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UMBILICS OF CONFORMALLY FLAT SUBMANIFOLDS IN EUCLIDEAN SPACE

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0. Introduction. Let f be an isometric immersion of an n-dimensional conformally flat Riemannian manifold M, with $n \geqq 4$, into the $(n+p)$-dimensional Euclidean space E^{n+p}. We shall investigate the character of isometric immersions f for which all sectional curvatures of M are positive. In this paper we generalize results due to O'Neill [4]. In Theorem 1, we give a fairly complete description of the second fundamental form tensor of f when $p \leqq n-3$, which shows that each tangent space $T_x(M)$, $x \in M$, contains an umbilic space $\mathscr{U}_x$ of dimension $r \geqq n-p$ (that is, all directions in $\mathscr{U}_x$ have the same normal curvature). Theorem 2 asserts that the umbilic distribution $\mathscr{U}$ in some open set may be integrated to give submanifolds umbilic in M and in E^{n+p}.

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1. Notation and some formulas of Riemannian geometry. Let $f: M \to E^{n+p}$ be an isometric immersion of an n-dimensional conformally flat Riemannian manifold M, with $n \geqq 4$, into the $(n+p)$-dimensional Euclidean space E^{n+p}. For all local formulas and computation we may consider f as an imbedding and thus identify $x \in M$ with $f(x) \in E^{n+p}$. The tangent space $T_x(M)$ is identified with a subspace of $T_x(E^{n+p})$. The normal space $T_x^\perp$ is the subspace of $T_x(E^{n+p})$ consisting of all $X \in T_x(E^{n+p})$ which are orthogonal to $T_x(M)$ with respect to the Euclidean metric $\langle \cdot, \cdot \rangle$. Let ∇ (respectively $\tilde{\nabla}$) denote the covariant differentiation in M (respectively E^{n+p}) and let $\nabla^\perp$ denote the covariant differentiation in the normal bundle. We will refer to ∇ as the tangential connection and to $\nabla^\perp$ as the normal connection.

The *second fundamental form* α is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) ,$$

where X and Y are vector fields tangent to M. Let R be the Riemannian curvature tensor of M. We then have the Gauss equation:

(G1) $\langle\alpha(Y, Z), \alpha(X, W)\rangle - \langle\alpha(X, Z), \alpha(Y, W)\rangle = \langle R(X, Y)Z, W\rangle$

for all $X, Y, Z, W \in T_x(M)$. Let Q and k be the Ricci tensor of type (1, 1) and the scalar curvature of M, respectively, and ψ the tensor defined by

$$\psi(X, Y) = (1/(n-2))\{\langle QX, Y\rangle - (k/2(n-1))\langle X, Y\rangle\}$$

for $X, Y \in T_x(M)$. Since M is a conformally flat Riemannian manifold, the Gauss equation may be written as

(G2) $$\begin{aligned}\langle\alpha(Y, Z), \alpha(X, W)\rangle - \langle\alpha(X, Z), \alpha(Y, W)\rangle \\ = \psi(Y, Z)\langle X, W\rangle - \psi(X, Z)\langle Y, W\rangle \\ + \langle Y, Z\rangle\psi(X, W) - \langle X, Z\rangle\psi(Y, W)\end{aligned}$$

for $X, Y, Z, W \in T_x(M)$. We define the *difference function* Δ of α by

$$\Delta(X, Y) = \Delta(\pi) = \{\langle\alpha(X, X), \alpha(Y, Y)\rangle - \|\alpha(X, Y)\|^2\}/\|X \wedge Y\|^2 ,$$

where X and Y are linearly independent vectors in a plane π tangent to M at x, and $\|X \wedge Y\|$ is the area of the parallelogram spanned by X and Y. The difference function is given by the Gauss equation:

(G3) $$\begin{aligned}\|X \wedge Y\|^2\Delta(X, Y) &= \langle\alpha(X, X), \alpha(Y, Y)\rangle - \|\alpha(X, Y)\|^2 \\ &= \psi(X, X)\|Y\|^2 + \psi(Y, Y)\|X\|^2 - 2\psi(X, Y)\langle X, Y\rangle ,\end{aligned}$$

where X and Y span a plane tangent to M at x.

For the second fundamental form α we define the covariant derivative, denoted by $\nabla^*_X\alpha$, to be

$$(\nabla^*_X\alpha)(Y, Z) = \nabla^\perp_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) ,$$

where X, Y, Z are vector fields of M. Then the Codazzi equation is

(C1) $$(\nabla^*_X\alpha)(Y, Z) = (\nabla^*_Y\alpha)(X, Z)$$

for all $X, Y, Z \in T_x(M)$. Let $\xi_1, \cdots, \xi_p$ be orthonormal normal vectors at x. Extend ξ_k's $(1 \leqq k \leqq p)$ to orthonormal normal vector fields defined in a neighborhood of x and define $-A_kX$ to be the tangential component of $\tilde{\nabla}_X\xi_k$ for $X \in T_x(M)$. A_kX depends only on ξ_k at x and X. We call the A_k's the *second fundamental forms associated with* $\xi_1, \cdots, \xi_p$. If $\xi_1, \cdots, \xi_p$ are orthonormal normal vector fields in a neighborhood of x, they determine normal connection forms $s_{kl}(1 \leqq k, l \leqq p)$, in a neighborhood of x, by

