \geqq 2$, and it is generated by two elements $(0, 1)$ and $(1, -1)$ if $n = 1$. Set $x = e((0, 1))$, $y = e((n, -1))$ and $z = e((1, 0))$, with the notation in §3. Then $k[M \cap \pi^\vee]$ is equal to $k[x, y]$ if $n = 1$. It is equal to $k[x, y, z]/(xy - z^n)$ if $n \geqq 2$. Thus every 2-dimensional Gorenstein semigroup ring is isomorphic to one of these. In particular, it is a complete intersection. However, in dimension 3, most of Gorenstein semigroup rings are not complete intersections, as we see in the next section.

8. Complete intersections of dimension 3. According to Theorem 7.7, 3-dimensional Gorenstein semigroup rings are classified by convex polygons in $\boldsymbol{R}^2$ with vertices in $\boldsymbol{Z}^2$ modulo $\boldsymbol{Z}$-rational affine transformations. In this section, we prove a theorem (Theorem 8.1) which determines 3-dimensional semigroup rings which are complete intersections.

Recall that a noetherian local ring R is a *complete intersection* if there exists a regular local ring R' such that $R \simeq R'/(f_1, \cdots, f_q)$ for a set of elements $\{f_1, \cdots, f_q\} \subset R'$ with $q = \dim R' - \dim R$. A noetherian ring A is said to be a *local complete intersection* if, for every prime ideal $\mathfrak{p}$ of A, the localization $A_{\mathfrak{p}}$ is a complete intersection. We say a ring A of finite type over a field k is a *global complete intersection* if $A \simeq k[x_1, \cdots, x_n]/(f_1, \cdots, f_q)$ for a positive integer n and for some elements $f_1, \cdots, f_q \in k[x_1, \cdots, x_n]$ with $q = n - \dim A$.

It is well-known that if a noetherian ring A is a local complete intersection, then A is Gorenstein.

THEOREM 8.1. *Let P be a convex polygon in $\boldsymbol{R}^2$ with vertices in $\boldsymbol{Z}^2$. Then the associated 3-dimensional semigroup ring S is a local complete intersection if and only if P is equivalent to the polygon $P_{a,b,c}$ for some $a, b, c \in \boldsymbol{Z}$ with* (1) $a > 0$ *and* $b, c \geqq 0$, (2) $b \neq 0$ *or* $c \neq 0$ *and* (3) $c \geqq a$ *if* $b = 0$, *where $P_{a,b,c}$ is the convex hull of the set* $\{(0, 0), (a, 0), (0, c + ba), (a, c)\}$. *Furthermore, if P is equivalent to $P_{a,b,c}$, then S is isomorphic to $k[x, y, z, w, u]/(xz - w^b u^c, yw - u^a)$. In particular, S is a global complete intersection.* (*See Figure* 2.)

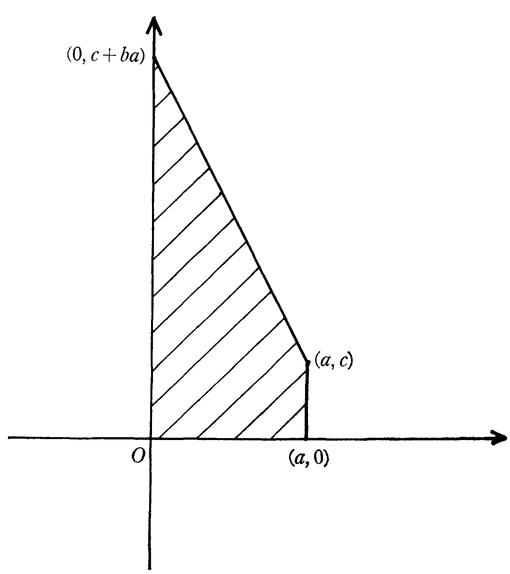

FIGURE 2. The polygon $P_{a,b,c}$

REMARK 8.2. In the above theorem, $P_{a,b,c}$ is a triangle if $c = 0$ and a quadrangle if $c > 0$. By conditions (1), (2) and (3), $P_{a,b,c}$ is not equivalent to $P_{a',b',c'}$ if $(a, b, c) \neq (a', b', c')$.

The rest of this section is devoted to the proof of Theorem 8.1.

PROPOSITION 8.3. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring, and let* $d = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. *If* R *is a complete intersection, then the inequality*

$$d - \dim R \geqq \binom{d+1}{2} - \dim_{R/\mathfrak{m}}(\mathfrak{m}^2/\mathfrak{m}^3)$$

holds.

