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STABILIZATION OF A CLASS OF BILINEAR CONTROL SYSTEMS WITH APPLICATIONS TO STEAM TURBINE REGULATION

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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We show by using a suitable Lyapunov function that a bilinear system may be stabilized by saturated linear feedback if some of the state variables remain positive. In the specific application to the steam turbine regulation a singular perturbations analysis is also performed.

1. **The control problem.** Consider a bilinear control system of the form

$$\begin{aligned} dx/dt &= A_{11}x + A_{12}y + A_{13}z + \textstyle\sum_1^k a_i^0 u^i + a \\ dy/dt &= A_{22}y + A_{23}z + \textstyle\sum_1^k b_i^0 u^i + \sum_{k+1}^r u^i b_i y^{l_i} + b \\ \varepsilon dz/dt &= A_{32}y + A_{33}z + \textstyle\sum_1^k c_i^0 u^i + \sum_{k+1}^r u^i c_i y^{l_i} + c\ . \end{aligned}$$

We have in the system r scalar controls; the first k appear in the usual linear way while the last $r-k$ appear multiplied by some state variables.

The specific form of the system corresponds to the model problem of steam turbine regulation we shall consider later. The admissible values of the controls are

$$0 \leqq u_{\min}^i \leqq u^i \leqq u_{\max}^i\ .$$

We assume further A_{23}, A_{32}, b_i^0, c_i^0, b, c, c_i with nonnegative elements, b_i^j nonnegative for $j \neq l_i$ A_{22}, A_{33} with nonnegative off-diagonal elements.

Under such assumptions if $y(0)$, $z(0)$ have nonnegative elements, then $y(t)$, $z(t)$ will have nonnegative elements for all $t \geqq 0$. A simple argument is obtained by multiplying y and z with a suitable exponential in order to make also the diagonal elements positive and then constructing the solution by successive approximations.

We shall assume further that A_{33} is Hurwitz.

REMARK. If A_{33} is Hurwitz and the off-diagonal elements of A_{33} are nonnegative, then all elements of A_{33}^{-1} are negative. Indeed since the off-diagonal elements of A_{33} are nonnegative, all elements of $e^{A_{33}t}$ are positive, hence all elements of $\int_0^\infty e^{A_{33}t}dt$ are positive; but

$$\int_0^\infty e^{A_{33}t}dt = A_{33}^{-1}\int_0^\infty A_{33}e^{A_{33}t}dt = -A_{33}^{-1}\,.$$

From this remark we deduce that if we simplify the model by letting $\varepsilon = 0$ the reduced system will preserve the invariance property. We have indeed

$$z = -A_{33}^{-1}(A_{32}y + \textstyle\sum_1^k c_i^0 u^i + \sum_{k+1}^r u^i c_i y^{l_i} + c)$$

and

$$\begin{aligned} dy/dt = (A_{22} - A_{23}A_{33}^{-1}A_{32})y + \textstyle\sum_1^k (b_i^0 - A_{23}A_{33}^{-1}c_i^0)u^i \\ + \textstyle\sum_{k+1}^r u^i(b^i - A_{23}A_{33}^{-1}c_i)y^{l_i} + b - A_{23}A_{33}^{-1}c\,, \end{aligned}$$

hence if $y(0) \geqq 0$ we still have $y(t) \geqq 0$ for all $t \geqq 0$ and also $z(t) \geqq 0$ for all $t \geqq 0$ (if z is defined from the equality above corresponding to the fast motion).

The control problem starts with the design of certain stationary solution and then to a construction of linear feedback corrections in order to keep this stationary solution.

2. Steady state design and the system for corrections. The steady state—constant solution of the system—is designed in order to keep given values for r outputs of the form $p_j^* x + q_j^* y$; we shall have thus the equations

$$\begin{aligned} &A_{11}x + A_{12}y + A_{13}z + \textstyle\sum_1^k a_i^0 u^i + a = 0 \\ &A_{22}y + A_{23}z + \textstyle\sum_1^k b_i^0 u^i + \sum_{k+1}^r u^i b_i y^{l_i} + b = 0 \\ &A_{32}y + A_{33}z + \textstyle\sum_1^k c_i^0 u^i + \sum_{k+1}^r u^i c_i y^{l_i} + c = 0 \\ &p_j^* x + q_j^* y = \rho_j^0\,, \qquad j = 1, \cdots, r\,. \end{aligned}$$

The number of equations is equal to the number of unknowns; A_{33} is invertible and we may start by solving with respect to z; we may then use the resulting equations and the conditions for the outputs to solve with respect to x, y and to the controls u^i. It is important to observe that if among the outputs we have the coordinates y^{l_i} the system to be solved is linear and has a unique solution. This is the practically important case as we shall see in the model problem of steam turbine.