$$\nabla^\perp_X\xi_k = \sum_{l=1}^{p} s_{kl}(X)\xi_l$$

for X tangent to M. s_{kl}'s are skew-symmetric with respect to indices k

and l. The Codazzi equation (C1) is also expressed as

(C2) $(\nabla_X A_k)Y - \sum_l s_{kl}(X)A_l Y = (\nabla_Y A_k)X - \sum_l s_{kl}(Y)A_l X \quad (1 \leqq k \leqq p)$

for X and Y tangent to M.

2. The second fundamental form at one point. The following useful result is due to Chern and Kuiper [1].

LEMMA 1. *Let $\mathscr{N}_x$ be the subspace $\{X \in T_x(M) \mid \alpha(X, Y) = 0$ for all $Y \in T_x(M)\}$ of $T_x(M)$. If $\Delta = 0$, then there exists a vector $Z \in \mathscr{N}_x^{\perp}$ such that $\alpha(Z, \cdot)$ is one-to-one from $\mathscr{N}_x^{\perp}$ to $T_x^{\perp}$. Hence* $\dim \mathscr{N}_x \geqq n - p$.

We now define a subspace $\mathscr{U}_x$ of $T_x(M)$ to be *umbilic* relative to α provided $\dim \mathscr{U}_x \geqq 2$ and $\alpha(X, X)$ is constant for all unit vectors X in $\mathscr{U}_x$. If the whole space $T_x(M)$ is umbilic relative to α, we say that the point x is *umbilic*. It is easy to see that if $\mathscr{U}_x$ is an umbilic subspace, then $\alpha(X, Y) = 0$ for any two orthogonal vectors X, Y in $\mathscr{U}_x$, and thus the difference function Δ is constant and non-negative on planes in $\mathscr{U}_x$. Recall that a non-zero vector $X \in T_x(M)$ is *asymptotic* provided $\alpha(X, X) = 0$. As is well known, if all sectional curvatures of M are positive, then α has no asymptotic vector.

Denote by h the real-valued function

$$h: X \to \|\alpha(X, X)\|^2 - 2\psi(X, X)$$

on the unit sphere in $T_x(M)$.

LEMMA 2. *If U is a critical point of the function h, then we have*

$$\langle \alpha(U, U), \alpha(U, X) \rangle - \psi(U, X) = 0$$

for all vectors $X \in T_x(M)$ orthogonal to U.

PROOF. Let Y be the curve in the unit sphere in $T_x(M)$ such that $Y(t) = \cos t U + \sin t X$. Then we have $(d/dt)h(Y(t))|_{t=0} = 4(\langle \alpha(U, U), \alpha(U, X) \rangle - \psi(U, X))$. Since U is a critical point of h, our assertion is proved.

LEMMA 3. *If U is a minimam point of h, and X is a unit vector orthogonal to U. Then we have*

$$3\|\alpha(U, X)\|^2 \geqq h(U) .$$

PROOF. For the curve Y as above, we have $(d^2/dt^2)h(Y(t))|_{t=0} = 4(3\|\alpha(U, X)\|^2 - h(U))$ by using (G3). Now U is a minimum point of h, hence we have the desired inequality.

LEMMA 4. *Let U be a critical point of h. Suppose that the sub-*

space $\mathscr{C} = \operatorname{Ker} \alpha(U, \cdot) \cap U^{\perp}$ *of* $T_x(M)$ *has dimension at least two. Let* γ *be the (umbilic) symmetric bilinear function such that* $\gamma(X, X) = \alpha(U, U)$ *for all unit vectors* X *in* $T_x(M)$. *Then*

(1) *the symmetric bilinear function* $\alpha^* = \alpha - \gamma$ *on* $\mathscr{C}$ *has its values in* $\alpha(U, \mathscr{P})^{\perp} \subset T_x^{\perp}$, *where* $\mathscr{P} = U^{\perp} \cap \mathscr{C}^{\perp}$, *and*

(2) *the difference function* Δ^* *of* α^* *on* $\mathscr{C}$ *has the constant value* $h(U)$.

PROOF. (1) It suffices to prove that if X and Y are two orthogonal unit vectors in $\mathscr{C}$, and $Z \in \mathscr{P}$, then $\alpha(U, Z)$ is orthogonal to both $\alpha^*(X, X)$ and $\alpha^*(X, Y)$. By (G2) and Lemma 2, we have $\langle \alpha^*(X, X), \alpha(U, Z) \rangle = 0$. Since $\mathscr{C}$ is umbilic relative to γ, we have $\alpha^*(X, Y) = \alpha(X, Y)$. Then $\langle \alpha^*(X, Y), \alpha(U, Z) \rangle = 0$ follows from (G2).