PROOF. Let $(R', \mathfrak{n})$ be a regular local ring with $R \simeq R'/(f_1, \cdots, f_q)$ and $q = \dim R' - \dim R$. If f_i is not in $\mathfrak{n}^2$ for an i, then $R'/(f_i)$ is a regular local ring, and we can replace R' by $R'/(f_i)$. Hence we may assume $f_i \in \mathfrak{n}^2$ for every $i = 1, \cdots, q$. Then $\mathfrak{n}/\mathfrak{n}^2 \simeq \mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}^2/\mathfrak{m}^3 \simeq \mathfrak{n}^2/(\mathfrak{n}^3 + (f_1, \cdots, f_q))$. Thus we have

$$\begin{aligned}\dim_{R/\mathfrak{m}}(\mathfrak{m}^2/\mathfrak{m}^3) &\geqq \dim_{R'/\mathfrak{n}}(\mathfrak{n}^2/\mathfrak{n}^3) - q \\ &= \binom{d+1}{2} - (d - \dim R)\,.\end{aligned}$$

q.e.d.

COROLLARY 8.4. *Let* $(R, \mathfrak{m})$ *be a noetherian local ring, and let* $\{x_1, \cdots, x_d\}$ *be a minimal set of generators of the maximal ideal* $\mathfrak{m}$. *If* R *is a complete intersection, then the number of pairs* (i, j) *with* $1 \leqq i \leqq j \leqq d$ *and* $x_i x_j = 0$ *is at most* $d - \dim R$.

PROOF. Since $\{x_1, \cdots, x_d\}$ is a minimal set of generators of $\mathfrak{m}$, we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = d$ and $\mathfrak{m}^2$ is generated by the set $\{x_i x_j;\, x_i x_j \neq 0,\, 1 \leqq i \leqq j \leqq d\}$. The assertion follows from the inequality of Proposition 8.3.

DEFINITION 8.5. For a strongly convex rational polyhedral cone π and for a face σ of π, a subset A of $M \cap \pi^\vee \cap \sigma^\perp$ is said to be a *set of generators* of the semigroup $M \cap \pi^\vee \cap \sigma^\perp$ if $M \cap \pi^\vee \cap \sigma^\perp = \sum_{x \in A} \boldsymbol{Z}_0 x$, where we understand $\sum_{x \in A} \boldsymbol{Z}_0 x = \{0\}$ if $A = \varnothing$.

When $\dim \pi = r = \operatorname{rank} N$, we denote by $A_{\pi,\sigma}$ the set of *irreducible* elements in $M \cap \pi^\vee \cap \sigma^\perp$, where we say an element x in $M \cap \pi^\vee \cap \sigma^\perp$ is *reducible* if either $x = 0$ or there exist non-zero elements $x_1, x_2 \in M \cap \pi^\vee \cap \sigma^\perp$ with $x = x_1 + x_2$. It is easy to see that $A_{\pi,\tau} = A_{\pi,\sigma} \cap \tau^\perp$ for any faces σ, τ of π with $\tau \succ \sigma$.

LEMMA 8.6. *If* $\dim \pi = r$, *then* $A_{\pi,\sigma}$ *is a set of generators of* $M \cap \pi^\vee \cap \sigma^\perp$, *and every set of generators of* $M \cap \pi^\vee \cap \sigma^\perp$ *includes* $A_{\pi,\sigma}$, *i.e.*, $A_{\pi,\sigma}$ *is the smallest set of generators of* $M \cap \pi^\vee \cap \sigma^\perp$.

PROOF. Since $\dim \pi = r$, we can take an element a in $N \cap (\operatorname{int} \pi)$. Then $\langle x, a \rangle$ is a positive integer for every x in $M \cap \pi^{\vee} \backslash \{0\}$. Hence if $x \neq 0$ in $M \cap \pi^{\vee} \cap \sigma^{\perp}$ is reducible, then $x = x' + x''$ with $x', x'' \in M \cap \pi^{\vee} \cap \sigma^{\perp}$ and $\langle x', a \rangle$, $\langle x'', a \rangle < \langle x, a \rangle$. By induction on the number $\langle x, a \rangle$, we know that x is a finite sum of irreducible elements. Hence $A_{\pi,\sigma}$ is a set of generators. The rest of the assertion is clear since every irreducible element is contained in any set of generators. q.e.d.

Let P be a convex n-gon ($n \geqq 3$) with vertices $v_1, \cdots, v_n$ in $\boldsymbol{Z}^2$ in clockwise order and sides $s_1 = \overline{v_1 v_2}, \cdots, s_{n-1} = \overline{v_{n-1} v_n}$ and $s_n = \overline{v_n v_1}$. (See Figure 3.)

