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REMARKS ON THE FORMULATION OF THE CAUCHY PROBLEM FOR GENERAL SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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1. **Introduction and statement of results.** In his paper [4], Wagschal proved that for a given non-degenerate system of (partial) differential equations various formulations of the (non-characteristic) Cauchy problem are well-defined. The purpose of this paper is to give a characterization of the well-defined Cauchy problem for general system of ordinary differential equations. It will be shown, in conclusion, that the well-defined Cauchy problem is nothing but the classical Cauchy problem for a normal system.

Let $A(x; D) = (a_{ij}(x; D))_{1 \leqq i,j \leqq N}$ be a system of ordinary differential operators with holomorphic coefficients in $\Omega \subset \boldsymbol{C}$, where $D = d/dx$. Let $T = (t_1, \cdots, t_N)$ be a pair of non-negative integers. We shall consider the following Cauchy problem $(A(x; D), T)$:

$$\sum_{j=1}^{N} a_{ij}(x; D)u_j(x) = f_i(x) , \qquad 1 \leqq i \leqq N , \tag{1.1}$$

$$D^k u_i(x_0) = w_{i,k} \in \boldsymbol{C} , \qquad 0 \leqq k < t_i , \qquad 1 \leqq i \leqq N , \tag{1.2}$$

where $x_0 \in \Omega$.

In order to clarify our problem we give some definitions. We say that the Cauchy problem $(A(x; D), T)$ is *well-defined* at x_0 if the problem (1.1)-(1.2) has a unique holomorphic solution $\{u_i(x)\}$ at x_0 for any holomorphic functions $\{f_i(x)\}$ at x_0 and any Cauchy data $\{w_{i,k}\}$. A system $A(x; D)$ is said to be *in a normal form with respect to* $T = (t_1, \cdots, t_N)$ or simply *T-normal* if $a_{ij} = \delta_{ij} D^{t_i} + b_{ij}(x; D)$ and order $b_{ij} < t_j$ for any i and j, where δ_{ij} is Kroneker's δ.

It is well-known that for a T-normal system $A(x; D)$ the Cauchy problem $(A(x; D), T)$ is well-defined at every point in Ω. Our purpose is to show the converse. In order to state our results we need the following definition. A system $P(x; D)$ of differential operators is said to be *invertible* if there exists a system $P^{-1}(x; D)$ of differential operators

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satisfying $PP^{-1} = P^{-1}P = I_N$, where I_N is the identity matrix of size N.
Now the main result is the following

THEOREM I. *The Cauchy problem* $(A(x; D), T)$ *is well-defined at every point in* Ω *if and only if there exists an invertible system* $P(x; D)$ *of differential operators with holomorphic coefficients in* Ω *such that* PA *is in a T-normal form. Moreover, the inverse system* $P^{-1}(x; D)$ *has also holomorphic coefficients in* Ω.

Next we give

THEOREM II. *Assume that the coefficients of* $A(x; D)$ *are meromorphic in* Ω. *Let us consider the Cauchy problem* $(A(x; D), T)$ *at every point in* Ω *with the exception of the points in a discrete subset of* Ω. *Then in order that the Cauchy problem* $(A(x; D), T)$ *may have at least one solution for any* $\{f_i(x)\}$ *and any Cauchy data* $\{w_{i,k}\}$, *it is necessary and sufficient that there exists an invertible system* $P(x; D)$ *with meromorphic coefficients in* Ω *such that* PA *is in a* $\tilde{T}$*-normal form with respect to some* $\tilde{T} = (\tilde{t}_1, \cdots, \tilde{t}_N)$ *with* $\tilde{t}_i \geqq t_i$.

REMARK 1. In the above theorem, a discrete subset of Ω is not given a priori.

REMARK 2. In the above theorem, if we demand the uniqueness of the solution, then we have $\tilde{T} = T$. In fact, the Cauchy problem $(A(x; D), \tilde{T})$ is well-defined at every point in Ω with the exception of the points in a discrete subset.

We give here a remark on the invertible system of differential operators. As in the theory of matrices, we define *elementary operations* on the system of differential operators $P(x; D) = (p_{ij}(x; D))$ with meromorphic coefficients in Ω.

(a) Multiplication of any row (resp. column) by a meromorphic function $c(x) \not\equiv 0$.

(b) Addition to any row (resp. column) of any other row (resp. column) multiplied by any arbitrary differential operator $b(x; D)$ with meromorphic coefficients.

(c) Interchange of any two rows (resp. columns).