(2) For X and Y as above, $\alpha(U, X) = \alpha(U, Y) = 0$. Making use of (G3), we have

$$\Delta^*(X, Y) = \langle \alpha^*(X, X), \alpha^*(Y, Y) \rangle - \|\alpha^*(X, Y)\|^2 = \Delta(X, Y) - \Delta(U, X) - \Delta(U, Y) + \|\alpha(U, U)\|^2 = \|\alpha(U, U)\|^2 - 2\psi(U, U) = h(U) .$$

LEMMA 5. *Let* U *be a minimum point of the function* h. *Then*

(1) *we have* $\psi(U, U) \geqq \psi(X, X)$ *for any unit vector* $X \in T_x(M)$ *such that* $\|\alpha(U, U)\| = \|\alpha(X, X)\|$, *and*

(2) *if* $p \leqq n - 3$, *we have* $\psi(U, U) = \psi(X, X)$ *for any unit vector* $X \in \mathscr{C}$ *such that* $\alpha(U, U) = \alpha(X, X)$.

PROOF. (1) Since U is a minimum point of h, we have $\|\alpha(U, U)\|^2 - 2\psi(U, U) \leqq \|\alpha(X, X)\|^2 - 2\psi(X, X)$ for $X \in T_x(M)$. Thus $\psi(U, U) \geqq \psi(X, X)$ follows from $\|\alpha(U, U)\| = \|\alpha(X, X)\|$.

(2) Let X be a unit vector in $\mathscr{C}$. The condition $p \leqq n - 3$ implies $\dim \operatorname{Ker} \alpha(X, \cdot) \geqq 3$. Hence there exists a unit vector $Y \in \operatorname{Ker} \alpha(X, \cdot)$ orthogonal to both U and X. Using the assumption $\alpha(U, U) = \alpha(X, X)$ and (G3), we have $\psi(X, X) + \psi(Y, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle = \langle \alpha(U, U), \alpha(Y, Y) \rangle = \|\alpha(U, Y)\|^2 + \psi(U, U) + \psi(Y, Y)$. Hence $0 \leqq \|\alpha(U, Y)\|^2 = \psi(X, X) - \psi(U, U)$. But $\psi(X, X) - \psi(U, U) \leqq 0$ by (1) above. Thus $\psi(X, X) = \psi(U, U)$.

LEMMA 6. *Suppose that* M *has positive sectional curvatures and that* $p \leqq n - 3$. *If* U *is a minimum point of* h, *then* $h(U) = 0$. *Furthermore,* $\alpha(U, \cdot)$ *is zero on the orthogonal complement of* U *in* $T_x(M)$.

PROOF. Since all sectional curvatures of M are positive, α has no asymptotic vector. The condition $p \leqq n - 3$ implies $\dim \operatorname{Ker} \alpha(U, \cdot) \geqq 3$. Hence we have $\dim \mathscr{C} \geqq 2$ and there exists a unit vector X orthogonal to U and such that $\alpha(U, X) = 0$. Using Lemma 3 we find that

$h(U) \leqq 0$. Hence, by Lemma 4, α^* on $\mathscr{Q}$ has $\Delta^* \leqq 0$. Now by Lemma 4 again, the values of α^* on $\mathscr{Q}$ lie in $\alpha(U, \mathscr{P})^\perp \subset T_x^\perp$. Since $\alpha(U, \cdot)$ is one-to-one on $\mathscr{P}$ and $T_x(M) = \mathscr{Q} + [U] + \mathscr{P}$, we have $n^* = \dim \mathscr{Q} > p - \dim \alpha(U, \mathscr{P}) = \dim \alpha(U, \mathscr{P})^\perp = p^*$. We may now apply the last assertion in Lemma 3 of O'Neill [4] to α^*, concluding that α^* has an asymptotic vector $Y \in \mathscr{Q}$. Thus $\alpha(Y, Y) = \alpha(U, U)$, and since $\alpha(U, Y) = 0$, the equation (G3) implies $\|\alpha(U, U)\|^2 - \psi(U, U) - \psi(Y, Y) = 0$. But we have $\psi(Y, Y) = \psi(U, U)$ by (2) of Lemma 5. Hence $h(U) = 0$.

We now show that $\alpha(U, U^\perp) = 0$, or equivalently that $\mathscr{P} = U^\perp \cap \mathscr{Q}^\perp$ is zero. Assume that there is a unit vector Z in $\mathscr{P}$, we shall derive a contradiction by a dimension argument. Let $\mathscr{E} = \{E \in \mathscr{Q} \mid \alpha^*(E, \cdot) = 0\}$. Note that $\Delta^* = h(U) = 0$ on $\mathscr{Q}$.