Let x_0, y_0, z_0, u_0 be the steady state values, assumed to be unique and admissible, that is

$$u_{\min}^i \leqq u_0^i \leqq u_{\max}^i\,.$$

Denote

$$\tilde{x} = x - x_0\,, \qquad \tilde{y} = y - y_0\,, \qquad \tilde{z} = z - z_0\,, \qquad \tilde{u} = u - u_0$$

and write down the equations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ with $\tilde{u}$ as control; we have

$$\begin{aligned} d\tilde{x}/dt &= A_{11}\tilde{x} + A_{12}\tilde{y} + A_{13}\tilde{z} + \textstyle\sum_1^k a_i^0\tilde{u}^i \\ d\tilde{y}/dt &= (A_{22} + \textstyle\sum_{k+1}^r u_0^i B_i)\tilde{y} + A_{23}\tilde{z} + \sum_1^k b_i^0\tilde{u}^i + \sum_{k+1}^r \tilde{u}^i b_i(\tilde{y}^{l_i} + y_0^{l_i}) \\ \varepsilon d\tilde{z}/dt &= (A_{32} + \textstyle\sum_{k+1}^r u_0^i C_i)\tilde{y} + A_{33}\tilde{z} + \sum_1^k c_i^0\tilde{u}^i + \sum_{k+1}^r \tilde{u}^i c_i(\tilde{y}^{l_i} + y_0^{l_i})\,, \end{aligned}$$

where B_i has columns b_i on position l_i and zero for other positions and C_i has columns c_i on position l_i and zero for other positions.

We shall stabilize the reduced system obtained for $\varepsilon = 0$

$$\begin{aligned} \tilde{z} &= -A_{33}^{-1}[(A_{32} + \textstyle\sum_{k+1}^r u_0^i C_i)\tilde{y} + \sum_1^k c_i^0\tilde{u}^i + \sum_{k+1}^r \tilde{u}^i c_i(\tilde{y}^{l_i} + y_0^{l_i})] \\ d\tilde{x}/dt &= A_{11}\tilde{x} + [A_{12} - A_{13}A_{33}^{-1}(A_{32} + \textstyle\sum_{k+1}^r u_0^i C_i)]\tilde{y} \\ &\quad + \textstyle\sum_1^k (a_i^0 - A_{13}A_{33}^{-1}c_i^0)\tilde{u}^i - A_{13}A_{33}^{-1}\sum_{k+1}^r \tilde{u}^i c_i(\tilde{y}^{l_i} + y_0^{l_i}) \\ d\tilde{y}/dt &= [A_{22} - A_{23}A_{33}^{-1}A_{32} + \textstyle\sum_{k+1}^r u_0^i(B_i - A_{23}A_{33}^{-1}C_i)]\tilde{y} \\ &\quad + \textstyle\sum_1^k (b_i^0 - A_{23}A_{33}^{-1}c_i^0)\tilde{u}^i + \sum_{k+1}^r \tilde{u}^i(b_i - A_{23}A_{33}^{-1}c_i)(\tilde{y}^{l_i} + y_0^{l_i})\,. \end{aligned}$$

To simplify notation let

$$\begin{aligned} \tilde{A}_{12} &= A_{12} - A_{13}A_{33}^{-1}(A_{32} + \textstyle\sum_{k+1}^r u_0^i C_i) \\ \tilde{A}_{22} &= A_{22} - A_{23}A_{33}^{-1}A_{32} + \textstyle\sum_{k+1}^r u_0^i(B_i - A_{23}A_{33}^{-1}C_i) \\ \tilde{a}_i^0 &= a_i^0 - A_{13}A_{33}^{-1}c_i^0\,, \qquad \tilde{a}_i = -A_{13}A_{33}^{-1}c_i \\ \tilde{b}_i^0 &= b_i^0 - A_{23}A_{33}^{-1}c_i^0\,, \qquad \tilde{b}_i = b_i - A_{23}A_{33}^{-1}c_i \end{aligned}$$