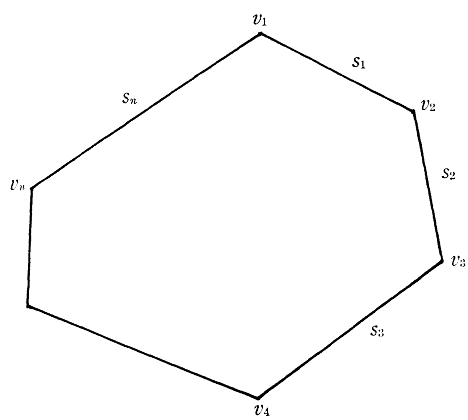

FIGURE 3

Then the associated cone π in $N_{\boldsymbol{R}} = \boldsymbol{R}^3$ is generated by n elements $(1, v_1), \cdots, (1, v_n)$ in $\boldsymbol{Z}^3$. Let γ_i and σ_i be the faces of π generated by $(1, v_i)$ and $\{1\} \times s_i \subset \boldsymbol{R}^3$, respectively, for every $i = 1, \cdots, n$. Then $\{\gamma_1, \cdots, \gamma_n\}$ and $\{\sigma_1, \cdots, \sigma_n\}$ are the set of 1- and 2-dimensional faces of π, respectively. Let $x_i \in M = (\boldsymbol{Z}^3)^{\vee}$ be the generator of the semigroup $M \cap \sigma_i^* = M \cap \pi^{\vee} \cap \sigma_i^{\perp}$ which is isomorphic to $\boldsymbol{Z}_0$. The two dimensional cone $\gamma_i^* = \pi^{\vee} \cap \gamma_i^{\perp}$ has 1-dimensional faces σ_{i-1}^* and σ_i^* for every $i = 1, \cdots, n$ with the convention $\sigma_0^* = \sigma_n^*$. Hence the smallest set of generators A_{π,γ_i} of the semigroup $M \cap \gamma_i^*$ contains x_{i-1} and x_i with the convention $x_0 = x_n$.

Let $A_{\pi,\gamma_i} = \{x_{i-1}, x_i, y_{i,1}, \cdots, y_{i,q_i}\}$ for every $i = 1, \cdots, n$.

PROPOSITION 8.7. *The set* $G = \{(1, 0, 0), x_1, \cdots, x_n, y_{i,j}; 1 \leqq i \leqq n, 1 \leqq j \leqq q_i\}$ *is a set of generators of* $M \cap \pi^{\vee}$, *and the smallest set of generators of* $M \cap \pi^{\vee}$ *is of the form* $A_{\pi,\{0\}} = G$ *or* $G \backslash \{(1, 0, 0)\}$.

PROOF. By Propositions 7.1 and 7.4, we have $M \cap (\operatorname{int} \pi^{\vee}) = (1, 0, 0) +$

$M \cap \pi^\vee$. Hence any $x \neq (1, 0, 0)$ in $M \cap (\operatorname{int} \pi^\vee)$ is reducible. On the other hand, any x in $M \cap (\pi^\vee \backslash \operatorname{int} \pi^\vee)$ is in $M \cap \pi^\vee \cap \gamma_i^\perp$ for an i, and we know $A_{\pi,\{0\}} \cap \gamma_i^\perp = A_{\pi,\gamma_i}$. Hence $A_{\pi,\{0\}} = G$ if $(1, 0, 0)$ is irreducible and $A_{\pi,\{0\}} = G \backslash \{(1, 0, 0)\}$ if not. q.e.d.

The ideal $\mathfrak{m}$ of $k[M \cap \pi^\vee]$ generated by $\{e(m)\}_{m \in G}$ is a maximal ideal of $k[M \cap \pi^\vee]$.

LEMMA 8.8. *If the localization $k[M \cap \pi^\vee]_\mathfrak{m}$ is a complete intersection, then the following inequality holds.*

$$(*) \qquad n(n-3)/2 + (n-2)\left(\sum_{i=1}^n q_i\right) + \sum_{i<i'} q_i q_{i'} \leqq n + \sum_{i=1}^n q_i - 2\,.$$

PROOF. The left hand side of this inequality is the number of pairs $\{m, m'\} \subset G \backslash \{(1, 0, 0)\}$ with $m + m' \in M \cap (\operatorname{int} \pi^\vee)$. Let R be the quotient of $k[M \cap \pi^\vee]_\mathfrak{m}$ by the ideal generated by $e((1, 0, 0))$. Then clearly R is also a complete intersection, and the maximal ideal $\bar{\mathfrak{m}}$ of R has a minimal set of generators $\{\bar{e}(m)\}_{m \in G \backslash \{(1,0,0)\}}$, where $\bar{e}(m)$ is the image of $e(m) \in k[M \cap \pi^\vee]$ in R. Since $M \cap (\operatorname{int} \pi^\vee) = (1, 0, 0) + M \cap \pi^\vee$, the condition $m + m' \in M \cap (\operatorname{int} \pi^\vee)$ implies $\bar{e}(m)\bar{e}(m') = 0$. Hence by Corollary 8.4, we have $n(n-3)/2 + (n-2)(\sum_{i=1}^n q_i) + \sum_{i<i'} q_i q_{i'} \leqq {}^\#(G \backslash \{(1, 0, 0)\}) - \dim R = n + \sum_{i=1}^n q_i - 2$. q.e.d.

