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NONLINEARLY PERTURBED VOLTERRA EQUATIONS*

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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Abstract. Boundedness and asymptotic behavior as $t \to \infty$ of solutions of nonlinearly perturbed Volterra equations are studied. Equations of both convolution and nonconvolution type are considered. An auxiliary equation plays an important role in the analysis of each type.

1. Introduction and summary. We investigate the boundedness and behavior as $t \to \infty$ of the solution(s), $x \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$, of the nonlinear Volterra equation

$$x(t) + \int_0^t a(t-s)[x(s) + h(s, x(s))]ds = q(t) \quad (0 \leqq t < \infty) , \tag{1.1}$$

where $\boldsymbol{R}^+ = [0, \infty)$ and $\boldsymbol{C}^N = \{z = \mathrm{col}(z_1 \cdots z_N) \mid z_i \in \boldsymbol{C}^1\}$. The prescribed functions a, h and q are assumed to satisfy

$$a \in L^1_{\mathrm{loc}}(\boldsymbol{R}^+, \boldsymbol{C}^{N^2}) \tag{1.2}$$

$$h \in C(\boldsymbol{R}^+ \times \boldsymbol{C}^N, \boldsymbol{C}^N) \tag{1.3}$$

$$q \in C(\boldsymbol{R}^+, \boldsymbol{C}^N) . \tag{1.4}$$

The norms

$$|x| = \sum_{i=1}^N |x_i| \quad (x \in \boldsymbol{C}^N) , \quad |a| = \sum_{i,j=1}^N |a_{ij}| \quad (a = (a_{ij})) \tag{1.5}$$

are employed. Thus, (1.2) means that

$$\int_0^t |a(s)| ds = \int_0^t \sum_{i,j=1}^N |a_{ij}(s)| ds < \infty \quad (0 \leqq t < \infty) .$$

In Section 7 we indicate how the results which follow are extended to the equation

$$x(t) + \int_0^t a(t, s)[x(s) + h(s, x(s))]ds = q(t) \quad (0 \leqq t < \infty) , \tag{1.6}$$

which, of course, reduces to (1.1) when

$$a(t, s) = a(t-s) . \tag{1.7}$$

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There are many papers devoted to perturbation problems of the type discussed here. Among these are [3], [4], [9]-[12], and [15]. The novelty of the present approach lies in the use of a priori bounds and of auxiliary equations (which generalize the notion of limit equations). This results in some new and sharpened forms of known results as well as in considerably simpler proofs.

Equation (1.1) is regarded as a perturbed form of the linear equation

$$x(t) + \int_0^t a(t-s)x(s)ds = q(t) \quad (0 \leqq t < \infty) . \tag{1.8}$$

Thus, we will usually assume, in addition to (1.3), that

$$h(t, 0) = 0 \quad (0 \leqq t < \infty) , \tag{1.9}$$

(1.10) for each $\varepsilon > 0$ there exists $\nu(\varepsilon) > 0$ such that $|x| \leqq \nu$ and $t \in \boldsymbol{R}^+$ imply that $|h(t, x)| \leqq \varepsilon |x|$,

and sometimes that

(1.11) for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|x_1| \leqq \delta$, $|x_2| \leqq \delta$, and $t \in \boldsymbol{R}^+$ imply that $|h(t, x_1) - h(t, x_2)| \leqq \varepsilon |x_1 - x_2|$.

Conditions (1.9)-(1.11) state that h vanishes faster than linearly at $x = 0$. Clearly, (1.10) implies (1.9), while (1.9) and (1.11) imply (1.10) with $\nu = \delta$.

The forcing term, q, will usually be assumed to satisfy

$$q \in L^\infty(\boldsymbol{R}^+, \boldsymbol{C}^N) \tag{1.12}$$

in addition to (1.4). Moreover, a restriction on the size of the norm

$$\|q\|_\infty = \sup_{0 \leqq t < \infty} |q(t)| , \tag{1.13}$$

where $|\ |$ is defined in (1.5), will be made. Sometimes it will be further assumed that

$$q = \omega + f , \tag{1.14}$$

where ω is regarded as the principal component of q and f as a perturbation. Thus, for example,

$$\omega \in C(\boldsymbol{R}^1, \boldsymbol{C}^N) , \quad \omega(t) = \omega(t + \rho) \quad (-\infty < t < \infty) \tag{1.15}$$

or

$$\omega \in \mathrm{AP}(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.16}$$

will later be assumed as well as

$$f \in L^\infty \cap C(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.17}$$

$$\lim_{t \to \infty} f(t) = 0 . \tag{1.18}$$

If ω is not identically constant in (1.15), ρ is taken as the least positive period of ω. In (1.16), AP denotes almost periodic in the sense of Bohr.

The resolvent, r, of the kernel, a, is defined to be the solution of

$$r(t) + \int_0^t a(t-s)r(s)ds = a(t) \quad (0 \leqq t < \infty) . \tag{1.19}$$

Relevant known results concerning r are collected in Section 2. One of these implies

LEMMA 1.1. *If* (1.2) *holds, then* (1.19) *has a unique solution*, r, *which satisfies*

$$r \in L^1_{\text{loc}}(\boldsymbol{R}^+, \boldsymbol{C}^{N^2}) . \tag{1.20}$$

The critical assumption of our main results is that

$$r \in L^1(\boldsymbol{R}^+, \boldsymbol{C}^{N^2}) , \tag{1.21}$$

that is

$$\|r\|_1 = \int_0^\infty |r(s)|ds < \infty , \tag{1.22}$$

where $|\ |$ is defined in (1.5), which is much stronger than (1.20). A famous theorem of Paley and Wiener [13, p. 60] gives a necessary and sufficient condition for the satisfaction of (1.21) when

$$a \in L^1(\boldsymbol{R}^+, \boldsymbol{C}^{N^2}) \tag{1.23}$$

holds. Some important kernels satisfying (1.2) but not (1.23), have the property that their associated resolvent satisfies (1.21), see, e.g., [7] and [14].