We say that systems $A(x; D)$ and $B(x; D)$ are *equivalent* if one of them can be obtained from the other by means of elementary operations. Especially, we say that they are *left-equivalent* (resp. *right-equivalent*) if they are equivalent by means of elementary operations only by use of rows (resp. columns). Now we have

THEOREM III. *A system $P(x; D)$ is invertible if and only if $P(x; D)$ can be expressed as a product of elementary operations.*

The proof is the same as in the theory of elementary divisor of matrices (see Gantmacher [1]). It suffices to see that $P(x; D) = (p_{ij}(x; D)\delta_{ij})$ is invertible if and only if $p_{ii}(x; D) \equiv c_i(x) \not\equiv 0$.

We note that for a holomorphically invertible system $P(x;D)$ its elements of elementary operations are not necessarily holomorphic. In fact, the following example shows this.

EXAMPLE. Let

$$P(x, D) = \begin{pmatrix} x^2/2 & -(x/2)D + 1 \\ -xD - 3 & D^2 \end{pmatrix}.$$

Then we have

$$P^{-1} = \begin{pmatrix} D^2 & (x/2)D \\ xD + 1 & x^2/2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & (2/x)D \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^2/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -(1/x)D \\ 0 & 1 \end{pmatrix}.$$

Wagschal proved that for a given non-degenerate system $A(x; D)$ at x_0 with holomorphic coefficients there exists at least one T such that the Cauchy problem $(A(x; D), T)$ is well-defined at every point in a neighbourhood of x_0 [4, Th. 4.1]. The definition of a non-degenerate system will be given in §3. Hence, by Theorem I, there exists a holomorphically invertible system $P(x; D)$ of differential operators such that PA is in a T-normal form. On the other hand, as is shown by the above example, the elements of elementary operations are in general meromorphic. Concerning this we have

THEOREM IV. *Let $A(x; D)$ be a non-degenerate system at x_0 with holomorphic coefficients. Then there exists at least one T such that $A(x; D)$ can be reduced to a T-normal system $B(x; D)$ by holomorphic left-elementary operations in a neighbourhood of x_0.*

The remainder of this paper is organized as follows. Theorem I will be proved in §2 by means of Theorem II. §3 will be devoted to preliminary considerations for the proof of Theorem II. Then Theorem II will be proved in §4. In §5, we shall prove Theorem IV. In §6, we shall give an example which indicates the difference between Theorems I and II. Moreover, the existence will be shown of a well-defined Cauchy problem $(A(x; D), T)$ for such a system $A(x; D)$ which is not reduced to a T-normal system by holomorphic left-elementary operations.

We note that the idea of the proof of this paper was given in the previous paper of the author [2] (see also [2]′).

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2. Proof of Theorem I. We prove Theorem I by means of Theorem II. We have only to prove the necessity, since the sufficiency is obvious. Let $P(x; D)$ be an invertible system with meromorphic coefficients in Ω given by Theorem II and Remark 2. In order to prove the necessity, it suffices to show that $P(x; D)$ is holomorphically invertible in Ω. Now assume that P has singular coefficients at x_0. First, we consider the case where PA is holomorphic at x_0. Let $f(x) = {}^t(f_1(x), \cdots, f_N(x))$ be a vector of holomorphic functions at x_0 such that Pf is singular at x_0. Then it is easy to see that the Cauchy problem $(A(x; D), T)$ for the equation $Au = f$ has no holomorphic solution at x_0, where $u = {}^t(u_1, \cdots, u_N)$. Next, we consider the case where PA is singular at x_0. Let $f(x) = {}^t(f_1(x), \cdots, f_N(x))$ be a vector of holomorphic functions such that Pf is holomorphic at x_0. Then the Cauchy problem $(A(x; D), T)$ for the equation $Au = f$ has no holomorphic solution at x_0 for a suitable choice of the Cauchy data $\{w_{i,k}\}$. In fact, it suffices to choose the Cauchy data so that $(PAU)(x)$ is singular at x_0, where $U(x) = {}^t(U_1(x), \cdots, U_N(x))$ with $U_i(x) = \sum_{k=0}^{t_i-1} w_{i,k}(x - x_0)^k/k! + O((x - x_0)^{t_i})$. Hence $P(x; D)$ is holomorphic in Ω. Note that the Cauchy problem (PA, T) is well-defined at every point in Ω, since PA is a holomorphic T-normal system in Ω. Thus $P^{-1}(x; D)$ is also holomorphic in Ω. q.e.d.