and consider finally the system

$$(\Sigma) \qquad \begin{cases} d\tilde{x}/dt = A_{11}\tilde{x} + \tilde{A}_{12}\tilde{y} + \sum_1^k \tilde{a}_i^0\tilde{u}^i + \sum_{k+1}^r \tilde{a}_i\tilde{u}^i(\tilde{y}^{l_i} + y_0^{l_i}) \\ d\tilde{y}/dt = \tilde{A}_{22}\tilde{y} + \sum_1^k \tilde{b}_i^0\tilde{u}^i + \sum_{k+1}^r \tilde{b}_i\tilde{u}^i(\tilde{y}^{l_i} + y_0^{l_i})\,. \end{cases}$$

We introduce now the *main stability assumption.*

The matrix A_{22} is Hurwitz and A_{11} has all eigenvalues on the imaginary axis with simple elementary divisors.

3. The stabilizing feedback. We choose first

$$T = \begin{pmatrix} I & T_{12} \\ 0 & I \end{pmatrix}$$

such that

$$T\begin{pmatrix} A_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & \tilde{A}_{22} \end{pmatrix} T\,.$$

Here T_{12} is defined by the equation

$$T_{12}\widetilde{A}_{22} - A_{11}T_{12} = -\widetilde{A}_{12} .$$

This equation has a solution since $\widetilde{A}_{22}$ and $-A_{11}$ have no common eigenvalues.

The main stability assumption shows now that there exist $P_1 > 0$, $P_2 > 0$ such that

$$P_1A_{11} + A_{11}^*P_1 = 0$$

$$P_2\widetilde{A}_{22} + \widetilde{A}_{22}^*P_2 = -I$$

($P_1 = S^*S$ where S is such that $SA_{11}S^{-1}$ is diagonal). Define the Lyapunov function

$$V(\widetilde{x}, \widetilde{y}) = (\widetilde{x}^*, \widetilde{y}^*)T^*\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} T\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = (\widetilde{x}^* + \widetilde{y}^*T_{12}^*)P_1(\widetilde{x} + T_{12}\widetilde{y}) + \widetilde{y}^*P_2\widetilde{y} .$$

We have $V(\widetilde{x}, \widetilde{y}) > 0$ for $|\widetilde{x}| + |\widetilde{y}| \neq 0$. Compute the derivative with respect to our system (Σ)

$$\begin{aligned} dV/dt = {} & \{\widetilde{x}^*A_{11}^* + \widetilde{y}^*\widetilde{A}_{12}^* + \textstyle\sum_1^k \widetilde{u}^i\widetilde{a}_i^{0*} + \sum_{k+1}^r \widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i})\widetilde{a}_i^* + [\widetilde{y}^*\widetilde{A}_{22}^* \\ & + \textstyle\sum_1^k \widetilde{u}^i\widetilde{b}_i^{0*} + \sum_{k+1}^r \widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i})\widetilde{b}_i^*]T_{12}^*\}P_1(\widetilde{x} + T_{12}\widetilde{y}) \\ & + (\widetilde{x}^* + \widetilde{y}^*T_{12}^*)P_1\{A_{11}\widetilde{x} + \widetilde{A}_{12}\widetilde{y} + \textstyle\sum_1^k \widetilde{a}_i^0\widetilde{u}^i + \sum_{k+1}^r \widetilde{a}_i\widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i}) \\ & + T_{12}[\widetilde{A}_{22}\widetilde{y} + \textstyle\sum_1^k \widetilde{b}_i^0\widetilde{u}^i + \sum_{k+1}^r \widetilde{b}_i\widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i})]\} + [\widetilde{y}^*\widetilde{A}_{22}^* + \sum_1^k \widetilde{u}^i\widetilde{b}_i^{0*} \\ & + \textstyle\sum_{k+1}^r \widetilde{u}^i(\widetilde{y}^{l_i}+y_0^{l_i})\widetilde{b}_i^*]P_2\widetilde{y}+\widetilde{y}^*P_2[\widetilde{A}_{22}\widetilde{y}+\sum_1^k \widetilde{b}_i^0\widetilde{u}^i+\sum_{k+1}^r \widetilde{b}_i\widetilde{u}^i(\widetilde{y}^{l_i}+y_0^{l_i})] \\ = {} & -\widetilde{y}^*\widetilde{y} + \textstyle\sum_1^k \widetilde{u}^i[(\widetilde{a}_i^{0*} + \widetilde{b}_i^{0*}T_{12}^*)P_1(\widetilde{x} + T_{12}\widetilde{y}) + (\widetilde{x}^* + \widetilde{y}^*T_{12}^*)P_1(\widetilde{a}_i^0 + T_{12}b_i^0) \\ & + \widetilde{b}_i^{0*}P_2\widetilde{y} + \widetilde{y}^*P_2\widetilde{b}_i^0] + \textstyle\sum_{k+1}^r \widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i})[(\widetilde{a}_i^* + \widetilde{b}_i^*T_{12}^*)P_1(\widetilde{x} + T_{12}\widetilde{y}) \\ & + (\widetilde{x}^* + \widetilde{y}^*T_{12}^*)P_1(\widetilde{a}_i + T_{12}\widetilde{b}_i) + \widetilde{b}_i^*P_2\widetilde{y} + \widetilde{y}^*P_2\widetilde{b}_i] . \end{aligned}$$