In most of the proofs the equation

$$x(t) + \int_0^t r(t-s)h(s, x(s))ds = \theta(t) \quad (0 \leqq t < \infty) , \tag{1.24}$$

where

$$\theta(t) = q(t) - \int_0^t r(t-s)q(s)ds \quad (0 \leqq t < \infty) , \tag{1.25}$$

plays an essential role. Lemma 2.1(ii) states that θ is the solution of (1.8). Lemma 2.1(iv) states that a solution of (1.1) is also a solution of (1.24). Under hypothesis (1.21), this has the effect of replacing a kernel, a, which need not be in L^1 with one that is. If assumption (1.10) or (1.11) is also made, there is the further effect of having the unknown appear in the integral term in a "small" manner.

We shall employ the

DEFINITION. A nondecreasing function $\Gamma: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is called an a priori bound for (1.1) if

$$x \in C([0, \hat{t}), \boldsymbol{C}^N) \text{ satisfies (1.1) on } [0, \hat{t}) , \tag{1.26}$$

for some $\hat{t} \in (0, \infty]$, implies that

$$|x(t)| \leqq \Gamma(t) \quad (0 \leqq t < \hat{t}) . \tag{1.27}$$

The first result establishes an a priori bound for (1.1). Its proof is given in Section 3.

THEOREM 1. *Let* (1.2)-(1.4), (1.10), (1.12), *and* (1.21) *hold. Furthermore, let*

$$||q||_\infty \leqq (1-\alpha)\nu(\alpha/||r||_1)/(1+||r||_1) \quad (if \quad ||r||_1 > 0) \tag{1.28}$$

hold for some $\alpha \in (0, 1)$. *Then*

$$\Gamma_1(t) \equiv \begin{cases} (1+||r||_1)||q||_\infty/(1-\alpha) & (if \quad ||r||_1 > 0) \\ ||q||_\infty & (if \quad ||r||_1 = 0) \end{cases} \tag{1.29}$$

is an a priori bound for (1.1).

An immediate consequence of Theorem 1 and a well known existence theorem and continuation procedure (see, e.g., [5, p. 877]) is:

COROLLARY 1a. *Let the hypothesis of Theorem* 1 *hold. Then* (1.1) *has a continuous solution on* $\boldsymbol{R}^+$, *and* (1.26) *with* $\hat{t} < \infty$ *implies that* x *can be continuously extended to* $\boldsymbol{R}^+$ *as a solution of* (1.1).

Combining (1.27) and (1.29) yields

$$|x(t)| \leqq (1+||r||_1)||q||_\infty/(1-\alpha) \qquad (0 \leqq t < \hat{t}) \tag{1.30}$$

whenever (1.26) holds. If $\hat{t} = \infty$, (1.30) may be regarded as a stability result for (1.1), in the sense that it shows that $||x||_\infty \to 0$ as $||q||_\infty \to 0$.

Another byproduct of Theorem 1 is the following asymptotic equivalence result between solutions of (1.1) and (1.8). Its proof, given in Section 3, is an immediate consequence of (1.24) and Lemma 2.2 below.

COROLLARY 1b. *Let the hypothesis of Theorem* 1 *as well as*

$$\lim_{t\to\infty} \left\{\max_{|x| \leqq K} |h(t, x)|\right\} = 0 , \tag{1.31}$$

for each $K \in \boldsymbol{R}^+$, *hold. Then*

$$\lim_{t\to\infty} [x(t) - \theta(t)] = 0 \tag{1.32}$$

is satisfied by every solution $x \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$ *of* (1.1).

We now consider the asymptotic behavior as $t \to \infty$ of the solution of (1.1) without hypothesis (1.31). Since (1.11) is assumed here, (1.1) has a unique solution. Assumption (1.21) is also made. The central tool in this study is the auxiliary equation

$$(1.33) \qquad y(t) + \int_0^\infty r(s)H(t-s,\, y(t-s))ds = \psi(t) \quad (-\infty < t < \infty)\ .$$

Unlike (1.1), (1.33) is defined on $\boldsymbol{R}^1$. Later it will be seen that in certain special cases, which are not the only ones investigated here, (1.33) is reduced to the limit equation, in the sense of [6], associated with (1.24).

It is assumed that

(1.34) $H(t, x) = h(t, x)$ $(0 \leqq t < \infty,\ \ x \in \boldsymbol{C}^N)$, $H \in C(\boldsymbol{R}^1 \times \boldsymbol{C}^N, \boldsymbol{C}^N)$, $H(t, 0) = 0$ $(-\infty < t < \infty)$, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|x_1| \leqq \delta(\varepsilon)$, $|x_2| \leqq \delta(\varepsilon)$, and $t \in \boldsymbol{R}^1$ imply that $|H(t, x_1) - H(t, x_2)| \leqq \varepsilon|x_1 - x_2|$ where δ is as in (1.11) .

These conditions state that H is an extension of h, which is assumed to satisfy (1.3), (1.9), and (1.11), to $\boldsymbol{R}^1 \times \boldsymbol{C}^N$ in such a manner that the analogous conditions hold on $\boldsymbol{R}^1 \times \boldsymbol{C}^N$. For any h satisfying (1.3), (1.9), and (1.11) it is obvious that setting $H(t, x) = h(t, x)$ $(t \geqq 0)$ and, for example, any of

$$H(t, x) = 0, \quad H(t, x) = h(-t, x), \quad H(t, x) = -h(-t, x) \quad (t \leqq 0)$$

defines an H for which (1.34) holds. Thus, (1.34) is not an additional restriction on h beyond (1.3), (1.9), and (1.11). In the important special case

$$(1.35) \qquad h(t, x) = h(x) \quad (0 \leqq t < \infty)\ ,$$

the most "natural" definition of H is

$$(1.36) \qquad H(t, x) = h(x) \quad (-\infty < t < \infty)\ .$$

Other special cases, which will be discussed later, in which there is a natural extension of h are described by

(1.37) (1.34) holds and $H(t, x) = H(t + \rho, x)$ $(-\infty < t < \infty)$,

where $\rho > 0$, and by

(1.38) (1.34) holds and $H(t, x)$ is almost periodic with respect to t, uniformly with respect x, for x in any compact subset of $\boldsymbol{C}^N$.