3. Preliminary considerations. We begin by summarizing the work of Volevič [3]. Let $A(x; D) = (a_{ij}(x; D))_{1 \leq i, j \leq N}$ and let $m_{ij} = \text{order}\, a_{ij}(x; D)$ if $a_{ij} \not\equiv 0$ and $\text{order}\, a_{ij} = -\infty$ if $a_{ij} \equiv 0$. Then the *total order* m of the system $A(x; D)$ is defined by

$$m = \max_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{N} m_{i\sigma(i)}\ , \tag{3.1}$$

where $\mathfrak{S}_N$ denotes the permutation group of $\{1, 2, \cdots, N\}$ and $-\infty + r = -\infty$ for any $r \in \boldsymbol{Z}_+ = \{0, 1, 2, \cdots\}$. A system $A(x; D)$ of total order m is said to be *non-degenerate* (at x_0) if $m = \deg_\xi \{\det A(x; \xi)\}$ $(m = \deg_\xi \{\det A(x_0; \xi)\})$. Then we have

THEOREM ([3, Th. 1]). *A system $A(x; D)$ of size N with meromorphic coefficients in Ω is left-equivalent to a non-degenerate system $B(x; D)$ or to a system $B(x; D)$ of* rank $B < N$, *where* rank $B < N$ *means that $B = (b_{ij})$ satisfies $b_{i_0 j} \equiv 0$ for some i_0 and any j.*

It is obvious that in order that the Cauchy problem $(A(x; D), T)$ may be well-defined, it is necessary that the system $B(x; D)$ is non-degenerate in the above theorem. Hence in the following, we consider a non-degenerate system with meromorphic coefficients in Ω.

Now our first purpose is to reduce a non-degenerate system to a normal system by left-elementary operations. In order to do so, we need the following

LEMMA 3.1. *Assume that $A(x; D)$ is a non-degenerate system of total order $m(\geqq 0)$. Then $A(x; D)$ is left-equivalent to a system $B(x; D) = (b_{ij})$ such that*

$$(3.2) \qquad m = \sum_{i=1}^{N} n_{ii} > \sum_{i=1}^{N} n_{i\sigma(i)} \quad \textit{for any} \quad \sigma \neq 1 \, ,$$

where $n_{ij} = \text{order}\, b_{ij}$.

PROOF. By a suitable interchange of rows, we may assume that $m = \sum_{i=1}^{N} m_{ii}$, where $m_{ij} = \text{order}\, a_{ij}$. Therefore, there exists a system of integers $\{s_i\}_{i=1}^{N}$ such that $m_{ij} \leqq s_i - s_j + m_{jj}$ (see [2] or [2]'). We may assume without loss of generality that $s_1 \leqq s_2 \leqq \cdots \leqq s_N$ by a suitable interchange of rows and columns. Note that the interchange of columns is permitted in our problem. We put $\gamma_{ij} = s_i - s_j + m_{jj}$. Let $i_1 = \min\{i; m_{i1} = \gamma_{i1}\}$. Obviously $i_1 = 1$, since $m_{11} = \gamma_{11}$. Then we lower the order of the $(i, 1)$-component for $i \neq i_1$ to be less than γ_{i1}, using the i_1-th row. Thus we obtain a left equivalent system $B(x; D) = (b_{ij})$ satisfying $\text{order}\, b_{ij} \leqq \gamma_{ij}$, $\text{order}\, b_{i_1 1} = \gamma_{i_1 1}$ and $\text{order}\, b_{i1} < \gamma_{i1}$ for $i \neq i_1$. Next, we put $i_2 = \min\{i; \text{order}\, b_{i2} = \gamma_{i2}, i \neq i_1\}$. The existence of such $i_2(\neq i_1)$ is guaranteed by the non-degeneracy of $A(x; D)$, a fortiori of $B(x; D)$. Then we lower the order of the $(i, 2)$-component for $i \neq i_1, i_2$ to be less than γ_{i2}, using the i_2-th row. Continuing these operations, we finally obtain a left-equivalent system $E(x; D) = (e_{ij})$ and $\{i_1, \cdots, i_N\} = \{1, \cdots, N\}$ such that $\text{order}\, e_{ij} \leqq \gamma_{ij}$, $\text{order}\, e_{i_j j} = \gamma_{i_j j}$ and $\text{order}\, e_{ij} < \gamma_{ij}$ for $i \notin \{i_1, \cdots, i_j\}$. q.e.d.