We choose now for $i = 1, \cdots, k$

$$\widetilde{u}^i = -\beta^i[(\widetilde{a}_i^{0*} + \widetilde{b}_i^{0*}T_{12}^*)P_1(\widetilde{x} + T_{12}\widetilde{y}) + \widetilde{b}_i^{0*}P_2\widetilde{y}] , \qquad \beta^i > 0$$

if this value is admissible (i.e., belongs to $[u_{\min}^i - u_0^i,\ u_{\max}^i - u_0^i]$), $\widetilde{u}^i = u_{\max}^i - u_0^i$ if the value exceeds $u_{\max}^i - u_0^i$, and $\widetilde{u}^i = u_{\min}^i - u_0^i$ if the value is less than $u_{\min}^i - u_0^i$.

For $i = k + 1, \cdots, r$ we choose

$$\widetilde{u}^i = -\gamma^i[(\widetilde{a}_i^* + \widetilde{b}_i^*T_{12}^*)P_1(\widetilde{x} + T_{12}\widetilde{y}) + \widetilde{b}_i^*P_2\widetilde{y}] , \qquad \gamma^i > 0$$

if this value is admissible, and as above if not. With such choice of the controls we get $dV/dt \leqq 0$; *we used here the fact that on the invariant set we are interested in we have* $\widetilde{y}^{l_i} + y_0^{l_i} = y^{l_i} \geqq 0$.

On the set where $dV/dt = 0$ we have clearly

$$\widetilde{y} = 0 ; \quad \widetilde{u}^i = 0 , \quad i = 1, \cdots, k ; \quad \widetilde{u}^i(\widetilde{y}^{l_i} + y_0^{l_i}) = 0 , \quad i = k + 1, \cdots, r$$

hence for any trajectory on this set

$$(*) \qquad (\tilde{a}_i^{0*} + \tilde{b}_i^{0*} T_{12}^*) P_1 e^{A_{11}t} \tilde{x}_0 \equiv 0 \qquad i = 1, \cdots, k .$$

Denote by Q the matrix with columns $P_1(\tilde{a}_i^0 + T_{12}\tilde{b}_i^0)$.

Assume that (A_{11}^*, Q) *is controllable*; this is a necessary and sufficient condition for $(*)$ to imply $\tilde{x}_0 = 0$. We see thus that this complementary assumption gives (by using Barbashin-Krasovski-La Salle theorem) global asymptotic stability.

In this way we have stabilized the system on the invariant set defined by $\tilde{y}^i + y_0^i \geqq 0$.

Several specific features do not allow us to use directly known results for singularly perturbed systems as the ones of [1], [2], [3], [4]. Among them the fact that the controls are only continuous functions of the state hence we do not have the required smoothness, and the global character of the stability we want to see preserved. It is why specific arguments have to be produced and we shall produce them for the model problem of steam turbine regulation.