Case 1. $\mathscr{E} \neq \mathscr{Q}$. Then there exists, by Lemma 1, a unit vector W in $\mathscr{Q} \cap \mathscr{E}^\perp$ such that $\alpha^*(W, \cdot)$ is one-to-one on $\mathscr{Q} \cap \mathscr{E}^\perp$. Let $\mathscr{F}$ consist of the vectors in $\mathscr{Q}$ orthogonal to W and to $\mathscr{E}$. Since W is orthogonal to $\mathscr{E} + \mathscr{F}$, we have $\alpha(W, \cdot) = \alpha^*(W, \cdot)$ on $\mathscr{E} + \mathscr{F}$. Hence $\alpha(W, \cdot)$ is one-to-one on $\mathscr{F}$, and $\alpha(W, \mathscr{E}) = 0$. We have expressed $T_x(M)$ now as a sum of mutually orthogonal subspaces, thus: $T_x(M) = [U] + \mathscr{P} + (\mathscr{E} + \mathscr{F} + [W])$. Since $\alpha(U, \mathscr{E} + \mathscr{F}) = 0$ and $\alpha(W, \mathscr{E}) = 0$, we see from (G2) that subspaces $\alpha(U, \mathscr{P})$, $\alpha(W, \mathscr{F})$, $\alpha(Z, \mathscr{E})$ of $T_x^\perp$ are mutually orthogonal. Thus, counting dimensions, we find $p \geqq \dim \alpha(U, \mathscr{P}) + \dim \alpha(W, \mathscr{F}) + \dim \alpha(Z, \mathscr{E}) = \dim \mathscr{P} + \dim \mathscr{F} + \dim \alpha(Z, \mathscr{E})$, since $\alpha(U, \cdot)$ is one-to-one on $\mathscr{P}$ and $\alpha(W, \cdot)$ is one-to-one on $\mathscr{F}$. By the decomposition of $T_x(M)$ given above, and the fact that $p \leqq n - 3$, we conclude that $\dim \alpha(Z, \mathscr{E}) < \dim \mathscr{E}$. Let $E \in \mathscr{E}$ be a unit vector such that $\alpha(Z, E) = 0$. Using (G3) and $\alpha(E, E) = \alpha(U, U)$ which follows from $\alpha^*(E, E) = 0$, we have $\psi(E, E) + \psi(Z, Z) = \langle \alpha(E, E), \alpha(Z, Z) \rangle = \langle \alpha(U, U), \alpha(Z, Z) \rangle = \|\alpha(U, Z)\|^2 + \psi(U, U) + \psi(Z, Z)$. Hence we have $\|\alpha(U, Z)\|^2 = \psi(E, E) - \psi(U, U)$. Since $\psi(E, E) - \psi(U, U) = 0$ by (2) of Lemma 5, we have $\alpha(U, Z) = 0$. This is a contradiction.

Case 2. $\mathscr{E} = \mathscr{Q}$. Here the proof by contradiction is a simplification of the argument above, based on the orthogonal decomposition $[U] + \mathscr{P} + \mathscr{E}$ of $T_x(M)$, and the mutually orthonormal subspaces $\alpha(U, \mathscr{P})$, $\alpha(Z, \mathscr{E})$ in $T_x^\perp$.

Reduction to the flat case is completed by

LEMMA 7. *Suppose that M has positive sectional curvatures and $p \leqq n - 3$. Let U be a minimum point of h, with γ defined as in Lemma 4. Then the symmetric bilinear function α^* from $T_x(M) \times T_x(M)$ to $T_x^\perp$ has $\Delta^* = 0$.*

PROOF. By the preceding lemma, $T_x(M)$ is spanned by U and the nullspace $\mathscr{E}$ of $\alpha(U, \cdot)$. Furthermore, U is non-zero and orthogonal to $\mathscr{E}$. Since $h(U) = 0$, assertion (2) of Lemma 4 implies that Δ^* is zero on planes in $\mathscr{E}$. Since $\alpha^*(U, U) = 0$ and $\alpha^*(U, \mathscr{E}) = 0$, it follows easily that $\Delta^* = 0$ on all planes in $T_x(M)$.

We can now give the main result of this section.

THEOREM 1. *For $n \geqq 4$, let f be an isometric immersion of an n-dimensional conformally flat Riemannian manifold M of positive sectional curvatures into the $(n + p)$-dimensional Euclidean space, and $p \leqq n - 3$. Let α be the second fundamental form of the immersion f, and $\mathscr{U}_x$, $x \in M$, be the set of all vectors U in $T_x(M)$, such that $||\alpha(U, U)||^2 - 2||U||^2\psi(U, U) = 0$. Then $\mathscr{U}_x$ is the largest umbilic subspace of $T_x(M)$ relative to α, and has dimension $r \geqq n - p$. Furthermore, if V is a vector in $\mathscr{U}_x$, then*

$$\alpha(V, X) = \langle V, X\rangle\alpha(U, U) \quad \text{for any} \quad X \in T_x(M) ,$$

where U is a unit vector in $\mathscr{U}_x$ ($\alpha(U, U)$ is independent of the choice of U in $\mathscr{U}_x$).