PROPOSITION 3.1. *Assume that a non-degenerate system $A(x; D)$ satisfies*

$$(3.3) \qquad \sum_{i=1}^{N} m_{ii} > \sum_{i=1}^{N} m_{i\sigma(i)} \quad \textit{for any} \quad \sigma \neq 1 \, .$$

Then $A(x; D)$ is left-equivalent to an $(m_{11}, \cdots, m_{NN})$-normal system $B(x; D) = (b_{ij})$ with the following property:

(*) *If for some i_0 we have $m_{i_0 j} < m_{jj}$ for any $j(\neq i_0)$, then $b_{i_0 j} \equiv a_{i_0 j}$ for any j.*

For the proof of this proposition, we need some lemmas.

LEMMA 3.2. *Assume that* $A(x; D)$ *satisfies the condition* (3.3). *Then if* $m_{i_0j_0} \geqq m_{j_0j_0}$ *for some* $i_0 \neq j_0$, $A(x; D)$ *is left-equivalent to a system* $B(x; D) = (b_{ij})$ *such that* $\operatorname{order} b_{ii} = m_{ii}$, $\operatorname{order} b_{i_0j_0} < m_{j_0j_0}$ *and* $B(x; D)$ *satisfies the condition corresponding to* (3.3).

PROOF. Let $a_{i_0j_0}(x; D) = -c(x; D)a_{j_0j_0}(x; D) + b_{i_0j_0}(x; D)$, where $\operatorname{order} c(x; D) = m_{i_0j_0} - m_{j_0j_0}$ and $\operatorname{order} b_{i_0j_0} < m_{j_0j_0}$. Then adding to the i_0-th row the j_0-th row multiplied by $c(x; D)$, we obtain a system $B(x; D)$ which satisfies the desired properties. It is easy to see that $\operatorname{order} b_{ii} = m_{ii}$ and $\operatorname{order} b_{i_0j_0} < m_{j_0j_0}$, since $m_{i_0i_0} + m_{j_0j_0} > m_{i_0j_0} + m_{j_0i_0}$. In order to prove that $B(x; D)$ satisfies the condition corresponding to (3.3), we have only to prove it under the assumption that order $b_{i_0\sigma(i_0)} = m_{j_0\sigma(i_0)} + m_{i_0j_0} - m_{j_0j_0}$. Hence, it suffices to show

$$(3.4) \qquad \sum_{i=1}^{N} m_{ii} + m_{j_0j_0} > \sum_{i\neq i_0} m_{i\sigma(i)} + m_{i_0j_0} + m_{j_0\sigma(i_0)} , \qquad \sigma \neq 1 .$$

First, we consider the case when $\{\sigma(i_0), \sigma(j_0)\} \cap \{i_0, j_0\} \neq \emptyset$. We examine only the case when $\sigma(j_0) = i_0$. It is easy to see that $\sum_{i\neq i_0} m_{i\sigma(i)} + m_{i_0j_0} + m_{j_0\sigma(i_0)} < \sum_{i\neq i_0, j_0} m_{i\sigma(i)} + m_{j_0\sigma(i_0)} + (m_{i_0i_0} + m_{j_0j_0})$. On the other hand, we have $\sum_{i\neq i_0, j_0} m_{i\sigma(i)} + m_{j_0\sigma(i_0)} \leqq \sum_{i\neq i_0} m_{ii}$, since $\{1,2,\cdots,N\}\backslash\{i_0\} = \{\sigma(1),\cdots,\sigma(N)\}\backslash\{\sigma(j_0)\}$. This proves (3.4). Next, we consider the other case. Choose $k_0(\neq i_0, j_0)$ so that $\sigma(k_0) = j_0$. Let $A = \{\sigma^k(j_0);\ 0 \leqq k \leqq l\}$, $B = \{\sigma^k(i_0);\ 0 < k \leqq m\}$ and $C = \{1, \cdots, N\}\backslash A \cup B$, where $l = \min\{n;\ \sigma^n(j_0) = k_0 \text{ or } i_0\}$ and $m = \min\{n;\ \sigma^n(i_0) = i_0 \text{ or } k_0\}$. It is easy to see that $\sigma^l(j_0) \neq \sigma^m(i_0)$, $A \cap B = \emptyset$ and $j_0 \notin B$. Hence, $\{1, 2, \cdots, N\}$ is expressed as a disjoint union of A, B and C. We have $\sum_{i\in C} m_{i\sigma(i)} \leqq \sum_{i\in C} m_{ii}$, since $\sigma(C) = C$. On the other hand, we have $\sum_{i\in A} m_{i\sigma(i)} < \sum_{i\in A} m_{ii}$ and $m_{j_0\sigma(i_0)} + \sum_{i\in B} m_{i\sigma(i)} < \sum_{i\in B\cup\{j_0\}} m_{ii}$, where we redefine $\sigma^{l+1}(j_0) = j_0$ and $\sigma^{m+1}(i_0) = j_0$. q.e.d.