PROPOSITION 8.9. *If $k[M \cap \pi^\vee]_\mathfrak{m}$ is a complete intersection, then by renumbering $v_1, \cdots, v_n$, if necessary, the possibilities are reduced to the following three cases.*

(1) $n = 4$ *and* $q_i = 0,\ i = 1, \cdots, 4$.

(2) $n = 3,\ q_1 = q_2 = 0$ *and* $q_3 \geqq 0$.

(3) $n = 3,\ q_1 = 0$ *and* $q_2 = q_3 = 1$.

PROOF. If $n \geqq 5$, then $n(n-3)/2 > n - 2$ and $(n-2)(\sum_{i=1}^n q_i) \geqq \sum_{i=1}^n q_i$. Hence the inequality $(*)$ does not hold. If $n = 4$, then $(*)$ becomes $2 + 2(\sum_{i=1}^4 q_i) + \sum_{i<i'} q_i q_{i'} \leqq 2 + \sum_{i=1}^4 q_i$. Clearly, this inequality holds if and only if $q_i = 0$ for every $i = 1, \cdots, 4$. If $n = 3$, then $\sum_{i=1}^3 q_i + \sum_{i<i'} q_i q_{i'} \leqq \sum_{i=1}^3 q_i + 1$. Hence we have $\sum_{i<i'} q_i q_{i'} \leqq 1$, and only the cases (2) and (3) are possible. q.e.d.

For an n-gon P with vertices $v_1, \cdots, v_n$, say clockwise, in this order in $\boldsymbol{Z}^2$, we denote by u_i the element in $\boldsymbol{Z}^2$ with $\boldsymbol{R}_0(v_{i+1} - v_i) \cap \boldsymbol{Z}^2 = \boldsymbol{Z}_0 u_i$ for every $i = 1, \cdots, n$ with the convention $v_{n+1} = v_1$. (See Figure 4.)

For a vertex v_i of an n-gon P, we denote by C_{P,v_i} the cone in $\boldsymbol{R}^2$ generated by $P - v_i$. Obviously, C_{P,v_i} is the cone generated by two elements u_i and $-u_{i-1}$ with the convention $u_0 = u_n$. Hence this is a strongly convex rational polyhedral cone in $\boldsymbol{R}^2$.

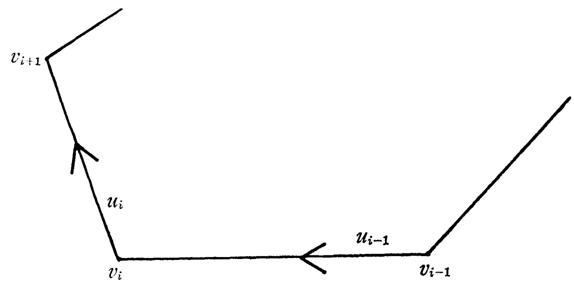

FIGURE 4

PROPOSITION 8.10. *The semigroups* $(\boldsymbol{Z}^2)^\vee \cap C^\vee_{P,v_i}$ *and* $M \cap \gamma_i^*$ *are isomorphic.*

PROOF. Recall that γ_i is a 1-dimensional face of π generated by $(1, v_i)$. Hence if we define a homomorphism $\phi: N \to \boldsymbol{Z}^2$ by $\phi(a_1, a_2, a_3) = (a_2 - a_1 v_i^{(1)}, a_3 - a_1 v_i^{(2)})$, where $v_i = (v_i^{(1)}, v_i^{(2)})$, then ϕ induces an isomorphism $N^{(\gamma_i)} \xrightarrow{\sim} \boldsymbol{Z}^2$ and $\pi^{(\gamma_i)} \xrightarrow{\sim} C_{P,v_i}$. (For the definitions of $N^{(\gamma_i)}$ and $\pi^{(\gamma_i)}$, see §1.) Hence $M^{(\gamma_i)}$ and $(\boldsymbol{Z}^2)^\vee$ are naturally isomorphic. Since it $M^{(\gamma_i)} \cap (\pi^{(\gamma_i)})^\vee = M \cap \pi^\vee \cap \gamma_i^\perp$ by Proposition 1.3, we are done. q.e.d.