The function ψ in (1.33) will, ultimately, be related to θ and, therefore, to q. However, it is convenient to first regard ψ as an independent function satisfying

$$\psi \in L^\infty(\mathbf{R}^1, \mathbf{C}^N) \tag{1.39}$$

with, in contrast to (1.13),

$$\|\psi\|_\infty = \operatorname*{ess\,sup}_{-\infty<t<\infty} |\psi(t)| , \tag{1.40}$$

where $|\ \ |$ is defined in (1.5).

For small $\|\psi\|_\infty$, the existence and uniqueness of solutions of (1.33) is established in the following result, whose proof is sketched in Section 4. The notation $y = y_\psi$, below, emphasizes the dependence of the solution of (1.33) on ψ.

LEMMA 1.2. *Let* (1.21), (1.34), *and* (1.39) *hold. Furthemore, let*

$$\|\psi\|_\infty \leqq (1-\alpha)\delta(\alpha/\|r\|_1) \quad (\textit{if} \quad \|r\|_1 > 0) \tag{1.41}$$

hold for some $\alpha \in (0, 1)$. *Then* (1.33) *has a solution* $y_\psi \in L^\infty(\mathbf{R}^1, \mathbf{C}^N)$. *Moreover,*

$$\|y_\psi - y_n\|_\infty \leqq \alpha^{n+1}\|\psi\|_\infty/(1-\alpha) \quad (n = 0, 1, \cdots) \tag{1.42}$$

$$\|y_\psi\|_\infty \leqq \|\psi\|_\infty/(1-\alpha) , \tag{1.43}$$

where

$$\begin{aligned} &y_0(t) = \psi(t) \\ &y_{n+1}(t) = \psi(t) - \int_0^\infty r(s)H(t-s,\ y_n(t-s))ds \\ &\qquad\qquad\qquad\qquad (-\infty < t < \infty, \quad n = 0, 1, \cdots) . \end{aligned} \tag{1.44}$$

Also, $y = y_\psi$ *is the unique solution of* (1.33) *such that* $\|y\|_\infty \leqq \delta(\alpha/\|x\|_1)$.

Some properties of y_ψ, which are useful in the study of (1.1), are listed in the following corollaries. Thus, for example, if ψ and H are almost periodic (the latter in the sense of (1.38)), then so is y_ψ. C_u denotes the set of uniformly continuous functions. The proofs are almost immediate consequences of Lemma 1.2. In connection with (1.48), note that

$$\int_0^\infty r(s)H(t-s,\ y_n(t-s))ds = \int_{-\infty}^t r(t-s)H(s,\ y_n(s))ds ,$$

and in connection with (1.52) see [1. Chap. 2].

COROLLARY 1.2a. *Let the hypothesis of Lemma* 1.2 *hold. Then*

$$\psi \in C(\mathbf{R}^1, \mathbf{C}^N) \tag{1.45}$$

implies that

$$y_n,\ y_\psi \in C(\mathbf{R}^1, \mathbf{C}^N) \quad (n = 0, 1, \cdots) \tag{1.46}$$

and

$$\psi \in C_u(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.47}$$

implies that

(1.48) y_n $(n = 0, 1, \cdots)$, y_ψ *are uniformly equicontinuous on* $\boldsymbol{R}^1$.

COROLLARY 1.2b. *Let the hypothesis of Lemma* 1.2 *hold. Then*
(i) (1.37), (1.45), *and*

$$\psi(t) = \psi(t + \rho) \quad (-\infty < t < \infty)\,, \tag{1.49}$$

where $\rho > 0$, *imply* (1.46) *and*

$$y_n(t) = y_n(t + \rho), \quad y_\psi(t) = y_\psi(t + \rho) \quad (-\infty<t<\infty, \quad n = 0, 1, \cdots)\,. \tag{1.50}$$

(ii) (1.38) *and*

$$\psi \in \mathrm{AP}(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.51}$$

imply that

$$y_n, y_\psi \in \mathrm{AP}(\boldsymbol{R}^1, \boldsymbol{C}^N)\,. \tag{1.52}$$

The asymptotic relationship between (1.1) and (1.33) is given in the next result, whose proof is given in Section 5.

THEOREM 2. *Let the hypotheses of Theorem* 1 *and Lemma* 1.2 *hold* (*for the same* $\alpha \in (0, 1)$). *In addition, let*

$$\lim_{t\to\infty} [\theta(t) - \psi(t)] = 0\,, \tag{1.53}$$

where θ *is defined by* (1.25). *Then*

$$\lim_{t\to\infty} [x(t) - y_\psi(t)] = 0\,, \tag{1.54}$$

where x *and* y_ψ *are the unique solutions of* (1.1) *and* (1.33), *respectively.*

We now show that (1.53) is not a further restriction on the prescribed functions a, h and q of (1.1). Let the assumptions of Theorem 2, excluding those relating to ψ (i.e., (1.39), (1.41) and (1.53)), hold. For simplicity also let $\|r\|_1 > 0$, although this is not necessary. Then (1.25), (1.28), and the remark following (1.11) imply that

$$\|\theta\|_\infty \leqq (1 - \alpha)\nu(\alpha/\|r\|_1)) = (1 - \alpha)\delta(\alpha/\|r\|_1)\,. \tag{1.55}$$

Hence, if ψ is defined by

$$\psi = \theta\,, \tag{1.56}$$

then (1.25) and (1.55) obviously imply that (1.39), (1.41) and (1.53) hold.

In Section 6 it is shown that the hypothesis of Theorem 2 does not imply that (1.1) and (1.33) are asymptotically equivalent in a sense comparable to that of (1.1) and (1.8) in Corollary 1b.

The following result, which is an immediate corollary of Theorem 2 and Corollaries 1.2a and 1.2b, shows how asymptotic properties of q (and corresponding additional assumptions on h) imply asymptotic properties of the solution, x, of (1.1). For example, if q is asymptotically periodic of period ρ, in the sense of (1.14), (1.15), (1.17), and (1.18), and if h is periodic of period ρ (or autonomous), in the sense of (1.37), then x is also asymptotically periodic of period ρ. Moreover, if the ψ in (1.33) is taken to be an appropriate periodic function, θ_0 below, rather than θ as in (1.56), then the periodic component of x is given by y_{θ_0}.