LEMMA 3.3. *Suppose that* $A(x; D)$ *satisfies the condition* (3.3) *and that* $m_{ij} < m_{jj}$ *for any* $i \neq j$ *with* $2 \leqq i, j \leqq N$. *Then* $A(x; D)$ *is left-equivalent to a system* $B(x; D)$ *which satisfies* $b_{ij} \equiv a_{ij}$ *for* $i \neq 1$, $\operatorname{order} b_{11} = m_{11}$ *and* $\operatorname{order} b_{1j} < m_{jj}$ *for* $j \neq 1$.

PROOF. Let $r = \max_{2\leqq j\leqq N}\{m_{1j} - m_{jj}\} \geqq 0$, and put $J = \{j;\ m_{1j} = m_{jj} + r\}$. Then for $j_0 \in J$, we can lower the order of the $(1, j_0)$-component to be less than $m_{j_0j_0}$, using the j_0-th row as in Lemma 3.2. Let $B(x; D)$ be a system obtained by this operation. Then, $b_{ij} \equiv a_{ij}$ for $i \neq 1$ and $\operatorname{order} b_{1j} \leqq \max\{m_{1j}, m_{j_0j} + r\}$ for $j \neq 1, j_0$. Hence, by the assumption of the lemma, we have $\operatorname{order} b_{1j} = m_{jj} + r$ for $j \in J$ with $j \neq j_0$ and $\operatorname{order} b_{1k} < m_{kk} + r$ for $k \notin J$. Continuing these operations, we finally obtain a desired system. q.e.d.

PROOF OF PROPOSITION 3.1. We prove the proposition by induction on N. First, the case $N=2$ is obvious. Assume that the proposition is true for $N-1$ and that the condititions in $(*)$ are true for $1 \leqq i \leqq k-1$. Then applying the induction assumption to the system of size $N-1$ obtained by removing the k-th row and k-th column from $A(x; D)$, we obtain a left-equivalent system $B(x; D) = (b_{ij})_{1\leqq i,j\leqq N}$ such that $b_{ij} \equiv a_{ij}$ for $i \leqq k$ and $m_{jj} =$ order $b_{jj} > m_{ij}$ for $i \neq j$ with $i \neq k$ or $j \neq k$. Note that $B(x; D)$ also satisfies the condition (3.3), in view of Lemma 3.2. We have to mention that the left-elementary operation in the proof of Lemma 3.2 is only used in our proof. Then applying Lemma 3.3 to the system $B(x; D)$, we obtain a left-equivalent system $C(x; D)$ which satisfies the conditions in $(*)$ for $1 \leqq i \leqq k$. Continuing these operations, we finally obtain a desired system. q.e.d.

Our next purpose is to reduce a normal system to another.

PROPOSITION 3.2. *Assume that an $(m_{11}, \cdots, m_{NN})$-normal system $A(x; D)$ satisfies*

$$m_{12} + \sum_{i=3}^{N} m_{ii} > \sum_{i\neq 2} m_{ij_i}\ , \tag{3.5}$$

where $\{j_1, j_3, \cdots, j_N\} = \{2, 3, \cdots, N\}$ and $(j_1, j_3, \cdots, j_N) \neq (2, 3, \cdots, N)$. Then $A(x; D)$ is left-equivalent to an $(m_{11}+m_{22}-m_{12}, m_{12}, m_{33}, \cdots, m_{NN})$-normal system $B(x; D)$.

For the proof we need the following

LEMMA 3.4. *Assume that $A(x; D)$ satisfies the condition* (3.3) *and $m_{jj} > m_{ij}$ for $i \neq j$ with $j \neq 1$. Then for any $k \neq 1$ we have*

$$\sum_{i\neq k} m_{i\sigma(i)} + m_{1\sigma(k)} < \sum_{i=1}^{N} m_{ii} \quad \textit{for any } \sigma\ . \tag{3.6}$$