Since the semigroup ring $k[M \cap \gamma_i^*]$ is generated over k by $q_i + 2$ elements, $q_i = 0$ implies $k[(\boldsymbol{Z}^2)^\vee \cap C^\vee_{P,v_i}] \cong k[x, y]$ and $q_i = 1$ implies $k[(\boldsymbol{Z}^2)^\vee \cap C^\vee_{P\ v_i}] \simeq k[x, y, z]/(xy - z^n)$ for an integer $n \geqq 2$ as we saw in Example 7.8.

LEMMA 8.11. *If* $q_i = 0$, *then the area of the parallelogram with sides* $\overline{O(-u_{i-1})}$ *and* $\overline{Ou_i}$ *is equal to* 1. *If* $q_i = 1$, *then the area of the parallelogram is an integer* $n \geqq 2$, *and the point* $(u_{i-1} + u_i)/n$ *is in* $\boldsymbol{Z}^2$.

PROOF. Since $k[(\boldsymbol{Z}^2)^\vee \cap C^\vee_{P\ v_i}]$ is Gorenstein as above, there exists, by Theorem 7.7, a linear transformation f; $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $f(\boldsymbol{Z}^2) = \boldsymbol{Z}^2$ which sends the cone C_{p,v_i} onto the cone generated by $\{(1, 0), (1, n)\}$ for a positive integer n. As we saw in Example 7.8, we have $q_i = 0$ if $n = 1$ and $q_i = 1$ if $n \geqq 2$. Since the assertions are invariant under linear transformations which send $\boldsymbol{Z}^2$ onto itself, we may assume $-u_{i-1} = (1, 0)$ and $u_i = (1, n)$. Then the assertion is obvious. q.e.d.

PROOF OF THEOREM 8.1. Assume P satisfies the condition (1) of Proposition 8.9. Then by Lemma 8.11, there exists a $\boldsymbol{Z}$-rational affine transformation which sends v_1, $-u_4$ and u_1 to $(0, 0)$, $(1, 0)$ and $(0, 1)$, respectively. Thus we may assume $v_1 = (0, 0)$, $v_2 = (0, t_1)$ and $v_4 = (s_1, 0)$ for positive integers t_1, s_1. Let $v_3 = (s_2, t_2)$ for positive integers s_2 and

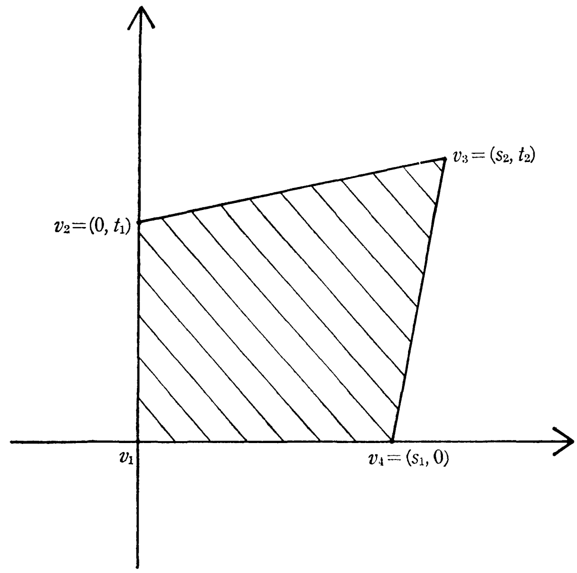

FIGURE 5

t_2. (See Figure 5.) Then since $u_4 = (-1, 0)$ and $u_1 = (0, 1)$, we know that $u_2 = (1, d_1)$ and $u_3 = (d_2, -1)$ for integers d_1, d_2 by Lemma 8.11 for $i = 2$ and 4. If $i = 3$ in Lemma 8.11, we have $1 - d_1d_2 = 1$. Thus we have $d_1 = 0$ or $d_2 = 0$, and $t_2 = t_1$ if $d_1 = 0$ and $s_2 = s_1$ if $d_2 = 0$. In both cases P is equivalent to $P_{a,b,c}$ for integers a, b, c with $a, c > 0$ and $b \geqq 0$. Assume P satisfies the condition (2) or (3) of Proposition 8.9. Similarly as in (1), we may assume $v_1 = (0, 0)$, $v_2 = (0, s)$ and $v_3 = (a, 0)$ for posi-