COROLLARY 2. *Let the hypothesis of Theorem* 1 *as well as* (1.34), (1.14), (1.7) *and* (1.18) *hold. In addition, assume that*

$$\omega \in L^\infty \cap C(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.57}$$

$$\|\omega\|_\infty \leqq (1-\alpha)\delta(\alpha/\|r\|_1)/(1+\|r\|_1) \quad (if \quad \|r\|_1 > 0)\,, \tag{1.58}$$

where $\alpha \in (0, 1)$ *is the same in* (1.58) *and* (1.28). *Further, let*

$$\theta_0(t) = \omega(t) - \int_0^\infty r(s)\omega(t-s)ds \quad (-\infty < t < \infty)\,. \tag{1.59}$$

$$\theta_1(t) = f(t) - \int_0^t r(s)f(t-s)ds + \int_t^\infty r(s)\omega(t-s)ds \quad (0 \leqq t < \infty)\,. \tag{1.60}$$

Then

$$\theta(t) = \theta_0(t) + \theta_1(t) \tag{1.61}$$

$$\theta_0 \in L^\infty \cap C(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{1.62}$$

$$\|\theta_0\|_\infty \leqq (1-\alpha)\delta(\alpha/\|r\|_1) \quad (if \quad \|r\|_1 > 0) \tag{1.63}$$

$$\theta_1 \in L^\infty \cap C(\boldsymbol{R}^+, \boldsymbol{C}^N)\,, \quad \lim_{t\to\infty} \theta_1(t) = 0 \tag{1.64}$$

$$\lim_{t\to\infty} [x(t) - y_{\theta_0}(t)] = 0\,, \tag{1.65}$$

where x *and* y_{θ_0} *are the unique solutions of* (1.1) *and* (1.33) (*with* $\psi = \theta_0$ *in the latter*), *respectively.*

Moreover,

(i) (1.15) *and* (1.37) (*or* (1.36)) *imply that*

$$y_{\theta_0} \in C(\boldsymbol{R}^1, \boldsymbol{C}^N)\,, \quad \theta_0(t) = \theta_0(t+\rho)\,, \quad y_{\theta_0}(t) = y_{\theta_0}(t+\rho) \quad (-\infty < t < \infty)\,, \tag{1.66}$$

(ii) (1.16) *and* (1.38) (*or* (1.36)) *imply that*

$$\theta_0,\, y_{\theta_0} \in \mathrm{AP}(\boldsymbol{R}^1, \boldsymbol{C}^N)\,. \tag{1.67}$$

In Corollary 2(i), x is the unique solution of

$$(1.68)\quad x(t)+\int_0^t r(s)H(t-s,\,x(t-s))ds=\theta_0(t)+\theta_1(t)\quad(0\leqq t<\infty)$$

as well as of (1.1). Also, y_{θ_0} is the unique solution of

$$(1.69)\quad y(t)+\int_0^\infty r(s)H(t-s,\,y(t-s))ds=\theta_0(t)\quad(-\infty<t<\infty)\,.$$

It follows from (1.62), (1.64), $\theta_0(t)\equiv\theta_0(t+\rho)$, (1.37), and Theorem 14b of [6] that (1.69) is the limit equation associated with (1.68). (Note, in [6] it is explained how Theorem 14a is modified in order to obtain Theorem 14b.) In this case the conclusion (1.65), thus, also follows from Theorem 14b of [6] and the present Lemma 1.2; the latter is required for the uniqueness of the solution of (1.69). However, in Corollary 2(ii), (1.69) is only one of a family of limit equations associated with (1.68); see [6, p. 567] for more on this point. Hence, the results of [6] do not yield (1.65) in this case. Of course, in the general situation, to which Theorem 2 and the first part of Corollary 2 pertain, the results of [6] are not applicable.

2. **Preliminaries.** Parts (i)-(iii) of the following lemma contain well known results for (1.8) and (1.19); see, e.g., [8, Chap. 4]. Note that (1.4) and (1.12) are not hypotheses of (i) and that Lemma 1.1 is an immediate consequence of (i). Also, while (iv) is concerned with (1.1), it follows from (ii), (iii), (1.19), and a straightforward calculation. None of (i)-(iv) invoke (1.21).

LEMMA 2.1. *Let* (1.2) *be satisfied. Then*
(i)

$$(2.1)\qquad q\in L^p_{\mathrm{loc}}(\boldsymbol{R}^+,\,\boldsymbol{C}^N)$$

for some $p\in[1,\infty]$, *implies that* (1.8) *has a unique solution* $x\in L^p_{\mathrm{loc}}(\boldsymbol{R}^+,\,\boldsymbol{C}^N)$.
(ii) (2.1) *implies that the solution of* (1.8) *may be written as*

$$(2.2)\qquad x(t)=q(t)-\int_0^t r(t-s)q(s)ds\quad(0\leqq t<\infty)\,.$$

(iii)

$$(2.3)\qquad \int_0^t a(t-s)r(s)ds=\int_0^t r(t-s)a(s)ds\quad(0\leqq t<\infty)\,.$$

(iv) (1.3), (1.4), *and* (1.26) *imply that*

$$(2.4)\qquad x(t)+\int_0^t r(t-s)h(s,\,x(s))ds=\theta(t)\quad(0\leqq t<\hat{t})\,,$$

where θ *is defined by* (1.25).

The following elementary lemma is employed in several proofs.

LEMMA 2.2. *Let* $\alpha \in L^1(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *and let* $\beta \in L^\infty(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *be satisfied. Then*

$$\limsup_{t\to\infty} \int_0^t \alpha(s)\beta(t-s)ds \leqq \|\alpha\|_1 \limsup_{t\to\infty} \beta(t) .$$

To verify this, let $\lambda = \limsup_{t\to\infty} \beta(t)$ and let $\varepsilon > 0$. Then $\beta(t) \leqq \lambda + \varepsilon$ on $[T, \infty)$ for some $T = T(\varepsilon)$. Hence, on $[T, \infty)$,

$$\int_0^t \alpha(s)\beta(t-s)ds \leqq (\lambda+\varepsilon)\int_0^{t-T} \alpha(s)ds + \|\beta\|_\infty \int_{t-T}^t \alpha(s)ds .$$

Letting $t \to \infty$ implies that

$$\limsup_{t\to\infty} \int_0^t \alpha(s)\beta(t-s)ds \leqq (\lambda+\varepsilon)\|\alpha\|_1 .$$

Since ε is arbitrary, the result now follows.