PROOF. Without loss of generality, we may assume that $k=N$. We consider the case when $\{\sigma(1), \sigma(N)\} \cap \{1\} = \emptyset$, since in the other case it is obvious. Let $l = \min\{s; \sigma^s(1) = 1\}$. First, consider the case when $\sigma^j(1) \neq \sigma(N)$ for any $j < l$. In view of (3.3), we have $\sum_{j=1}^{l} m_{\sigma^{j-1}(1)\sigma^j(1)} < \sum_{j=0}^{l-1} m_{\sigma^j(1)\sigma^j(1)}$. Next, consider the case when $\sigma^{j_0}(1) = \sigma(N)$ for some $j_0 < l$. Let $k = \min\{j; \sigma^j(N) = 1\}$. Then we have

$$m_{1\sigma(N)} + \sum_{j=2}^{k} m_{\sigma^{j-1}(N)\sigma^j(N)} < m_{11} + \sum_{j=1}^{k-1} m_{\sigma^j(N)\sigma^j(N)}\ .$$

Hence, by the assumptions of the lemma, we obtain (3.6). q.e.d.

PROOF OF PROPOSITION 3.2. Note that $a_{12} \not\equiv 0$. Now we lower the order of the (2, 2)-component to be less than m_{12}, using the first row.

Then we obtain a left-equivalent system $B(x; D) = (b_{ij})$ such that $b_{ij} \equiv a_{ij}$ for $i \neq 2$, order $b_{21} = m_{11} + m_{22} - m_{12}$, order $b_{22} < m_{12}$ and order $b_{2j} \leqq \max\{m_{2j}, m_{1j} + m_{22} - m_{12}\}$ for $j = 3, \cdots, N$. By Proposition 3.1, we have only to show that order $b_{2\sigma(2)} + \sum_{i\neq 2} m_{i\sigma(i)} < \sum m_{ii}$ for any σ such that $(\sigma(1), \cdots, \sigma(N)) \neq (2, 1, 3, \cdots, N)$. In the case when $\{\sigma(1), \sigma(2)\} \cap \{1, 2\} \neq \emptyset$, it is obvious by the assumptions and the construction of $B(x; D)$. Let us consider the other case. Moreover, it suffices to show the inequality under the assumption that order $b_{2\sigma(2)} = m_{1\sigma(2)} + m_{22} - m_{12}$. We choose $k(\in\{3, \cdots, N\})$ so that $\sigma(k) = 1$. Considering that $m_{k\sigma(k)} + m_{22} - m_{11} <$ order b_{21}, it suffices to show

$$m_{1\sigma(2)} + \sum_{i\neq 2,k} m_{i\sigma(i)} < m_{12} + \sum_{i=3}^{N} m_{ii} \,. \tag{3.7}$$

On the other hand, for the matrix of size $N-1$ which is obtained by removing the second row and the first column from $A(x; D)$, the assumptions of Lemma 3.4 are satisfied, in view of (3.5). This implies (3.7).

q.e.d.

4. Proof of Theorem II. Without loss of generality, we may assume that $A(x; D)$ is in a normal form with respect to $(m_{11}, \cdots, m_{NN})$. We put $s_i = m_{ii}$. If $s_i \geqq t_i$ for any i, there is nothing to prove. Suppose $s_{i_0} < t_{i_0}$ for some i_0. Then there exists $j_0(\neq i_0)$ such that $m_{i_0 j_0} \geqq t_{j_0}$, for otherwise, in the i_0-th equation $\sum_{j=1}^{N} a_{i_0 j}(x; D)u_j(x) = f_{i_0}(x)$, there should exist compatibility conditions between the Cauchy data $\{w_{i,k}\}$ and $f_{i_0}(x)$. Here we note that we consider the Cauchy problem at the point where the coefficients of the system are holomorphic. If

$$m_{i_0 j_0} + \sum_{i\neq i_0, j_0} m_{ii} > \sum_{i\neq j_0} m_{i j_i} \,, \tag*{$(4.1)_{i_0}$}$$

for $\{j_1, \cdots, j_{j_0-1}, j_{j_0+1}, \cdots, j_N\} = \{1, \cdots, N\} \backslash \{i_0\}$ and $(j_1, \cdots, j_{j_0-1}, j_{j_0+1}, \cdots, j_N) \neq (1, \cdots, i_0-1, j_0, i_0+1, \cdots, j_0-1, j_0+1, \cdots, N)$, then applying Proposition 3.2, we obtain a left-equivalent system $B(x; D)$ which is in a normal form with respect to

$$(s_1, \cdots, s_{i_0-1}, s_{i_0} + s_{j_0} - m_{i_0 j_0}, s_{i_0+1}, \cdots, s_{j_0-1}, m_{i_0 j_0}, s_{j_0+1}, \cdots, s_N) \,. \tag*{$(4.2)_{i_0}$}$$