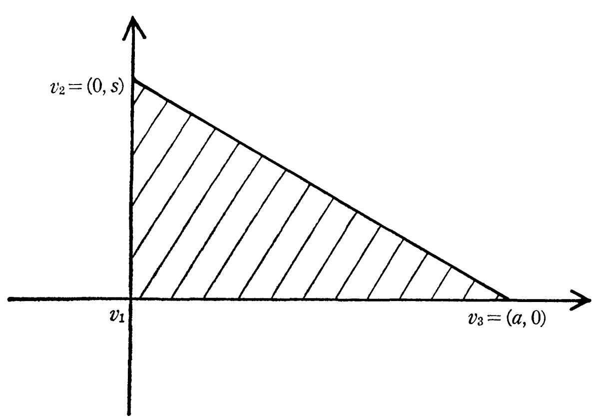

FIGURE 6

tive integers s, a. (See Figure 6.) Then we have $u_3 = (-1, 0)$, $u_1 = (0, 1)$ and $u_2 = (a', -s') = (a/d, -s/d)$, where d is the greatest common divisor of s and a. Now assume P satisfies (2). Then, by Lemma 8.11 for $i = 2$, we have $a' = 1$. Thus P is equivalent to $P_{a,b,0}$ for $b = s' > 0$. If P satisfies the condition (3), then by Lemma 8.11, we have $a', s' \geqq 2$, $(u_1 + u_2)/a' = (1, -(s'-1)/a') \in \boldsymbol{Z}^2$ and $(u_2 + u_3)/s' = ((a'-1)/s', -1) \in \boldsymbol{Z}^2$. Hence $(s'-1)/a'$ and $(a'-1)/s'$ are integers. This is impossible since $0 < (s'-1)(a'-1)/a's' < 1$. Hence the case (3) does not occur. Thus we know that P is equivalent to $P_{a,b,c}$ for integers a, b, c with $a > 0$, $b, c \geqq 0$ and $b \neq 0$ or $c \neq 0$ if the associated semigroup ring $k[M \cap \pi^\vee]$ is a local complete intersection. Hence it remains to show $k[M \cap \pi^\vee] \simeq k[x, y, z, w, u]/(xz - w^b u^c, yw - u^a)$ if $P = P_{a,b,c}$.

For the polygon $P_{a,b,c}$, the associated cone π is generated by $\{(1, 0, 0), (1, 0, c + ba), (1, a, c), (1, a, 0)\}$. Hence the dual cone $\pi^\vee$ is equal to

$$\{(t_1, t_2, t_3) \in (\boldsymbol{R}^3)^\vee;\ t_1 \geqq 0,\ t_1 + (c + ba)t_3 \geqq 0,\ t_1 + at_2 + ct_3 \geqq 0,\ t_1 + at_2 \geqq 0\} .$$

We define $m_0, \cdots, m_4 \in (\boldsymbol{Z}^3)^\vee (= M)$ by

$$m_0 = (1, 0, 0)\ ,\quad m_1 = (0, 0, 1)\ ,\quad m_2 = (0, 1, 0)\ ,$$
$$m_3 = (c + ba, -b, -1)\ ,\quad m_4 = (a, -1, 0)\ .$$

It is easy to check that $m_i \in M \cap \pi^\vee$ for every $i = 0, \cdots, 4$. We need the following.

LEMMA 8.12. *Every element m in $M \cap \pi^\vee$ is expressed uniquely in the form*

$$m = c_0 m_0 + \cdots + c_4 m_4$$

for non-negative integers $c_0, \cdots, c_4$ with $c_1 c_3 = 0$ and $c_2 c_4 = 0$.

PROOF. Let $C_1, \cdots, C_4$ be cones in $M_{\boldsymbol{R}}$ defined by

$$C_1 = \pi^\vee \cap \{t_3 \geqq 0\} \cap \{t_2 \geqq 0\}\ ,\quad C_2 = \pi^\vee \cap \{t_3 \leqq 0\} \cap \{t_2 - bt_3 \geqq 0\}\ ,$$
$$C_3 = \pi^\vee \cap \{t_3 \leqq 0\} \cap \{t_2 - bt_3 \leqq 0\}\ ,\quad C_4 = \pi^\vee \cap \{t_3 \geqq 0\} \cap \{t_2 \leqq 0\}\ .$$