PROOF. By Lemma 6, the set of unit vectors in $\mathscr{U}_x$ is precisely the set $h^{-1}(0)$ at which the function h takes its minimum value. For one such unit vector U, let $\mathscr{N}_x^*$ be the subspace of $T_x(M)$ consisting of all $X \in T_x(M)$ such that $\alpha^*(X, \cdot) = 0$, where as usual, $\alpha^* = \alpha - \gamma$. Since $\Delta^* = 0$, it follows from Lemma 1 that $\dim \mathscr{N}_x^* \geqq n - p$. We shall show that $\mathscr{U}_x = \mathscr{N}_x^*$. If X is a unit vector in $\mathscr{N}_x^*$, then $0 = \alpha^*(X, X) = \alpha(X, X) - \alpha(U, U)$, so $\mathscr{N}_x^*$ is umbilic relative to α. Since $\psi(X, X) = \psi(U, U)$ by the assertion (2) of Lemma 5, $||\alpha(X, X)||^2 - 2\psi(X, X) = ||\alpha(U, U)||^2 - 2\psi(U, U) = 0$. Hence X is in $\mathscr{U}_x$. Thus $\mathscr{N}_x^* \subset \mathscr{U}_x$.

Now assume that there exists a unit vector V in $\mathscr{U}_x$ which is not in $\mathscr{N}_x^*$. Without loss of generality we may suppose that V is orthogonal to U. (In fact, we can write $V = cU + sX$, where X is a unit vector orthogonal to U, and $c^2 + s^2 = 1$. Since $\alpha(U, X) = 0$ and $\psi(U, X) = 0$, we have $0 = ||\alpha(V, V)||^2 - 2\psi(V, V) = c^2||\alpha(U, U)||^2 + s^2||\alpha(X, X)||^2 - c^2s^2h(X) - 2(c^2\psi(U, U) + s^2\psi(X, X)) = s^4h(X)$. This shows that X is in $\mathscr{U}_x$. But U is evidently in $\mathscr{N}_x^*$, hence $V \notin \mathscr{N}_x^*$ implies $X \notin \mathscr{N}_x^*$). Thus V is in $\mathscr{E}$ which implies $\alpha(U, V) = 0$. By this and (G3), Schwartz's inequality $\langle\alpha(U, U), \alpha(V, V)\rangle^2 \leqq ||\alpha(U, U)||^2||\alpha(V, V)||^2$ is reduced $(\psi(U, U) - \psi(V, V))^2 \leqq 0$. Hence, $\psi(U, U) = \psi(V, V)$. Since U and V are unit vectors in $\mathscr{U}_x$, the vectors $\alpha(U, U)$ and $\alpha(V, V)$ have the same norm. Now $\alpha^*(V, \cdot) \neq 0$ since $V \notin \mathscr{N}_x^*$. Then $\Delta^* = 0$

implies $\alpha^*(V, V) \neq 0$, that is, $\alpha(U, U) \neq \alpha(V, V)$. Thus $\langle \alpha(U, U) - \alpha(V, V), \alpha(U, U) \rangle \gtrless 0$. But, by (G3), $\alpha(U, V) = 0$ and $\psi(U, U) = \psi(V, V)$, we see that $||\alpha(U, U)||^2 - \langle \alpha(U, U), \alpha(V, V) \rangle = 0$. Thus we reach a contradiction. This completes the proof that $\mathscr{U}_x = \mathscr{N}_x^*$. So, by the remark above, $\mathscr{U}_x$ is an umbilic subspace relative to α and has dimension $r \geqq n - p$.

If $\mathscr{Y}_x$ is any umbilic subspace, we have $\alpha(X, X) = \alpha(Y, Y)$ and $\alpha(X, Y) = 0$ for any orthogonal unit vectors X and Y in $\mathscr{Y}_x$. Then, by (G3), $||\alpha(X, X)||^2 = \psi(X, X) + \psi(Y, Y)$. Since $p \leqq n - 3$, as in the proof of (2) of Lemma 5, there exists a unit vector $Z \in T_x(M)$ such that $\alpha(X, Z) = 0$. Hence $\psi(X, X) - \psi(Y, Y) = ||\alpha(Y, Z)||^2 \geqq 0$. By the symmetry in X and Y, we have $\psi(X, X) = \psi(Y, Y)$. Thus we see that $||\alpha(X, X)||^2 - 2\psi(X, X) = 0$. Then, by the definition of $\mathscr{U}_x$, $\mathscr{Y}_x$ is a subspace of $\mathscr{U}_x$.

It remains to prove the final assertion of the theorem. If $X \in T_x(M)$ is orthogonal to $V \in \mathscr{U}_x$, the assertion follows immediately from Lemma 6. Let U, V be unit vectors in $\mathscr{U}_x$ and $X = aV$ $(a \in \boldsymbol{R})$. Then $\alpha(V, X) = a\alpha(V, V) = a\alpha(U, U)$, and $\langle V, X \rangle \alpha(U, U) = a\langle V, V \rangle \alpha(U, U) = a\alpha(U, U)$. Thus the assertion holds, and the proof is complete.

REMARK. The lower bound $n - p$ of the largest umbilic space $\mathscr{U}_x$ was obtained independently by Moore [3].