4. Steam turbine regulation. The mathematical model of a steam turbine with one regulated bleeding is the following

$$\begin{aligned} T_a ds/dt &= \alpha_1\pi_1 + \alpha_2\pi_2 - \nu_g \\ T_T d\pi_T/dt &= -\beta_T\pi_T + \beta_1\pi_1 - \beta_2\mu_2\pi_T \\ T_1 d\pi_1/dt &= -\beta_1\pi_1 + \mu_1 \\ T_2 d\pi_2/dt &= -\beta_3\pi_2 + \mu_2\pi_T \end{aligned}$$

with the following admissible values of the controls

$$0 \leqq \mu_1 \leqq 1 , \qquad 0 < \gamma_2 \leqq \mu_2 \leqq 1 .$$

The time constants T_1/β_1 and T_2/β_3 associated with the steam volumes enclosed in the turbine cylinders are small. It can be easily seen that the above system belongs to the class defined previously. The system's coefficients are the following

$$A_{11} = 0 , \quad A_{12} = 0 , \quad A_{13} = (\alpha_1/T_a \;\; \alpha_2/T_a) , \quad a_i^0 = 0 , \quad a = -\nu_g/T_a$$

$$A_{22} = -\beta_T/T_T , \quad A_{23} = (\beta_1/T_T \;\; 0) , \quad b_i^0 = 0 , \quad b_2 = -\beta_2/T_T , \quad b = 0$$

$$A_{32} = 0 , \quad A_{33} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \quad c_1^0 = \begin{pmatrix} 1/\beta_1 \\ 0 \end{pmatrix} , \quad c_2 = \begin{pmatrix} 0 \\ 1/\beta_3 \end{pmatrix} , \quad c = 0 .$$

We have here $k = 1$, $r = 2$. It is obvious that the assumptions concerning the coefficients are fulfilled in this case. Also is A_{33} Hurwitz.

It follows that $\pi_T(t)$, $\pi_1(t)$, $\pi_2(t)$ are nonnegative for all $t \geqq 0$. (This can be proved also directly.)

Now, if we simplify the model by letting $T_1/\beta_1 = T_2/\beta_3 = 0$, the reduced model will have the form

$$T_a ds/dt = (\alpha_1/\beta_1)\mu_1 + (\alpha_2/\beta_3)\mu_2\pi_T - \nu_g$$
$$T_T d\pi_T/dt = \mu_1 - \beta_T\pi_T - \beta_2\mu_2\pi_T$$

and the invariance property $\pi_T(t) \geqq 0$ if $\pi_T(0) \geqq 0$ holds.

The steady state solution of the system is designed in order to keep given values for s and π_T. More precisely we shall have

$$\alpha_1\pi_1 + \alpha_2\pi_2 - \nu_g = 0\ , \qquad -\beta_T\pi_T + \beta_1\pi_1 - \beta_2\mu_2\pi_T = 0\ ,$$
$$-\beta_1\pi_1 + \mu_1 = 0\ , \qquad -\beta_3\pi_2 + \mu_2\pi_T = 0\ , \qquad s = 0\ , \qquad \pi_T = \pi_T^0\ .$$

It can be seen that this system has a unique solution (π_T corresponds here to y^{l_i} from the general case). It must be mentioned that π_T^0 and ν_g are such that this unique solution should be admissible, that is

$$0 \leqq \mu_1^0 \leqq 1\ , \qquad \gamma_2 \leqq \mu_2^0 \leqq 1\ .$$

Let π_T^0, π_1^0, π_2^0, μ_1^0, μ_2^0 be the steady state values. Denote

$$x_1 = s\ , \qquad x_2 = \pi_T - \pi_T^0\ , \qquad x_3 = \pi_1 - \pi_1^0\ , \qquad x_4 = \pi_2 - \pi_2^0\ ,$$
$$u_1 = \mu_1 - \mu_1^0\ , \qquad u_2 = \mu_2 - \mu_2^0\ .$$

We have

$$(**) \qquad \begin{cases} T_a dx_1/dt = \alpha_1 x_3 + \alpha_2 x_4 \\ T_T dx_2/dt = -(\beta_T + \beta_2\mu_2^0)x_2 + \beta_1 x_3 - \beta_2\pi_T^0 u_2 - \beta_2 u_2 x_2 \\ T_1 dx_3/dt = -\beta_1 x_3 + u_1 \\ T_2 dx_4/dt = \mu_2^0 x_2 - \beta_3 x_4 + \pi_T^0 u_2 + u_2 x_2\ , \end{cases}$$

where

$$-\mu_1^0 \leqq u_1 \leqq 1 - \mu_1^0\ , \qquad \gamma_2 - \mu_2^0 \leqq u_2 \leqq 1 - \mu_2^0\ .$$

One can see that for $u_1 = u_2 = 0$ this system has a trivial solution which is not asymptotically stable. The simple stability is not satisfactory from the practical point of view. Therefore the system needs stabilization and, more precisely, feedback stabilization.