3. Proof of Theorem 1 and Corollary 1b. If $\|r\|_1 = 0$, then (1.19) implies that $\|a\|_\infty = 0$. Hence, (1.1) reduces to $x(t) = q(t)$ and (1.29) is obviously an a priori bound for (1.1).

Let $\|r\|_1 > 0$ and let (1.28) hold for some $\alpha \in (0, 1)$. Then (1.25) and (1.28) imply that

$$\|\theta\|_\infty \leqq (1 + \|r\|_1)\|q\|_\infty \leqq (1-\alpha)\nu(\alpha/\|r\|_1) . \tag{3.1}$$

Let (1.26) hold. This, by Lemma 2.1 (iv), (2.4) holds. From (2.4) and (3.1) it follows that

$$|x(0)| = |\theta(0)| \leqq (1-\alpha)\nu(\alpha/\|r\|_1) . \tag{3.2}$$

We first show that

$$|x(t)| < \nu(\alpha/\|r\|_1) \quad (0 \leqq t < \hat{t}) . \tag{3.3}$$

If (3.3) does not hold, then (3.2) and (1.26) imply that there exists a $\tilde{t} \in (0, \hat{t})$ such that

$$|x(t)| < \nu(\alpha/\|r\|_1) \quad (0 \leqq t < \tilde{t}) , \quad |x(\tilde{t})| = \nu(\alpha/\|r\|_1) . \tag{3.4}$$

Successively employing (3.4), (3.1), (2.4), (1.10), and (3.4) yields

$$\begin{aligned} \alpha\nu(\alpha/\|r\|_1) &= (1-(1-\alpha))\nu(\alpha/\|r\|_1) \leqq |x(\tilde{t})| - |\theta(\tilde{t})| \\ &\leqq |x(\tilde{t}) - \theta(\tilde{t})| \leqq \int_0^{\tilde{t}} |r(\tilde{t}-s)|\,|h(s, x(s))|ds \\ &\leqq (\alpha/\|r\|_1)\int_0^{\tilde{t}} |r(\tilde{t}-s)|\,|x(s)|ds < \alpha\nu(\alpha/\|r\|_1) , \end{aligned}$$

a contradiction. Thus (3.3) is established.

From (2.4), (3.3), (1.10), and (3.1) it follows that, for $t \in [0, \hat{t})$,

$$\begin{aligned} |x(t)| &\leqq \|r\|_1 \max_{0 \leqq s \leqq t} |h(s, x(s))| + \|\theta\|_\infty \\ &\leqq \|r\|_1(\alpha/\|r\|_1) \max_{0 \leqq s \leqq t} |x(s)| + (1 + \|r\|_1)\|q\|_\infty . \end{aligned}$$

Hence, (1.30) holds, which, in view of (1.26), (1.27) and (1.29), establishes Theorem 1.

In order to establish Corollary 1b, let $x \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$ be a solution of (1.1). From (1.24), Theorem 1, and (1.31) it follows that

$$|x(t) - \theta(t)| \leqq \int_0^t |r(s)|\,|h(t - s, x(t - s))|ds \quad (0 \leqq t < \infty) ,$$
$$\lim_{t \to \infty} |h(t, x(t))| = 0 ,$$

which together with (1.21) and Lemma 2.2 implies (1.32) and proves Corollary 1b.

4. **Proof of Lemma 1.2.** Since the lemma is trivially true if $\|r\|_1 = 0$, suppose that $\|r\|_1 > 0$. Let (1.41) hold and define $\{y_n\}$ by (1.44). An elementary induction employing (1.34), with $\varepsilon = \alpha/\|r\|_1$, shows that

$$y_n \in L^\infty(\boldsymbol{R}^1, \boldsymbol{C}^N) \tag{4.1}$$

$$\|y_{n+1} - y_n\|_\infty \leqq \alpha^{n+1}\|\psi\|_\infty \tag{4.2}$$

$$\|y_n\|_\infty \leqq \|\psi\|_\infty \sum_{k=0}^n \alpha^k \leqq \|\psi\|_\infty/(1 - \alpha) \tag{4.3}$$

for $n = 0, 1, \cdots$. By (4.1) and (4.2), $\{y_n\}$ is a Cauchy sequence in $L^\infty(\boldsymbol{R}^1, \boldsymbol{C}^N)$. Hence there exists a $y_\psi \in L^\infty(\boldsymbol{R}^1, \boldsymbol{C}^N)$ such that

$$\lim_{n \to \infty} \|y_n - y_\psi\|_\infty = 0 . \tag{4.4}$$

From (4.3) and (4.4), (1.43) follows immediately. From (4.3), (4.4), (1.44), (1.21), and (1.34) it follows that y_ψ is a solution of (1.33). The triangle inequality and (4.2) yield

$$\|y_{n+m} - y_n\|_\infty \leqq \alpha^{n+1}\|\psi\|_\infty/(1 - \alpha) \quad (n, m = 1, 2, \cdots) ,$$

which together with (4.4) implies (1.42).

Suppose that $\mu_j \in L^\infty(\boldsymbol{R}^1, \boldsymbol{C}^N)$ $(j = 1, 2)$ are solutions of (1.33) with $\|\mu_j\|_\infty \leqq \delta(\alpha/\|r\|_1)$. Then (1.33) implies

$$\mu_1(t) - \mu_2(t) = -\int_0^\infty r(s)[H(t - s, \mu_1(t - s)) - H(t - s, \mu_2(t - s))]ds ,$$

which together with (1.34) yields

$$\|\mu_1 - \mu_2\|_\infty \leqq \|r\|_1(\alpha/\|r\|_1)\|\mu_1 - \mu_2\|_\infty = \alpha\|\mu_1 - \mu_2\|_\infty .$$

Hence $\|\mu_1 - \mu_2\|_\infty = 0$, which together with (1.33) implies that $\mu_1 = \mu_2$ and completes the proof.