When the inequality $(4.1)_{i_0}$ does not hold, there exists $i(\neq i_0, j_0)$ such that $m_{ij_0} > m_{i_0 j_0}$. We choose i^* such that $m_{i^* j_0} = \max_{i\neq j_0}\{m_{ij_0}\}(> m_{i_0 j_0})$. Then the inequality $(4.1)_{i^*}$ holds instead of $(4.1)_{i_0}$. Hence we have a left-equivalent system $C(x; D)$ which is in a normal form with respect to $(4.2)_{i^*}$. Here we have to mention that in the system $C(x; D)$ it holds that $c_{i_0 j} \equiv a_{i_0 j}$ for any j in view of $(*)$ in Proposition 3.1. Now, for the system $C(x; D)$, if the condition corresponding to $(4.1)_{i_0}$ does not hold, then we continue the above operations. Finally we obtain a system $E(x; D)$ satisfying the following:

(i) $E(x; D)$ is in a normal form with respect to some $\tilde{S}$ with $\tilde{S} = (\tilde{s}_1, \cdots, \tilde{s}_{i_0-1}, s_{i_0}, \tilde{s}_{i_0+1}, \cdots, \tilde{s}_N)$, where $t_{j_0} \leqq \tilde{s}_{j_0} < s_{j_0}$ and $\tilde{s}_i \geqq s_i$ for $i \neq i_0, j_0$,

(ii) $e_{i_0 j}(x; D) \equiv a_{i_0 j}$ for any j,

(iii) The condition corresponding to $(4.1)_{i_0}$ holds for $E(x; D)$.

Therefore by the method of the first step, we have a left-equivalent system $F(x; D)$ in a normal form with respect to some $\tilde{T} = (\tilde{t}_1, \cdots, \tilde{t}_N)$ satisfying $\tilde{t}_{i_0} > s_{i_0}$, $t_{j_0} \leqq \tilde{t}_{j_0} < s_{j_0}$ and $\tilde{t}_i \geqq s_i$ for $i \neq i_0, j_0$. Hence, continuing these operations we obtain our proposition. q.e.d.

5. **Proof of Theorem IV.** The proof is similar to that of Theorem I, which, however, was carried out in the class of meromorphic functions. So we need more careful considerations.

Let $A(x; D)$ be a non-degenerate system of total order m at x_0 with holomorphic coefficients. Then we may assume without loss of generality that

$$m = \sum_{i=1}^{N} m_{ii}^{(0)} > \sum_{i=1}^{N} m_{i\sigma(i)}^{(0)} \quad \text{for any} \quad \sigma \neq 1\,, \tag{5.1}$$

where $m_{ij}^{(0)} = \text{order}\, a_{ij}(x_0; D)$.
In fact, the reduction to such a system by holomorphic left-elementary operations is the same as that of Lemma 3.1 with m_{ij} replaced by $m_{ij}^{(0)}$. In this case $\{s_i\} \subset \boldsymbol{Z}$ should be chosen so that $m_{ij} = \text{order}\, a_{ij}(x; D) \leqq s_i - s_j + m_{jj}^{(0)}$.

Now we have

LEMMA 5.1. *Assume that $A(x; D)$ satisfies the condition* (5.1). *Then $A(x; D)$ can be reduced to a system $B(x; D) = (b_{ij})$ by locally holomorphic left-elementary operations in a neighbourhood of x_0 with the following property*:

$$m = \sum_{i=1}^{N} n_{ii}^{(0)} > \sum_{i=1}^{N} n_{i\sigma(i)} \quad \textit{for any} \quad \sigma \neq 1\,, \tag{5.2}$$

where $n_{ij} =$ order $b_{ij}(x; D)$ and $n_{ij}^{(0)} =$ order $b_{ij}(x_0; D)$.

PROOF. By the condition (5.1), there exists $\{s_i\} \subset \boldsymbol{Z}$ such that $m_{ij} \leqq s_i - s_j + m_{jj}^{(0)}$, where $m_{ij} = \text{order}\, a_{ij}(x; D)$. Without loss of generality, we may assume that $s_1 \leqq s_2 \leqq \cdots \leqq s_N$. We put $\gamma_{ij} = s_i - s_j + m_{jj}^{(0)}$. Then we can show that $A(x; D)$ can be reduced to a system $B(x; D) = (b_{ij})$ by locally holomorphic left-elementary operations which satisfies order $b_{ii}(x_0; D) = m_{ii}^{(0)}$, order $b_{ij}(x; D) \leqq \gamma_{ij}$ and order $b_{ij}(x; D) < \gamma_{ij}$ for $i < j$.