We shall stabilize the reduced system obtained by neglecting the effect of the steam volumes ($T_1/\beta_1 = T_2/\beta_3 = 0$).

$$T_a dx_1/dt = (\alpha_2\mu_2^0/\beta_3)x_2 + (\alpha_1/\beta_1)u_1 + (\alpha_2\pi_T^0/\beta_3)u_2 + (\alpha_2/\beta_3)u_2 x_2$$
$$T_T dx_2/dt = -(\beta_T + \beta_2\mu_2^0)x_2 + u_1 - \beta_2\pi_T^0 u_2 - \beta_2 u_2 x_2\ .$$

We have

$$A_{11} = 0 , \qquad \tilde{A}_{22} = -(\beta_T + \beta_2\mu_2^0)/T_T ,$$

hence the main stability assumption is fulfilled.

Define the following Lyapunov function

$$V(x_1, x_2) = \frac{A}{2}T_a\Big(x_1 + \frac{T_T}{T_a}\frac{\alpha_2\mu_2^0}{\beta_3(\beta_T + \beta_2\mu_2^0)}x_2\Big)^2 + \frac{B}{2}T_T x_2^2$$

where $A > 0$, $B > 0$. Differentiating this function along the solutions of the reduced system we find

$$dV/dt = l_1(x_1(t), x_2(t))u_1 + (x_2 + \pi_T^0)l_2(x_1(t), x_2(t))u_2 - B(\beta_T + \beta_2\mu_2^0)[x_2(t)]^2 ,$$

where the linear forms $l_i(x_1, x_2)$ are given by

$$l_1(x_1, x_2) = A\Big(x_1 + \frac{T_T}{T_a}\frac{\alpha_2\mu_2^0}{\beta_3(\beta_T + \beta_2\mu_2^0)}x_2\Big)\Big(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2\mu_2^0}{\beta_3(\beta_T + \beta_2\mu_2^0)}\Big) + Bx_2$$

$$l_2(x_1, x_2) = A\frac{\alpha_2\beta_T}{\beta_3(\beta_T + \beta_2\mu_2^0)}\Big(x_1 + \frac{T_T}{T_a}\frac{\alpha_2\mu_2^0}{\beta_3(\beta_T + \beta_2\mu_2^0)}x_2\Big) - B\beta_2 x_2 .$$

We choose now

$$u_1(x_1, x_2) = -Cl_1(x_1, x_2) , \qquad C > 0$$

if this value is admissible, $u_1 = 1 - \mu_1^0$ if the value exceeds $1 - \mu_1^0$, and $u_1 = -\mu_1^0$ if the value is less than $-\mu_1^0$; $u_2(x_1, x_2) = -Dl_2(x_1, x_2)$, $D > 0$ if this value is admissible, $u_2 = 1 - \mu_2^0$ if this value exceeds $1 - \mu_2^0$, and $u_2 = \gamma_2 - \mu_2^0$ if the value is less than $\gamma_2 - \mu_2^0$.

Taking into account that on the invariant set we have $\pi_T^0 + x_2 \geqq 0$ it follows that

$$dV/dt \leqq -B(\beta_T + \beta_2\mu_2^0)[x_2(t)]^2 .$$

On the set where $dV/dt = 0$, $x_2 = 0$ and, because of the form of the equations, $x_1 = 0$. Therefore the global asymptotic stability with respect to the above mentioned invariant set follows from the theorem of Barbashin-Krasovski-La Salle.

In this way we obtained the stability of the reduced model. But the linear saturated feedback defined above must stabilize the complete model. In the following we shall prove, using the same Lyapunov function defined previously, that *the complete model with the feedback constructed for the reduced one, is globally asymptotically stable if the time constants T_1/β_1 and T_2/β_3 are small enough.*

The proof will be performed in several steps.