5. **Proof of Theorem 2.** We first note that hypothesis (1.34) implies that (1.11) also holds and that Lemma 2.1(iv) implies that $x \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$ satisfying (1.1) is also a solution of (1.24). It follows from (1.11), (1.24) and an argument very similar to one employed in Section 4 that x is the unique solution of (1.24) (and (1.1)). If $\|r\|_1 = 0$, then (1.24) and (1.33) imply that $x(t) \equiv \theta(t)$ and $y_\psi(t) \equiv \psi(t)$, respectively. In this case, (1.54) follows immediately from (1.53). Suppose, therefore, that $\|r\|_1 > 0$.

From (1.27)-(1.29) we have

$$|x(t)| \leqq \nu(\alpha/\|r\|_1) = \delta(\alpha/\|r\|_1) \quad (0 \leqq t < \infty) , \tag{5.1}$$

where the equality is a consequence of the remark following (1.11). From (1.41) and (1.43) it follows that

$$\|y_\psi\|_\infty \leqq \delta(\alpha/\|r\|_1) . \tag{5.2}$$

From (1.24), (1.33) and (1.34) we have

$$\begin{aligned} x(t) - y_\psi(t) = &- \int_0^t r(s)[H(t-s, x(t-s)) - H(t-s, y_\psi(t-s))]ds \\ &+ \int_t^\infty r(s)H(t-s, y_\psi(t-s))ds + \theta(t) - \psi(t) \end{aligned} \tag{5.3}$$

on $\boldsymbol{R}^+$. From (5.1)-(5.3) and (1.34) it follows that

$$\begin{aligned} |x(t) - y_\psi(t)| \leqq &(\alpha/\|r\|_1)\int_0^t |r(s)|\,|x(t, s) - y_\psi(t-s)|ds \\ &+ (\alpha/\|r\|_1)\delta(\alpha/\|r\|_1)\int_t^\infty |r(s)|ds + |\theta(t) - \psi(t)| \end{aligned} \tag{5.4}$$

on $\boldsymbol{R}^+$. Letting $t \to \infty$ in (5.4) and invoking (1.21), (1.53) and Lemma 2.2 yields

$$\limsup_{t\to\infty} |x(t) - y_\psi(t)| \leqq \alpha \limsup_{t\to\infty} |x(t) - y_\psi(t)| ,$$

which, since $\alpha \in (0, 1)$ implies (1.54) and completes the proof.

6. **The non-asymptotic equivalence of** (1.1) **and** (1.33). Corollary 1b may be regarded as providing sufficient conditions for the asymptotic equivalence of (1.1) and (1.8). In contrast, Theorem 2 only gives sufficient conditions for a partial asymptotic equivalence of (1.1) and (1.33). That is, for each appropriate q there exist, as has been shown in Section 1, appropriate ψ's such that (1.53) and, therefore, (1.54) hold.

However, under the hypothesis of Theorem 2, this procedure can not, in general, be reversed. That is, starting with a ψ which satisfies (1.41), there does not, in general, exist an appropriate q. In order to illustrate this phenomenon, it is convenient to make two preliminary observations.

Equation (1.25) has, thus far, only served to define θ in terms of q. It can also be regarded as an equation in which θ is the independent function, satisfying

$$\theta \in C(\boldsymbol{R}^+, \boldsymbol{C}^N) , \tag{6.1}$$

and q is the unknown function. The following lemma covers this situation.

LEMMA 6.1. *Let* (1.2) *and* (6.1) *hold. Then* (1.25) *has a unique solution* $q \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$. *Moreover,*

$$q(t) = \theta(t) + \int_0^t a(t-s)\theta(s)ds \quad (0 \leqq t < \infty) . \tag{6.2}$$

In view of Lemma 1.1, (1.20) holds here. Hence, the existence and uniqueness assertions follow from (6.1) and Lemma 2.1(i). Comparison with (1.19) shows the resolvent associated with $-r$, and (thus) with (1.25), is defined as the unique solution, r_1, of

$$r_1(t) - \int_0^t r(t-s)r_1(s)ds = -r(t) \quad (0 \leqq t < \infty) . \tag{6.3}$$

Since (1.19) may, in view of (2.3), be written as

$$-a(t) - \int_0^t r(t-s)[-a(s)]ds = -r(t) \quad (0 \leqq t < \infty) , \tag{6.4}$$

it follows from (6.3), (6.4), and the uniqueness assertion of Lemma 1.1 that

$$r_1 = -a . \tag{6.5}$$

Applying Lemma 2.1(ii) to (1.25) and invoking (6.5) yields (6.2).

If

$$h(t, x) \equiv 0 \quad (0 \leqq t < \infty, \quad x \in \boldsymbol{C}^N) , \tag{6.6}$$

then, of course, (1.1) and (1.24) are linear equations and

$$(6.6) \text{ implies that } x = \theta . \tag{6.7}$$

Let H be consistent with (6.6) and (1.34) and let ψ satisfy (1.39) and (1.41) for some $\alpha \in (0, 1)$. Then (1.33) implies

$$y_\psi(t) + \int_{\max(0,t)}^\infty r(s)H(t-s, y_\psi(t-s))ds = \psi(t) \quad (-\infty < t < \infty) . \tag{6.8}$$

It follows from (1.21), (1.34), (1.43) and (6.8) that

$$(6.9) \qquad (6.6) \text{ implies that } \lim_{t\to\infty}[y_\psi(t) - \psi(t)] = 0 .$$

Consider the special case of (1.1), (1.24) and (1.33) in which

$$(6.10) \qquad N = 1 , \quad a(t) = t^{-1/2} , \quad (6.6) \text{ holds} .$$

The r determined by (6.10) and (1.19) satisfies (1.21) [8, p. 209]. Let $\alpha \in (0, 1)$ and let

$$(6.11) \qquad \psi(t) \equiv c \neq 0 \quad (-\infty < t < \infty) , \quad |c| \leqq (1-\alpha)\delta(\alpha/\|r\|_1) ,$$

as in (1.41). It is evident from (6.2), (6.10), and (6.11) that there does not exist a $q \in L^\infty$ such that (6.1) and (1.53) hold. Hence, in view of (6.6), (6.7), and (6.9), if (6.10) and (6.11) hold, then

$$x(t) - y_\psi(t) \not\to 0 \quad (t \to \infty) \quad \text{for all} \quad q \in L^\infty \cap C .$$

Thus, (6.10) and (6.11) illustrate the phenomenon described in the first paragraph of this section.