COROLLARY. *Under the assumptions of Theorem* 1, *we have*

$$\psi(V, X) = (\lambda^2/2)\langle V, X \rangle ,$$

where V *is a vector in* $\mathscr{U}_x$, X *is a vector in* $T_x(M)$ *and* λ *is the length of* $\alpha(U, U)$ *for any unit vector in* $\mathscr{U}_x$.

PROOF. In the case of $X = V \in \mathscr{U}_x$ and $||V|| = 1$, we have

$$\psi(V, V) = ||\alpha(V, V)||^2/2 = \lambda^2/2 .$$

If X and V are orthogonal, then $\alpha(V, X) = 0$. Hence, by Lemma 2, $\psi(V, X) = 0$. These imply the assertion.

3. Local properties. We assume throughout that $f: M \to E^{n+p}$ is an isometric immersion of an n-dimensional conformally flat Riemannian manifold into an $(n + p)$-dimensional Euclidean space such that $n \geqq 4$, $p \leqq n - 3$ and that the sectional curvatures of M are positive. Denote by $Z(x)$ the common value of the normal curvature vectors $\alpha(U, U)$ for all unit vectors U in $\mathscr{U}_x$. We call Z the *normal curvature vector field* of f. Let $\rho(x)$ denote the dimension of $\mathscr{U}_x$ and call it the umbilic index of f at x.

By Corollary of Theorem 1, any vector in $\mathscr{U}_x$ is a proper vector of the Ricci tensor Q corresponding to the proper value $(n-2)\lambda^2/2 + k/2(n-1) = \mu$, say. Let $\mathscr{E}_x(\mu)$ be the proper space of the Ricci tensor Q corresponding to this proper value μ. We denote by $\mathscr{W}_x$ the subspace of $T_x^\perp$ generated by the vectors $\alpha(X(x), Y(x))$ for all vectors $X(x)$, $Y(x)$ in $\mathscr{E}_x(\mu)$. Let $\mathscr{O}$ be an open set in M on which the umbilic index takes a constant value and the multiplicity of each proper value of the Ricci tensor is constant (in the argument below, we only need the constancy of the multiplicity of the proper value μ of the Ricci tensor). Furthermore we assume that the field of space $\mathscr{W}$ has constant dimension on $\mathscr{O}$.

LEMMA 8. *The normal curvature vector field Z is differentiable on $\mathscr{O}$.*

PROOF. Let $A(X(x)) = \alpha(X(x), X(x))$ for all unit vectors $X(x)$ in $T_x(M)$, $x \in M$. By Theorem 1 and (G3), $\langle A(X(x)), Z(x)\rangle = \lambda^2(x)$ for all unit vectors $X(x)$ in $\mathscr{E}_x(\mu) \subset T_x(M)$, $x \in \mathscr{O}$. Let $\mathscr{F}$ be the plane in $T_x^\perp \subset E^p$ through the end points of $A(X(x))$ for $X(x) \in \mathscr{E}_x(\mu)$. Choose unit vectors $X_1(x), \cdots, X_q(x)$ in $\mathscr{E}_x(\mu)$ such that the vectors $A(X_1(x)), \cdots, A(X_q(x))$ are affinely independent and determine $\mathscr{F}$. We assert that $\mathscr{F}$ does not contain the zero vector. In fact, if $F(x) \in \mathscr{F}$, we can write $F(x) = \sum f_i(x)A(X_i(x))$ with $\sum f_i = 1$. By Theorem 1 and (G3), $\langle Z(x), F(x)\rangle = (\sum f_i(x))\lambda^2(x) = \lambda^2(x) > 0$, hence $F(x) \neq 0$. Thus the vectors $A(X_1(x)), \cdots, A(X_q(x))$ are linearly independent. Furthermore, they are a basis for the space $\mathscr{W}_x$. For, $\mathscr{W}_x$ is spanned by the set $\{A(X(x)) | X(x) \in \mathscr{E}_x(\mu)\}$ since $\alpha(X(x), Y(x))$ is a linear combination of $A(X(x))$, $A(Y(x))$ and $A(X(x) + Y(x))$.

By hypothesis the field of proper space $\mathscr{E}(\mu)$ of the Ricci tensor Q is differentiable on $\mathscr{O}$. Thus we can extend $X_i(x)$'s $(1 \leqq i \leqq q)$ differentiably to vector fields X_i's $(1 \leqq i \leqq q)$ on $\mathscr{O}$ such that they are in $\mathscr{E}(\mu)$ at each point of $\mathscr{O}$. If $\mathscr{O}$ is small enough, the vector fields $A(X_i)$'s $(1 \leqq i \leqq q)$ remain affinely independent and linearly independent at each point of $\mathscr{O}$. These vector fields are in $\mathscr{W}$ at each point of $\mathscr{O}$. By hypothesis we may suppose that $\dim \mathscr{W}$ is constant on $\mathscr{O}$. Thus the vector fields $A(X_i)$'s $(1 \leqq i \leqq q)$ form a basis for $\mathscr{W}$ at each point of $\mathscr{O}$. Since $Z = \alpha(U, U)$ $(U \in \mathscr{U} \subset \mathscr{E}(\mu))$ is in $\mathscr{W}$ at each point of $\mathscr{O}$, we have a unique expression $Z = \sum z_i A(X_i)$ for Z on $\mathscr{O}$. Since $\langle Z, A(X_j)\rangle = \lambda^2$ $(1 \leqq j \leqq q)$, we obtain $\sum z_i \langle A(X_i), A(X_j)\rangle = \lambda^2$ for each index j $(1 \leqq j \leqq q)$. By hypothesis the proper value μ of the Ricci tensor Q is differentiable on $\mathscr{O}$. So the function $\lambda^2 = (2\mu - k/(n-1))/(n-2)$ is differentiable on $\mathscr{O}$. Furthermore the $q \times q$ matrix $(\langle A(X_i), A(X_j)\rangle)$ is