In fact, assume that order $a_{i_0 1}(x; D) = \gamma_{i_0 1}$ for some $i_0 \neq 1$. Now we put

$$a_{ij}(x; D) = \sum_{k=0}^{\gamma_{ij}} a_{ijk}(x) D^{\gamma_{ij}-k} .$$

Note that $a_{ii0}(x_0) \neq 0$. Then adding to the i_0-th row the first row multiplied by $-(a_{i_0 10}(x)/a_{110}(x))D^{s_{i_0}-s_1}$, we obtain a system $E(x; D) = (e_{ij})$ which satisfies order $e_{ij} \leqq \gamma_{ij}$, order $e_{i_0 1} < \gamma_{i_0 1}$, order $e_{ii}(x_0; D) = m_{ii}^{(0)}$ and the condition corresponding to (5.1) holds for $E(x; D)$. In order to prove these facts, it suffices to show

$$\prod_{i \neq i_0} a_{i\sigma(i)0}(x_0) \times (a_{i_0 10}(x_0) \cdot a_{1\sigma(i_0)0}(x_0)) = 0 \tag{5.3}$$

for any $\sigma \in \mathfrak{S}_N$. We omit the proof. It is the same as that of Lemma 3.2. Continuing these operations, we finally obtain a desired system.

q.e.d.

PROOF OF THEOREM IV. We may assume that $A(x; D)$ satisfies the condition corresponding to (5.2). It is easy to see that for our system $A(x; D)$ Lemmas 3.2 and 3.3 hold by locally holomorphic left-elementary operations in a neighbourhood of x_0. Hence Proposition 3.1 holds for $A(x; D)$ by locally holomorphic left-elementary operations. q.e.d.

6. An example.

Let us consider the following system in $\boldsymbol{C}$:

$$A(x; D) = \begin{pmatrix} D^2 & D \\ xD + 1 & D^2 \end{pmatrix} .$$

The Cauchy problem $(A(x; D), T)$ is well-defined at every point in $\boldsymbol{C}$ if and only if $T = (3, 1)$ or $(2, 2)$ or $(0, 4)$. On the other hand, the Cauchy problem $(A(x; D), T)$ is well-defined at every point in $\boldsymbol{C}$ with the exception of the points in a discrete subset of $\boldsymbol{C}$ if and only if $T \neq (4, 0)$ and $|T| = t_1 + t_2 = 4$.

Let us prove these facts. First, the case $T = (2, 2)$ is obvious.

(i) Let us consider the case $T = (4, 0)$. Let

$$P_1(D) = \begin{pmatrix} D & -1 \\ 1 & 0 \end{pmatrix} .$$

Then we have

$$P_1^{-1}(D) = \begin{pmatrix} 0 & 1 \\ -1 & D \end{pmatrix}, \qquad P_1 A = \begin{pmatrix} D^3 - xD - 1 & 0 \\ D^2 & D \end{pmatrix} .$$

This proves immediately that the Cauchy problem is not well-defined at every point in $\boldsymbol{C}$.

(ii) In the case $T=(3,1)$, it is obvious that the Cauchy problem is well-defined at every point in $\boldsymbol{C}$ by (i).

(iii) In the case $T=(1,3)$, it is easy to see that the Cauchy problem is not well-defined at the origin. Let

$$P_2(x;D)=\begin{pmatrix}0 & 1/x\\ -x & D-2/x\end{pmatrix}.$$

Then we have

$$P_2^{-1}=\begin{pmatrix}D-1/x & -1/x\\ x & 0\end{pmatrix}, \qquad P_2A=\begin{pmatrix}D+1/x & (1/x)D^2\\ -2/x & D^3-(2/x)D^2-xD\end{pmatrix}.$$

This shows that the Cauchy problem is well-defined at every point in $\boldsymbol{C}$ except the origin.

(iv) Let us consider the case $T=(0,4)$. Let

$$P_3(x;D)=\begin{pmatrix}x^2/2 & -(x/2)D+1\\ -xD-3 & D^2\end{pmatrix}.$$

Then we have

$$P_3^{-1}=\begin{pmatrix}D^2 & (x/2)D\\ xD+1 & x^2/2\end{pmatrix}, \qquad P_3A=\begin{pmatrix}1 & -(x/2)D^3+D^2+(x^2/2)D\\ 0 & D^4-xD^2-3D\end{pmatrix}.$$

This implies that the Cauchy problem is well-defined at every point in $\boldsymbol{C}$. Note that the system $P_3(x;D)$ is the one given by Example in §1.

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