7. The nonconvolution equation. We now briefly indicate how the preceding considerations for (1.1) may be carried over to the nonconvolution equation (1.6).

The analogues of (1.8) and (1.19) are, respectively,

$$(7.1) \qquad x(t) + \int_0^t a(t, s)x(s)ds = q(t) \quad (0 \leqq t < \infty)$$

$$(7.2) \qquad r(t, s) + \int_s^t a(t, \xi)r(\xi, s)d\xi = a(t, s) \quad (0 \leqq s \leqq t < \infty) ,$$

where r is again called the resolvent of a. Unfortunately, there are no simple L^1 analogues of Lemmas 1.1 and 2.1 concerning (7.1) and (7.2). Since our concern here, as above, is with asymptotic behavior, we make the following common assumptions, which cover many interesting special cases:

(7.3) $a(t, s)$ is Lebesgue measurable in (t, s) and in t, s separately for $0 \leqq s \leqq t < \infty$; $a(t, \cdot) \in L^1([0, t], \boldsymbol{C}^{N^2})$ $(t \in \boldsymbol{R}^+)$; (7.2) has, with an absolutely convergent integral, a similarly measurable unique solution, $r(t, s)$; $r(t, \cdot) \in L^1([0, t], \boldsymbol{C}^{N^2})$ $(t \in \boldsymbol{R}^+)$;

$$\int_s^t a(t, \xi)r(\xi, s)d\xi = \int_s^t r(t, \xi)a(\xi, s)d\xi \quad (0 \leqq s \leqq t < \infty) ;$$

(7.1) has a unique solution for each q satisfying (1.4).

A sufficient condition for (7.3) is that $a(t, s)$ is continuous—for others see [8, Chap. 4] and [2]. Note that if (1.7) holds, the convolution case, it is easily shown that $r(t, s) = r(t - s)$.

From (7.3) it follows easily (as in the proof of Lemma 2.1) that, with (1.4) holding, the unique solution of (7.1) is given by

$$x(t) = q(t) - \int_0^t r(t, s)q(s)ds \quad (0 \leqq t < \infty) . \tag{7.4}$$

Moreover, with (1.3) also holding, (1.6) is equivalent to

$$x(t) + \int_0^t r(t, s)h(s, x(s))ds = \theta(t) \quad (0 \leqq t < \infty) , \tag{7.5}$$

where

$$\theta(t) = q(t) - \int_0^t r(t, s)q(s)ds \quad (0 \leqq t < \infty) . \tag{7.6}$$

Thus, (7.4), (7.5) and (7.6) are, respectively, the analogues of (2.2), (1.24) and (1.25).

Our key hypothesis, the analogue of (1.21), is that

$$\int_0^t |r(t, s)|ds = \int_0^t |r(t, t - s)|ds \leqq A < \infty \quad (0 \leqq t < \infty) \tag{7.7}$$

for some constant $A \in (0, \infty)$.

In order to guarantee that (7.1) (and (1.6), (7.5)) has continuous solutions when (1.4) holds, we further assume that

$$\begin{gathered} \text{for each } \varepsilon > 0 \text{ and } T \in \boldsymbol{R}^+ \text{ there exists } \delta_1(\varepsilon, T) > 0 \text{ such that} \\ \int_{t_1}^{t_2} |a(t_2, s)|ds + \int_0^{t_1} |a(t_2, s) - a(t_1, s)|ds \leqq \varepsilon \\ \text{whenever } 0 \leqq t_1 \leqq t_2 \leqq T, \quad t_2 \leqq t_1 + \delta_1 . \end{gathered} \tag{7.8}$$

Similar to Section 1, a nondecreasing function $\Gamma\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is called an a priori bound for (1.6) if

$$x \in C([0, \hat{t}), \boldsymbol{C}^N) \quad \text{satisfies (1.6) on } [0, \hat{t}) , \tag{7.9}$$

for some $\hat{t} \in (0, \infty]$, implies that

$$|x(t)| \leqq \Gamma(t) \quad (0 \leqq t < \hat{t}) . \tag{7.10}$$

Analogous to Theorem 1 we have

THEOREM 7.1. *Let* (1.3), (1.4), (1.10), (1.12), (7.3), (7.7) *and* (7.8) *hold. Furthermore, let*

$$(7.11)\qquad \|q\|_\infty \leqq (1-\alpha)\nu(\alpha/A)/(1+A)$$

hold for some $\alpha \in (0, 1)$. *Then*

$$(7.12)\qquad \Gamma_2(t) \equiv (1+A)\|q\|_\infty/(1-\alpha)$$

is an a priori bound for (1.6).

To see this observe that (7.3), (7.6), (7.7) and (7.11) imply

$$(7.13)\qquad \|\theta\|_\infty \leqq (1+A)\|q\|_\infty \leqq (1-\alpha)\nu(\alpha/A)\,.$$

Let (7.9) hold. Then (7.5) on $[0, \hat{t})$ and (7.13) imply

$$|x(0)| = |\theta(0)| \leqq (1-\alpha)\nu(\alpha/A)\,.$$

The same argument by contradiction of Section 3 now yields

$$|x(t)| < \nu(\alpha/A) \quad (0 \leqq t < \hat{t})\,,$$

which in turn similarly implies

$$|x(t)| \leqq (1+A)\|q\|_\infty/(1-\alpha) \quad (0 \leqq t < \hat{t})\,,$$

completing the proof.

The existence theorem and continuation procedure for (1.1) referred to in Section 1 is readily modified to apply to (1.6). It together with Theorem 1.1 readily yields

COROLLARY 7.1a. *Let the hypothesis of Theorem* 7.1 *hold. Then*: (1.6) *has a continuous solution on* $\boldsymbol{R}^+$, *and* (7.9) *with* $\hat{t} < \infty$ *implies that* x *can be continuously extended to* $\boldsymbol{R}^+$ *as a solution of* (1.6).