non-singular. Thus we may solve for z_i's $(1 \leqq i \leqq q)$ as differentiable function on $\mathcal{O}$. Hence Z is differentiable on $\mathcal{O}$.

LEMMA 9. *The distribution $\mathcal{U}: x \to \mathcal{U}_x$ is differentiable on $\mathcal{O}$.*

PROOF. At each point of $\mathcal{O}$ we define α^* to be $\alpha - \gamma$, where $\gamma(X, X) = Z(x)$ for all unit vectors $X \in T_x(M)$, $x \in \mathcal{O}$. Then, by Lemma 8, α^* is a differentiable tensor field on $\mathcal{O}$, and by the proof of Theorem 1, $\mathcal{U}_x$ is the null space of the linear transformation $X \to \alpha^*(X, \cdot)$. Since $\rho = \dim \mathcal{U}$ is constant on $\mathcal{O}$, it follows that $\mathcal{U}$ is differentiable on $\mathcal{O}$.

LEMMA 10. *The normal curvature vector field Z on $\mathcal{O}$ satisfies $\nabla^\perp_U Z = 0$ for all $U \in \mathcal{U}$.*

PROOF. Since $\rho \geqq n - p \geqq 3$, we can take orthonormal vector fields $U, V \in \mathcal{U}$ on $\mathcal{O}$. Applying the last assertion of Theorem 1, we have

$$(\nabla^*_U \alpha)(V, V) = \nabla^\perp_U(\alpha(V, V)) - 2\alpha(\nabla_U V, V) = \nabla^\perp_U Z ,$$
$$(\nabla^*_V \alpha)(U, V) = \nabla^\perp_V(\alpha(U, V)) - \alpha(\nabla_V U, V) - \alpha(U, \nabla_V V) = 0 .$$

But, by the Codazzi equation (C1), we have $\nabla^\perp_U Z = 0$ for all $U \in \mathcal{U}$.

LEMMA 11. *The umbilic distribution $\mathcal{U}$ is involutive on $\mathcal{O}$.*

PROOF. Let U and V be non-zero vectors in $\mathcal{U}$. Then $\alpha(V, X) = \langle V, X\rangle Z$ for all vector field X tangent to $\mathcal{O}$. Using Lemma 10, we have $(\nabla^*_U \alpha)(V, X) = (U\langle V, X\rangle)Z - \alpha(\nabla_U V, X) - \langle V, \nabla_U X\rangle Z = \langle \nabla_U V, X\rangle Z - \alpha(\nabla_U V, X)$. Hence, by (C1), we have $\alpha([U, V], X) = \langle [U, V], X\rangle Z$, which implies $[U, V]$ is in $\mathcal{U}$.

We consider now the configuration of a leaf L of $\mathcal{U}$ in $\mathcal{O}$.

LEMMA 12. *Each leaf L of $\mathcal{U}$ in $\mathcal{O}$ is umbilic in M and in E^{n+p} relative to f.*

PROOF. Let $\xi_1, \cdots, \xi_p$ be orthonormal normal vector fields in $\mathcal{O}$ such that $\lambda\xi_1$ $(\lambda = ||Z||)$ is the normal curvature vector field Z. Denote by $\mathfrak{X}(L)$ (resp. $\mathfrak{X}(M)$) the algebra of vector fields on L (resp. M). Let U and V denote vector fields in $\mathfrak{X}(L)$, and let X denote a vector field in $\mathfrak{X}(M)$. The second fundamental forms A_k's $(1 \leqq k \leqq p)$ satisfy $A_1 U = \lambda U$, $A_k U = 0$ $(2 \leqq k \leqq p)$. By Lemma 10, we have $U\lambda = 0$, that is, λ is constant on L. Hence we have $s_{1k}(U) = 0$ $(1 \leqq k \leqq p)$. Differentiating both sides of $A_1 U = \lambda U$ by X and using (C2), we have

$$(*) \qquad (\nabla_U A_1)X = -A_1 \nabla_X U + (X\lambda)U + \lambda \nabla_X U .$$

By a similar computation for $A_k U = 0$, we have

$$(\nabla_U A_k)X = \lambda s_{1k}(X)U + \sum_{l=1}^{p} s_{kl}(U)A_l X + A_k \nabla_X U \quad (2 \leqq k \leqq p) .$$