(a) We show that the complete system with u_i defined as above is locally asymptotically stable. In order to prove this we choose a neighborhood of the origin such that u_i should be linear functions of x_1 and x_2. Neglecting the higher order terms we get a fourth order linear system. Taking into account that T_1/β_1 and T_2/β_3 are small parameters and that the boundary layer system is stable, we need only the stability of the reduced second order system. Applying the Hurwitz criterion we get, after some lengthy but straightforward computation, that this second order linear system is stable. Using the first approximation stability theorem and singular perturbations, we get local stability for system (**), if T_1/β_1 and T_2/β_3 are sufficiently small.

(b) The solutions of (**) with the controls chosen as above are bounded. Indeed, using a method which is typical for singular perturbations we get

$$|x_3(t) - u_1(x_1(t), x_2(t))/\beta_1| \leqq k\sqrt{T_1/\beta_1} \qquad t \geqq t_0 .$$

From the equation of $\pi_1(t)$ it follows that

$$0 \leqq \pi_1(t) \leqq 2/\beta_1 \qquad t \geqq t_0$$

and from the equation of $\pi_T(t)$ we get

$$0 \leqq \pi_T(t) \leqq \pi_T(0) \exp(-(\beta_T + \beta_2\mu_2^0)(t/T_T)) + \beta_1(\beta_T + \beta_2\gamma_2)^{-1} \sup_{t>0} \pi_1(t)$$

hence $\pi_T(t)$ is bounded.

Using again the singular perturbations we find

$$|x_4(t) - (\mu_2^0 x_2(t) + u_2(x_1(t), x_2(t))\pi_T(t))/\beta_3| \leqq k\sqrt{T_2/\beta_3} \qquad t \geqq t_0 .$$

From the equation of $\pi_2(t)$ it follows that

$$0 \leqq \pi_2(t) \leqq \pi_2(0) \exp(-\beta_3 t/T_2) + (1/\beta_3) \sup_{t>0} \pi_T(t)$$

hence $\pi_2(t)$ is bounded.

It remains to show the boundedness of $x_1(t)$. This follows from the properties of the Lyapunov function. Taking into account the boundedness of $x_2(t)$ and the above relations, we have

$$\frac{dV}{dt} \leqq -A\left\{\left[\frac{\alpha_1}{\beta_1} + \frac{\alpha_2\mu_2^0}{\beta_3(\beta_T + \beta_2\mu_2^0)}\right]\min(\mu_1^0; 1-\mu_1^0) - \frac{\alpha_1}{\beta_1}k_1\sqrt{\frac{T_1}{\beta_1}} - \frac{\alpha_2}{\beta_3}k_2\sqrt{\frac{T_2}{\beta_3}}\right\}|x_1(t)| + \tilde{K} .$$

If T_1/β_1 and T_2/β_3 are sufficiently small, then

$$dV/dt \leqq -\hat{K}|x_1(t)| + \tilde{K}$$

for $t \geqq t_0$ and $|x_1| \geqq L$. Here L follows from the condition that $u_1(x_1(t), x_2(t))$ should be saturated (i.e., to equal either $-\mu_1^0$ or $1 - \mu_1^0$). If this control is not saturated, then $-\mu_1^0 \leqq l_1(x_1, x_2) \leqq 1 - \mu_1^0$ and, from the boundedness of $x_2(t)$ it follows that $x_1(t)$ is also bounded.

If dV/dt satisfies the above inequality, then there exists $t_1 > t_0$ such that

$$|x_1(t_1)| \leqq \tilde{K}/\hat{K} + L .$$

We prove this by contradiction, using the positivity of $V(x_1(t), x_2(t))$. Denote

$$l = \max_{|x_1| \leqq \tilde{K}/\hat{K}+L} V(x_1, x_2(t)) .$$

Obviously $l < \infty$ due to the boundedness of $x_2(t)$. We have also the inequality $V(x_1, x_2(t)) \geqq ax_1^2 - \hat{b}|x_1|$. Denote by $\hat{x}$ the positive solution of the equation