A trivial modification of the proof of Lemma 2.2 yieds

LEMMA 7.1. *Let* $\alpha(t, s) \geqq 0$ *be measurable with respect to* s *on* $[0, t]$ *for each* $t \in \boldsymbol{R}^+$ *and let*

$$(7.14)\qquad \sup_{0 \leqq t < \infty} \int_0^t \alpha(t, t-s)ds < \infty$$

$$(7.15)\qquad \lim_{t\to\infty} \int_{t-T}^t \alpha(t, t-s)ds = 0 \quad (0 \leqq T < \infty)\,.$$

Then, for each $\beta \in L^\infty(\boldsymbol{R}^+, \boldsymbol{R}^+)$,

$$\limsup_{t\to\infty} \int_0^t \alpha(t, t-s)\beta(t-s)ds \leqq \left[\sup_{0\leqq t<\infty} \int_0^t \alpha(t, t-s)ds\right] \limsup_{t\to\infty} \beta(t)\,.$$

It should be noted that if $\alpha(t, s) = |r(t, s)|$ and if (7.7) holds, then (7.14) is satisfied but (7.15) need not be. However, in the convolution case, $r(t, s) = r(t-s)$, (7.15) (with $\alpha(t, s) = |r(t, s)|$) is a consequence of (7.7). This accounts for the hypothesis (7.16) in the following corollary of Theorem 7.1 and Lemma 7.1, which is analogous to Corollary 1b.

COROLLARY 7.1b. *Let the hypothesis of Theorem* 7.1, (1.31), *and*

$$\lim_{t\to\infty}\int_{t-T}^{t}|r(t, t-s)|ds = 0 \quad (0 \leqq T < \infty) \tag{7.16}$$

hold. Then

$$\lim_{t\to\infty}[x(t) - \theta(t)] = 0$$

is satisfied by every solution $x \in C(\boldsymbol{R}^+, \boldsymbol{C}^N)$ *of* (1.6).

The auxiliary equation for (1.6) is

$$y(t) + \int_0^\infty R(t, t-s)H(t-s, y(t-s))ds = \psi(t) \quad (-\infty < t < \infty)\,. \tag{7.17}$$

It plays the same role for (1.6) that (1.33) does for (1.1). Moreover, the same hypotheses are made on H and ψ in (7.17) as were made on them in (1.33). Concerning R it is assumed that

$$\begin{aligned} &R(t, s) \text{ is measurable in } (t, s) \text{ and in } t, s \text{ separately for } t, s \in \boldsymbol{R}^1;\\ &R(t, s) = r(t, s) \text{ for } 0 \leqq s \leqq t < \infty;\\ &\int_{-\infty}^{t}|R(t, s)|ds = \int_0^\infty |R(t, t-s)|ds \leqq A < \infty \quad (-\infty < t < \infty)\,,\\ &\text{where } A \text{ is as in (7.7).}\end{aligned} \tag{7.18}$$

There is, obviously, no loss of generality in employing the same A in (7.7) and (7.18). Also, if (7.3) and (7.7) hold and if

$$R(t, s) = r(t, s) \text{ for } 0 \leqq s \leqq t < \infty\,, \quad R(t, s) = 0 \text{ otherwise}\,, \tag{7.19}$$

then $R(t, s)$ satisfies (7.18). Thus (7.18) is not an additional restriction on $r(t, s)$ beyond (7.3), (7.7). However, (7.19) won't be assumed.

Existence and uniqueness of solutions of (7.17) is considered in the following lemma, whose proof is completely analogous to that of Lemma 1.2.

LEMMA 7.2. *Let* (1.34), (1.39), *and* (7.18) *hold. Furthermore, let*

$$||\psi||_\infty \leqq (1-\alpha)\delta(\alpha/A)$$

hold for some $\alpha \in (0, 1)$. *Then* (7.17) *has a solution* $y_\psi \in L^\infty(\boldsymbol{R}^1, \boldsymbol{C}^N)$. *Moreover,* (1.42) *and* (1.43) *are satisfied, where*

$$\begin{aligned} y_0(t) &= \psi(t)\\ y_{n+1}(t) &= \psi(t) - \int_0^\infty R(t, t-s)H(t-s, y_n(t-s))ds\\ &\qquad\qquad (-\infty < t < \infty, \quad n = 0, 1, \cdots)\,.\end{aligned}$$

Also, $y = y_\psi$ *is the unique solution of* (7.17) *such that* $\|y\|_\infty \leqq \delta(\alpha/A)$.

Further assumptions on H, ψ, and R lead to additional properties on the y_n and y_ψ of Lemma 7.2. This is analogous to, but more technical than, Corollaries 1.2a and 1.2b; we omit the details.

In order to compare the solutions of (1.6) and (7.17) we need an additional assumption on R:

$$\lim_{t\to\infty}\int_t^\infty |R(t, t-s)|ds = 0 . \tag{7.20}$$

Of course, (7.20) is trivially satisfied if R satisfies (7.19). However, (7.19) is too restrictive an extension of r for some applications, e.g., when r is periodic in the sense that $r(t, s) = r(t+\rho, s+\rho)$ $(\rho > 0)$ (a consequence of such periodicity for a and (7.3)).

THEOREM 7.2. *Let* (7.16), (7.20), *and the hypotheses of Theorem* 7.1 *and Lemma* 7.2 *hold* (*for the same* $\alpha \in (0, 1)$). *In addition let* (1.53) *hold, where* θ *is defined by* (7.6). *Then* (1.54) *is satisfied, where* x *and* y_ψ *are the unique solutions of* (1.6) *and* (7.17), *respectively.*

The remarks in Section 1 concerning (1.53) not being an additional restriction also apply here—as does, with appropriate changes, the proof given in Section 5. Thus, for example, the analog of (5.3) on $\boldsymbol{R}^+$ is

$$\begin{aligned} x(t) - y_\psi(t) \\ &= -\int_0^t r(t, t-s)[H(t-s, x(t-s)) - H(t-s, y_\psi(t-s))]ds \\ &\quad + \int_t^\infty R(t, t-s)H(t-s, y_\psi(t-s))ds + \theta(t) - \psi(t) , \end{aligned}$$

which is analyzed with the aid of Lemma 7.1.

Corollaries of Theorem 7.2, analogous to Corollary 2 of Section 1, can be similarly formulated.

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