Then, clearly, $\pi^\vee$ is the union of these four cones. Since $\pi^\vee$ is defined by four linear inequalities, each cone C_i is defined by six inequalities. However it is easy to see that three of them are implied by the others. Namely we have $C_1 = \{t_1 \geqq 0\} \cap \{t_2 \geqq 0\} \cap \{t_3 \geqq 0\}$, $C_2 = \{t_1 + (c + ba)t_3 \geqq 0\} \cap \{t_2 - bt_3 \geqq 0\} \cap \{t_3 \leqq 0\}$, $C_3 = \{t_1 + at_2 + ct_3 \geqq 0\} \cap \{t_2 - bt_3 \leqq 0\} \cap \{t_3 \leqq 0\}$, $C_4 = \{t_1 + at_2 \geqq 0\} \cap \{t_2 \leqq 0\} \cap \{t_3 \geqq 0\}$. We see easily that the cones C_1, C_2, C_3 and C_4 are generated by $\{m_0, m_1, m_2\}$, $\{m_0, m_2, m_3\}$, $\{m_0, m_3, m_4\}$ and $\{m_0, m_4, m_1\}$, respectively. Since, as we see easily, each of these generators forms a basis of $M(=(\boldsymbol{Z}^3)^\vee)$, we have $M \cap C_i = \boldsymbol{Z}_0 m_0 + \boldsymbol{Z}_0 m_i + \boldsymbol{Z}_0 m_{i+1}$ for every $i = 1, \cdots, 4$ with the convention $m_5 = m_1$. Since any m in $M \cap \pi^\vee$ is in $M \cap C_i$ for an i, we know that m is equal to $c_0 m_0 + \cdots + c_4 m_4$ for some non-negative integers $c_0, \cdots, c_4$ with $c_1 c_3 = c_2 c_4 = 0$. This 5-ple of integers is unique for m since $C_1, \cdots, C_4$ form a subdivision of $\pi^\vee$. q.e.d.

We define a homomorphism of k-algebras $\psi: k[x, y, z, w, u] \to k[M \cap \pi^\vee]$ by $\psi(x) = e(m_1)$, $\psi(y) = e(m_2)$, $\psi(z) = e(m_3)$, $\psi(w) = e(m_4)$ and $\psi(u) = e(m_0)$. By Lemma 8.12, ψ is surjective. The proof of Theorem 8.1 is complete if we show

$$(\sharp) \qquad \operatorname{Ker} \psi = (xz - w^b u^c, yw - u^a) .$$

It is easy to check that $I = (xz - w^b u^c, yw - u^a)$ is contained in $\operatorname{Ker} \psi$. We define a homomorphism $s: k[M \cap \pi^\vee] \to k[x, y, z, w, u]$ of k-vector spaces by $s(e(m)) = x^{c_1} y^{c_2} z^{c_3} w^{c_4} u^{c_0}$, where $\{c_0, \cdots, c_4\}$ is a 5-ple of non-negative integers with $c_1 c_3 = c_2 c_4 = 0$ and $m = c_0 m_0 + \cdots + c_4 m_4$ uniquely determined by m by Lemma 8.12. Clearly, s is a section of ψ. Hence in order to prove $(\sharp)$ it is sufficient to show that $k[x, y, z, w, u] = s(k[M \cap \pi^\vee]) + I$. Let $f = x^{p_1} y^{p_2} z^{p_3} w^{p_4} u^{p_0}$ be an arbitrary monomial in $k[x, y, z, w, u]$. We have to show that

$$(\sharp\sharp) \qquad f \text{ is in } s(k[M \cap \pi^\vee]) + I .$$

We first show $(\sharp\sharp)$ to be the case provided p_1 or p_3 is equal to 0. Indeed suppose f is not in $s(k[M \cap \pi^\vee]) + I$ with $p_2 p_4$ minimal. If $p_2 p_4 > 0$, then

$$x^{p_1} y^{p_2} z^{p_3} w^{p_4} u^{p_0} - x^{p_1} y^{p_2-1} z^{p_3} w^{p_4-1} u^{p_0+a} = x^{p_1} y^{p_2-1} z^{p_3} w^{p_4-1} u^{p_0}(yw - u^a) \in I .$$

Hence the monomial $x^{p_1} y^{p_2-1} z^{p_3} w^{p_4-1} u^{p_0+a}$ is not in $s(k[M \cap \pi^\vee]) + I$, a contradiction to the minimality of $p_2 p_4$. If $p_2 p_4 = 0$, then since $p_1 p_3 = 0$, we know $f = s(e(m))$ for an m in $M \cap \pi^\vee$ by Lemma 8.12, again a contradiction. Thus f is in $s(k[M \cap \pi^\vee]) + I$ if $p_1 p_3 = 0$. Let us prove $(\sharp\sharp)$ in the general case. Assume f has the minimal $p_1 p_3$ among the monomials which are not in $s(k[M \cap \pi^\vee]) + I$. Then clearly $p_1 p_3 > 0$, and

$$x^{p_1} y^{p_2} z^{p_3} w^{p_4} u^{p_0} - x^{p_1-1} y^{p_2} z^{p_3-1} w^{p_4+b} u^{p_0+c} = x^{p_1-1} y^{p_2} z^{p_3-1} w^{p_4} u^{p_0}(xz - w^b u^c) \in I .$$

Hence the monomial $x^{p_1-1} y^{p_2} z^{p_3-1} w^{p_4+b} u^{p_0+c}$ is not in $s(k[M \cap \pi^\vee]) + I$. This is impossible since we assume $p_1 p_3$ is minimal. Hence f is in $s(k[M \cap \pi^\vee]) + I$. q.e.d.