Let β be the second fundamental form of L in M, and let P denote the orthogonal projection of $\mathfrak{X}(M)$ to $\mathfrak{X}(L)$. For any vector field ξ which is orthogonal to L and tangent to M, we denote by B_ξ the second fundamental form associated to ξ. Then we have $B_\xi U = -P\nabla_U \xi$. Since $A_1 X - \lambda X$ is in $\mathfrak{X}(L)^\perp \cap \mathfrak{X}(M)$, we may compute $B_{A_1X-\lambda X}$ as follows:

$$\begin{aligned} -B_{A_1X-\lambda X}(U) &= P((\nabla_U A_1)X + A_1\nabla_U X - \lambda \nabla_U X) \\ &= P(-A_1\nabla_X U + (X\lambda)U + \lambda\nabla_X U + A_1\nabla_U X - \lambda\nabla_U X) \\ &= P(A_1[U, X] + (X\lambda)U - \lambda[U, X]) \\ &= A_1 P[U, X] + (X\lambda)U - \lambda P[U, X] = (X\lambda)U , \end{aligned}$$

where we have used the equation (*) and the fact that $\mathfrak{X}(L)$ is a proper space of the symmetric operator A_1. Hence we have $\langle \beta(U, V), A_1X - \lambda X\rangle = -(X\lambda)\langle U, V\rangle$. Since A_kX's $(2 \leqq k \leqq p)$ are in $\mathfrak{X}(L)^\perp \cap \mathfrak{X}(M)$, a similar computation shows that $B_{A_kX}U = \lambda s_{1k}(X)U$ $(2 \leqq k \leqq p)$. Hence we have $\langle \beta(U, V), A_kX\rangle = \lambda s_{1k}(X)\langle U, V\rangle$ $(2 \leqq k \leqq p)$.

Let β^* be the symmetric bilinear function on $\mathfrak{X}(L)$ defined by

$$\beta^*(U, V) = \beta(U, V) - \langle U, V\rangle \beta(W, W) ,$$

where W is a fixed unit vector field tangent to L. Then the above two equations for β imply $\langle \beta^*(U, V), A_1X - \lambda X\rangle = 0$ and $\langle \beta^*(U, V), A_kX\rangle = 0$ $(2 \leqq k \leqq p)$. Hence we have $\langle \alpha(\beta^*(U, V), X), \xi_1\rangle = \lambda\langle \beta^*(U, V), X\rangle$ and $\langle \alpha(\beta^*(U, V), X), \xi_k\rangle = 0$ $(2 \leqq k \leqq p)$. These imply $\alpha(\beta^*(U, V), X) = \langle \beta^*(U, V), X\rangle Z$, that is, $\beta^*(U, V)$ is in $\mathfrak{X}(L)$. But, by definition, $\beta^*(U, V)$ is in $\mathfrak{X}(L)^\perp$. Thus we have $\beta^*(U, V) = 0$. So we have $\beta(U, V) = \langle U, V\rangle \beta(W, W)$ for all $U, V \in \mathfrak{X}(L)$ and a fixed unit vector field $W \in \mathfrak{X}(L)$. This also implies $\beta(W, W)$ is independent of the choice of a unit vector field W in $\mathfrak{X}(L)$. Hence L is umbilic in M. Let δ be the second fundamental form of L in E^{n+p} relative to $f|L$. Then $\delta = \alpha + \beta$ on $\mathfrak{X}(L)$. Thus we have $\delta(U, V) = \langle U, V\rangle(Z + \beta(W, W))$. Hence L is umbilic in E^{n+p}.

THEOREM 2. *Let $f: M \to E^{n+p}$, with $p \leqq n - 3$ and $n \geqq 4$, be an isometric immersion from an n-dimensional conformally flat Riemannian manifold M with positive sectional curvatures to the $(n + p)$-dimensional Euclidean space E^{n+p}. Let $\mathcal{O}$ be an open set on which the umbilic index takes constant value and the multiplicity of each proper value of the Ricci tensor is constant. Then the umbilic distribution $\mathcal{U}$ is involutive on $\mathcal{O}$ and each leaf L of $\mathcal{U}$ in $\mathcal{O}$ is umbilic in M and in*

E^{n+p} relative to f. Furthermore, L is a Riemannian manifold of constant curvature $\lambda^2 + \mu^2$, where λ (resp. μ) is the length of the normal curvature vector field of f (resp. $f|L$).

PROOF. All except the final assertion were already proved. Let δ be the second fundamental form of $f|L$, and K_L the sectional curvature function on L. Then we have $K_L(U, V) = \langle \delta(U, U), \delta(V, V) \rangle - \|\delta(U, V)\|^2 = \|Z\|^2 + \|\beta(U, U)\|^2 = \lambda^2 + \mu^2$ for all orthonormal vectors U and V tangent to L. Since $\dim L \geqq 3$ by Theorem 1, L is a Riemannian manifold of constant curvature $\lambda^2 + \mu^2$.

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