$$a\xi^2 - \hat{b}\xi = l + 1$$

where l has been defined above. In this case

$$|x_1(t)| < \hat{x} \quad \text{for all} \quad t \geqq t_1$$

where t_1 is the one defined above (the moment when $x_1(t)$ enters the interval $|x_1| \leqq \tilde{K}/\hat{K} + L$). Indeed, if this wouldn't be true, one could find $t_2 > t_1$ such that $|x_1(t_2)| > \hat{x}$, hence $V(x_1(t_2), x_2(t_2)) \geqq ax_1^2(t_2) - \hat{b}|x_1(t_2)| > l + 1$. But $|x_1(t_1)| \leqq \tilde{K}/\hat{K} + L$, hence $V(x_1(t_1), x_2(t_1)) \leqq l$. It follows that there exists $t_3 \in (t_1, t_2)$ such that

$$V(x_1(t_3), x_2(t_3)) = l + 1$$

and

$$V(x_1(t), x_2(t)) \geqq l + 1 \quad \text{for} \quad t_3 \leqq t \leqq t_2 .$$

Therefore $|x_1(t_3)| > \tilde{K}/\hat{K} + L$, $(dV/dt)(t_3) \geqq 0$ hence $|x_1(t_3)| \leqq \tilde{K}/\hat{K}$. This contradiction proves the boundedness of $x_1(t)$.

(c) Using again the properties of the Lyapunov function it is possible to show that for a given δ_0 there exists $\hat{t} \geqq t_0$ such that $|x_1(\hat{t})| + |x_2(\hat{t})| \leqq \delta_0$. Indeed, after some simple manipulation we find

$$\begin{aligned} dV/dt \leqq -B(\beta_T + \beta_2\mu_2^0)[x_2(t)]^2 - \min\{(1-\mu_1^0)^2/c, (\mu_1^0)^2/c\,, \\ cl_3^2(x_1(t), x_2(t))\} + \tilde{k}_1\sqrt{T_1/\beta_1} + \tilde{k}_2\sqrt{T_2/\beta_3} \qquad t > t_0 . \end{aligned}$$

Now, if our statement is not true, then

$$|x_1(t)| + |x_2(t)| > \delta_0 \quad \text{for all} \quad t > t_0 .$$

On the set $|x_1| + |x_2| > \delta_0$ we have

$$(d/dt)V^*(t) = (d/dt)V(x_1(t), x_2(t)) \leqq \tilde{k}_1\sqrt{T_1/\beta_1} + \tilde{k}_2\sqrt{T_2/\beta_3} - \alpha(\delta_0)$$

where $\alpha(\delta_0) > 0$. If T_1/β_1 and T_2/β_3 are small enough, e.g., such that $\tilde{k}_1\sqrt{T_1/\beta_1} + \tilde{k}_2\sqrt{T_2/\beta_3} < \alpha(\delta_0)/2$ then $(d/dt)V^*(t) \leqq -\alpha(\delta_0)/2$ and this inequality gives a contradiction. Hence our statement is true.

(d) Due to the fact that our system is locally asymptotically stable, there exists $\tilde{\delta}_0$ such that $|x_i^0|\leqq\tilde{\delta}_0$ should imply $\lim_{t\to\infty} x_i(t; t_0, x_1^0, x_2^0, x_3^0, x_4^0)=0$. Taking into account the estimates obtained from singular perturbations, we see that, if T_1/β_1 and T_2/β_3 are sufficiently small, one can find such a δ_0 in order that $|x_1(t)| \leqq \delta_0$, $|x_2(t)| \leqq \delta_0$ should imply $|x_i(t)| \leqq \tilde{\delta}_0$, $i = 1, \cdots, 4$. But, as it has been already shown, one can find a $\hat{t} \geqq t_0$ such that $|x_1(\hat{t})| + |x_2(\hat{t})| < \delta_0$. This ends the proof.

5. Concluding remarks. As it has been shown by several authors (e.g., [5]), global stabilization of some bilinear control systems can be obtained by saturated quadratic feedback. An engineering application (steam turbine regulation) where the quadratic feedback is difficult to implement lead up to a special class of bilinear control systems, with nonnegative state variables, which can be stabilized by saturated linear feedback. In order to simplify the design, singular perturbations theory was used, but the proof of the correctness of this procedure could be accomplished only for the application. The extension to the whole class of systems discussed here is an open problem.

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