Appendix. Let M be a free $\boldsymbol{Z}$-module of finite rank.

DEFINITION 1. For an M-graded ring $A = \bigoplus_{m \in M} A_m$, we call a homogeneous ideal $\mathfrak{p}$ *M-maximal* if $\mathfrak{p}$ is maximal in the set of proper homogeneous ideals.

DEFINITION 2. We call an M-graded ring A an *M-field* if the ideal $\{0\}$ is M-maximal.

It is clear that an M-graded ring A is an M-field if and only if every non-zero homogeneous element of A is invertible. The following lemma can be proved as in the non-graded case.

LEMMA 3. *Let A be an M-field and let E be an M-graded A-module. Then E is a free A-module with a basis consisting of homogeneous elements.*

DEFINITION 4. We call an M-graded ring A *M-local* if A has only one M-maximal ideal.

It is obvious that every homogeneous quotient ring of an M-local ring is also an M-local ring.

EXAMPLE 5. If $M=\boldsymbol{Z}$ and $A=\bigoplus_{n=0}^{\infty} A_n$ is a $\boldsymbol{Z}$-graded ring with non-negative degrees such that A_0 is a field, then $A^+=\bigoplus_{n=1}^{\infty} A_n$ is the unique $\boldsymbol{Z}$-maximal ideal of A. Hence A is a $\boldsymbol{Z}$-local ring.

EXAMPLE 6. Let M be the dual of a free $\boldsymbol{Z}$-module N and let π be a strongly convex rational polyhedral cone of $N_{\boldsymbol{R}}$ (see §1). Then, for a field k, the semigroup ring $k[M\cap\pi^{\vee}]$ is an M-local ring with the maximal ideal $P(\pi)=\bigoplus_{m\in M\cap(\pi^{\vee}\setminus\pi^{\perp})} ke(m)$. Hence every homogeneous quotient ring of $k[M\cap\pi^{\vee}]$ is M-local, too.

The following lemma is an analogue of Nakayama's Lemma.

LEMMA 7. *Let A be an M-local ring with the M-maximal ideal $\mathfrak{p}$. Then for an M-graded A-module E of finite type, $\mathfrak{p}E=E$ implies $E=0$.*

PROOF. Let $\{e_1,\cdots,e_n\}$ be a set of homogeneous generators for E and let m_i be the degree of e_i for $i=1,\cdots,n$. Then $\mathfrak{p}E=E$ implies that $e_i=\sum_{j=1}^{n} a_{i,j}e_j$ for some homogeneous elements $\{a_{i,j}\}_{1\le i,j\le n}$ of $\mathfrak{p}$ with $\deg a_{i,j}=m_i-m_j$. Then $d=\det(\delta_{i,j}-a_{i,j})\in 1+\mathfrak{p}$ is a homogeneous element of degree 0 and $de_i=0$ for every i. Since A is M-local, d is invertible and we have $e_i=0$ for every i. q.e.d.

We have the following proposition as in the non-graded case.

PROPOSITION 8. *Let A be a noetherian M-local ring with the M-maximal ideal $\mathfrak{p}$ and let P be an M-graded projective A-module of finite type. Then P is a free A-module with a basis consisting of homogeneous elements.*

PROOF. The quotient $P/\mathfrak{p}P$ is a free $A/\mathfrak{p}$-module by Lemma 3. Let

r be its rank and let $m_1, \cdots, m_r$ be the degrees of elements of a basis. Then there exists a homomorphism $f: \bigoplus_{i=1}^r Au_i \to P$ of degree 0 which induces isomorphism $\bigoplus_{i=1}^r (A/\mathfrak{p})u_i \to P/\mathfrak{p}P$, where u_i is an indeterminate of degree m_i for every $i = 1, \cdots, r$. Thus we have $\mathfrak{p}\,\mathrm{Coker}\, f = \mathrm{Coker}\, f$ and f is surjective by Lemma 7. Since P is a projective A-module, $\mathrm{Tor}_1^A(A/\mathfrak{p}, P) = 0$ and we have $(\mathrm{Ker}\, f) \otimes_A A/\mathfrak{p} = 0$. Again by Lemma 7, $\mathrm{Ker}\, f = 0$ and f is an isomorphism. q.e.d.

Let Σ be a star closed subset of $\Gamma(\pi)$. Then the ring S_Σ defined in §3 is a homogeneous quotient ring of $k[M \cap \pi^\vee]$. Hence S_Σ is M-local and this proposition is applicable to $A = S_\Sigma$. For the case $\dim \pi = r = \mathrm{rank}\, N$ and $\Sigma = \Gamma(\pi)$, this proposition was proved by Kaneyama [K1, Theorem 3.5].

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